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Generalizations Of An Inequality Of Ramanujan Concerning Prime Counting Function*

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Abstract

We study some generalizations of the inequality $\pi(x)^2 < \frac{ex}{\log x}\pi(\frac{x}{e})$, which is due to Ramanujan. We also obtain some reverses for it under various conditions including the Riemann Hypothesis.

1 Introduction

A part of various conjectures and results of Ramanujan on the theory of prime numbers, are about the prime counting function $\pi(x)$, which as usual denotes the number of primes not exceeding x. On page 310 in Ramanujan's second notebook, he asserts that the inequality

$$\pi(x)^2 < \frac{ex}{\log x}\pi\left(\frac{x}{e}\right) \tag{1}$$

holds for x sufficiently large (see also [2]). This is not very hard to verify. Let us set $\ell := \log x$, for the whole text. We let

$$\varpi(\ell) := \frac{\pi(e^\ell)}{e^\ell} = \frac{\pi(x)}{x}.$$

Then, the prime number theorem with error term gives the expansion

$$\varpi(\ell) = \frac{1}{\ell}\left(\sum_{k=0}^{n}\frac{k!}{\ell^k} + O\Big(\frac{1}{\ell^{n+1}}\Big)\right) \tag{2}$$

for any integer $n \geq 0$. We note that $\pi(xe^a)/(xe^a) = \varpi(\ell + a)$, and the inequality (1) is equivalent to the following one

$$\varpi(\ell)^2 < \frac{\varpi(\ell-1)}{\ell}. \tag{3}$$

Considering the expansion of $\varpi(\ell)$ with $n = 4$, we see that

$$\varpi(\ell)^2 - \frac{\varpi(\ell-1)}{\ell} = -\frac{1}{\ell^6} + O\left(\frac{1}{\ell^7}\right) \qquad (\text{as } \ell \to \infty),$$

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from which we obtain validity of (3) for ℓ sufficiently large; furthermore, we obtain

$$\pi(x)^2 - \frac{ex}{\log x}\pi\left(\frac{x}{e}\right) = -\frac{x^2}{\log^6 x} + O\left(\frac{x^2}{\log^7 x}\right) \qquad (\text{as } x \to \infty),$$

and this proves (1) for x sufficiently large. But how much large?

To answer this question, we need some "very good" bounds for the prime counting function. Let us clear what we mean by "very good"!

DEFINITION 1. Let $n \geq 0$ be an integer. We say the function $\pi(x)$ satisfies the "Condition U-n", if the inequality $\ell\varpi(\ell) - \sum_{k=0}^{n} k!/\ell^k < c/\ell^{n+1}$ holds true for some $c > (n+1)!$, and for $x \geq x_0$.

In search of the integer $x_{\mathcal{R}}$ in which the inequality (1) holds for $x \geq x_{\mathcal{R}}$, and fails for $x < x_{\mathcal{R}}$, recently the author has proved the following conditional results, which are Theorem 1.1 and Theorem 1.2 of [1], respectively.

PROPOSITION 1. Assume that the function $\pi(x)$ satisfies the Condition U-4, and let $\varepsilon \in (0, 1/25)$. Then $x_{\mathcal{R}} \leq e^{\lambda}$ with

$$\lambda = \max\{2b(1+\varepsilon) + 73 + \varepsilon, 2.2 + 132/\varepsilon, \log x_0\} > 530.2.$$

PROPOSITION 2. Let $h = 138766146692471228$, and assume that the Riemann Hypothesis is true. Then, we have $x_{\mathcal{R}} \leq h$.

Our aim in this paper is to obtain further generalizations of the Ramanujan's inequality, and examine validity of them under Condition U-4, and assuming validity of the Riemann Hypothesis. Implication of the first generalization is based on the appearance of the function $\pi(x)$ on both sides of it. Indeed, repeated use of itself on its right hand side leads us to the following result.

THEOREM 1. For given positive integer n, we set

$$\Xi_n(x) := \prod_{k=1}^{n} \left(1 - \frac{k-1}{\log x}\right)^{2^{n-k}}. \tag{4}$$

Consider the following generalization of Ramanujan's inequality

$$\pi(x)^{2^n} < \frac{e^n}{\Xi_n(x)}\left(\frac{x}{\log x}\right)^{2^n - 1}\pi\left(\frac{x}{e^n}\right). \tag{5}$$

(i) Assume that the function $\pi(x)$ satisfies the Condition U-4, and let $\varepsilon \in (0, 1/25)$. Then the inequality (5) is valid for $x > e^{\lambda+n-1}$, with λ defined as in Proposition 1.

(ii) Assume that the Riemann hypothesis is true. Then the inequality (5) is valid for $x > e^{n-1}h$, with h defined as in Proposition 2.

We note that for $n = 1$, the inequality (5) becomes the Ramanujan inequality (1). In continuation, we do some numerical observations, which lead us to more generalizations of (1), as well as, to some inverses for it.

2 Proof of Theorem 1

We define the function $\mathcal{R}_n^{\Xi}(x)$ by setting

$$\mathcal{R}_n^{\Xi}(x) := \frac{e^n}{\Xi_n(x)} \left(\frac{x}{\log x}\right)^{2^n-1} \pi\left(\frac{x}{e^n}\right) - \pi(x)^{2^n}.$$

First, we observe that $\mathcal{R}_1^{\Xi}(x) > 0$ (Ramanujan's inequality) is valid for $x \geq x_{\mathcal{R}}$. Then, by induction on n we show that $\mathcal{R}_n^{\Xi}(x) > 0$ is valid for $x \geq e^{n-1}x_{\mathcal{R}}$. To do this, we assume that $n \geq 2$, and we square both sides of $\mathcal{R}_{n-1}^{\Xi}(x) > 0$, which is valid for $x \geq e^{n-2}x_{\mathcal{R}}$. Thus, we obtain

$$\pi(x)^{2^n} < \frac{e^{2n-2}\left(\frac{x}{\log x}\right)^{2^n-2}}{\prod_{k=1}^{n-1}\left(1-\frac{k-1}{\log x}\right)^{2^{n-k}}}\pi\left(\frac{x}{e^{n-1}}\right)^2, \qquad (x \geq e^{n-2}x_{\mathcal{R}}).$$

Validity of $\mathcal{R}_1^{\Xi}(x) > 0$ for $x \geq x_{\mathcal{R}}$ implies that

$$\pi\left(\frac{x}{e^{n-1}}\right)^2 < \frac{\frac{x}{\log x}}{e^{n-2}\left(1-\frac{n-1}{\log x}\right)}\pi\left(\frac{x}{e^n}\right), \qquad \left(\frac{x}{e^{n-1}} \geq x_{\mathcal{R}}\right).$$

If we combine the above inequalities, then, for $x \geq e^{n-1}x_{\mathcal{R}}$ we obtain

$$\pi(x)^{2^n} < \left(\frac{e^{2n-2}\left(\frac{x}{\log x}\right)^{2^n-2}}{\prod_{k=1}^{n-1}\left(1-\frac{k-1}{\log x}\right)^{2^{n-k}}}\right)\left(\frac{\frac{x}{\log x}}{e^{n-2}\left(1-\frac{n-1}{\log x}\right)^{2^{n-n}}}\pi\left(\frac{x}{e^n}\right)\right).$$

The right hand side of the last inequality is actually the right hand side of (5). So, we have completed induction's steps. Now, we note that under assumptions of Proposition 1, $x_{\mathcal{R}} \leq e^{\lambda}$. This completes the proof of (i). Implication of (ii) is similar under assumptions of Proposition 2, where $x_{\mathcal{R}} \leq h$.

3 Numerical Experiments and Further Generalizations

The factor $\Xi_n(x)$ equals 1 only for $n = 1$, and for $n \geq 2$ it satisfies $0 < \Xi_n(x) < 1$ for $x > e^n$. More precisely, for $n \geq 2$, by using the inequality $\log(1-t) \leq -t$, which is valid for $0 \leq t < 1$, we have

$$\log \Xi_n(x) = \sum_{k=1}^{n} 2^{n-k}\log\left(1-\frac{k-1}{\log x}\right) \leq \sum_{k=1}^{n} -2^{n-k}\left(\frac{k-1}{\log x}\right) = -\frac{2^n-n-1}{\log x}.$$

Thus, it is natural to ask about the validity of the inequality

$$\pi(x)^{2^n} < \frac{e^n}{\Theta_n(x)}\left(\frac{x}{\log x}\right)^{2^n-1}\pi\left(\frac{x}{e^n}\right)$$

where

$$\Theta_n(x) = e^{-\frac{2^n - n - 1}{\log x}}. \tag{6}$$

We consider the function

$$\mathcal{R}_n^{\Theta}(x) := \frac{e^n}{\Theta_n(x)} \left(\frac{x}{\log x}\right)^{2^n - 1} \pi\left(\frac{x}{e^n}\right) - \pi(x)^{2^n}.$$

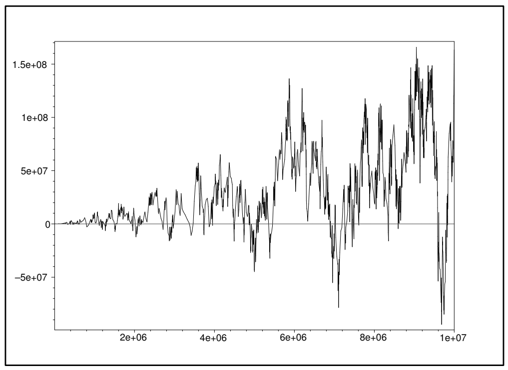

Figure 1: Graph of the function $\mathcal{R}_1^{\Theta}(x) = \mathcal{R}_1^{\Xi}(x)$ for $10^6 \le x \le 10^7$.

To determine the sign of $\mathcal{R}_n^{\Theta}(x)$, and also examine critical situation of (5), we study the functions $\mathcal{R}_n^{\Theta}(x)$ and $\mathcal{R}_n^{\Xi}(x)$ together. It is clear that $\mathcal{R}_1^{\Theta}(x) = \mathcal{R}_1^{\Xi}(x) := \mathcal{R}(x)$, say, and for $n \ge 2$ we have $\mathcal{R}_n^{\Theta}(x) < \mathcal{R}_n^{\Xi}(x)$. Figures 1 and 2 show the graph of functions $\mathcal{R}_n^{\Theta}(x)$ and $\mathcal{R}_n^{\Xi}(x)$ for some values of n.

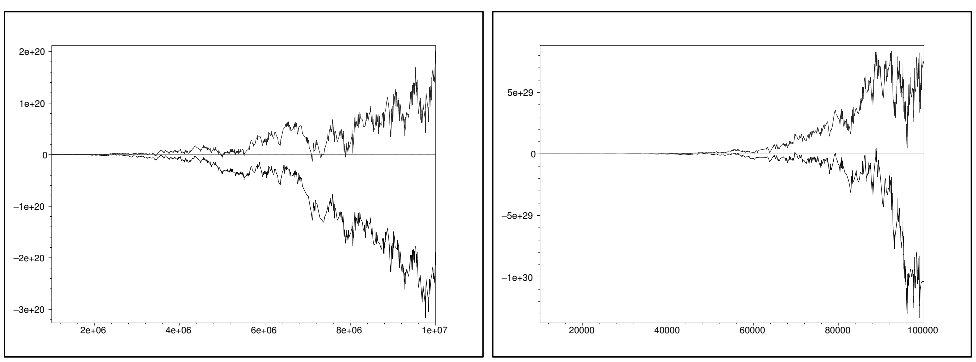

Figure 2: Graphs of the functions $\mathcal{R}_2^{\Theta}(x)$ and $\mathcal{R}_2^{\Xi}(x)$ for $10^6 \le x \le 10^7$ (left), and graphs of the functions $\mathcal{R}_3^{\Theta}(x)$ and $\mathcal{R}_3^{\Xi}(x)$ for $10^4 \le x \le 10^5$ (right).

For $n \ge 4$, the values of $\mathcal{R}_n^{\Theta}(x)$ become very large and some technical difficulties appear in generating figures. To cope with this problem, we consider some new functions involving logarithms of sides. Indeed, we define functions $\mathcal{L}_n^{\Xi}(x)$ and $\mathcal{L}_n^{\Theta}(x)$ by setting

$$\mathcal{L}_n^{\Xi}(x) := \log\left(\frac{e^n}{\Xi_n(x)}\left(\frac{x}{\log x}\right)^{2^n-1}\pi\left(\frac{x}{e^n}\right)\right) - 2^n\log\pi(x)$$

and

$$\mathcal{L}_n^{\Theta}(x) := \log\left(\frac{e^n}{\Theta_n(x)}\left(\frac{x}{\log x}\right)^{2^n-1}\pi\left(\frac{x}{e^n}\right)\right) - 2^n\log\pi(x).$$

As Figure 3 shows, it seems that $\mathcal{R}_n^{\Theta}(x) < 0$ for $n \geq 2$ and for x sufficiently large.

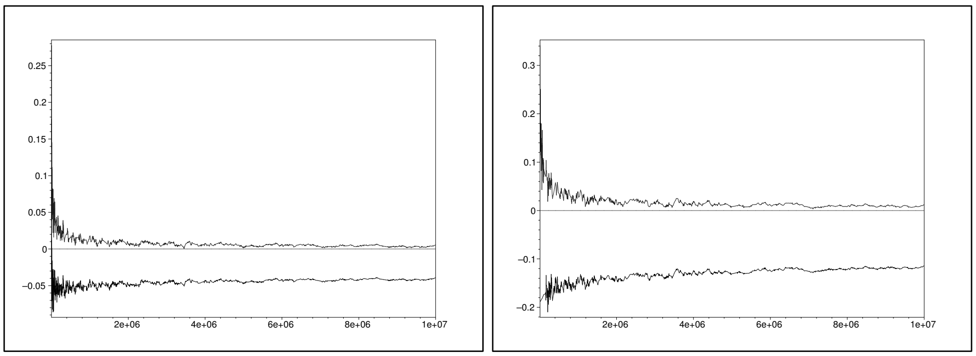

Figure 3: Graphs of the functions $\mathcal{L}_4^{\Theta}(x)$ and $\mathcal{L}_4^{\Xi}(x)$ for $10^6 \leq x \leq 10^7$ (left), and graphs of the functions $\mathcal{L}_5^{\Theta}(x)$ and $\mathcal{L}_5^{\Xi}(x)$ for $10^6 \leq x \leq 10^7$ (right).

The inequality $\mathcal{R}_2^{\Theta}(x) < 0$ is equivalent to

$$\frac{e^{\frac{1}{\ell}}}{\ell^3}\varpi(\ell-2) < \varpi(\ell)^4. \tag{7}$$

To verify the validity of this inequality, we consider the expansion of $\varpi(\ell)$ with $n = 2$, from which we obtain

$$\frac{e^{\frac{1}{\ell}}}{\ell^3}\varpi(\ell-2) - \varpi(\ell)^4 = -\frac{1}{2\ell^6} + O\left(\frac{1}{\ell^7}\right) \qquad (\text{as } \ell \to \infty).$$

Thus, we get (7) for ℓ sufficiently large, and consequently, we obtain validity of the inequality $\mathcal{R}_2^{\Theta}(x) < 0$ for x sufficiently large. More generally, the inequality $\mathcal{R}_n^{\Theta}(x) < 0$ is equivalent to

$$\frac{e^{\frac{2^n-n-1}{\ell}}}{\ell^{2^n-1}}\varpi(\ell-n) < \varpi(\ell)^{2^n}. \tag{8}$$

We deduce this general form by induction on n. To do this, we assume the validity of $\mathcal{R}_{n-1}^{\Theta}(x) < 0$ in the equivalent form

$$\frac{e^{\frac{2^{n-1}-(n-1)-1}{\ell}}}{\ell^{2^{n-1}-1}}\varpi(\ell-n-1) < \varpi(\ell)^{2^{n-1}} \qquad (n \geq 3),$$

and we square both sides to obtain

$$\frac{e^{\frac{2^n-2n-4}{\ell}}}{\ell^{2^n-2}}\varpi(\ell-n-1)^2 < \varpi(\ell)^{2^n},$$

from which we observe that to complete implication of (8), we should show

$$\frac{e^{\frac{2^n-n-1}{\ell}}}{\ell^{2^n-1}}\varpi(\ell-n) \le \frac{e^{\frac{2^n-2n-4}{\ell}}}{\ell^{2^n-2}}\varpi(\ell-n-1)^2,$$

which is equivalent to $e^{\frac{n+3}{\ell}}\varpi(\ell-n) \le \ell\varpi(\ell-n-1)^2$. By considering the truth of the following lemma, this last inequality is valid for ℓ (and consequently for x) sufficiently large.

LEMMA 1. Assume that $n \ge 1$ is an integer. There exists $x_0 > 0$ such that:

(i) For $x \ge x_0$, we have

$$\frac{x}{1+\log x}e^{\frac{4}{1+\log x}-2}\pi(x) \le \pi\left(\frac{x}{e}\right)^2. \tag{9}$$

(ii) For $x \ge e^n x_0$, we have

$$\frac{x}{\log x}e^{\frac{n+3}{\log x}-(n+2)}\pi\left(\frac{x}{e^n}\right) \le \pi\left(\frac{x}{e^{n+1}}\right)^2. \tag{10}$$

PROOF. *(i)* The inequality (9) is equivalent to $e^{\frac{4}{1+\ell}}\frac{\varpi(\ell)}{1+\ell} \le \varpi(\ell-1)^2$. By considering the expansion of $\varpi(\ell)$ with $n=2$ we see that

$$e^{\frac{4}{1+\ell}}\frac{\varpi(\ell)}{1+\ell} - \varpi(\ell-1)^2 = -\frac{8}{\ell^4} + O\left(\frac{1}{\ell^5}\right) \qquad (\text{as } \ell \to \infty).$$

Thus, (9) is valid for ℓ sufficiently large, and consequently for $x \ge x_0$, where $x_0 > 0$ is large enough.
(ii) Let us take $y := \frac{x}{e^n}$. Then, the inequality (10) is equivalent to

$$f(n,y)\pi(y) \le \pi\left(\frac{y}{e}\right)^2, \tag{11}$$

where

$$f(n,y) = \frac{y}{n+\log y}e^{\frac{n+3}{n+\log y}-2}.$$

We have

$$\frac{\partial}{\partial n}f(n,y) = -\frac{(n+3)ye^{-\frac{n-3+2\log y}{n+\log y}}}{(n+\log y)^3} < 0.$$

Thus, for $n \ge 1$ we imply $f(n,y) \le f(1,y)$, and this means that for proving (11), it is enough to show that $f(1,y)\pi(y) \le \pi\left(\frac{y}{e}\right)^2$, and this is what we have done in part (i) for $y \ge x_0$. This completes the proof of (10).

Therefore, we have proved the validity of the inequality $\mathcal{R}_n^{\Theta}(x) < 0$ for every $n \geq 2$, and for x sufficiently large. We state this result and a corollary of Theorem 1:

THEOREM 2. Assume that $n \geq 2$ is an integer. Recall the functions $\Xi_n(x)$ and $\Theta_n(x)$ defined by (4) and (6), respectively. Then, the following double side inequality

$$\frac{e^n}{\Theta_n(x)}\left(\frac{x}{\log x}\right)^{2^n-1}\pi\left(\frac{x}{e^n}\right) < \pi(x)^{2^n} < \frac{e^n}{\Xi_n(x)}\left(\frac{x}{\log x}\right)^{2^n-1}\pi\left(\frac{x}{e^n}\right)$$

is valid for x sufficiently large.

Finally, for arbitrary real number α, we consider the inequality

$$\frac{x}{\alpha+\log x}e^{\frac{4}{1+\log x}-2}\pi(x) \leq \pi\left(\frac{x}{e}\right)^2, \tag{12}$$

which is generalization of (9), and it is equivalent to $e^{\frac{4}{1+\ell}}\frac{\varpi(\ell)}{\alpha+\ell} \leq \varpi(\ell-1)^2$. Again, by considering the expansion of $\varpi(\ell)$ with $n=2$, we imply

$$e^{\frac{4}{1+\ell}}\frac{\varpi(\ell)}{\alpha+\ell} - \varpi(\ell-1)^2 = \frac{1-\alpha}{\ell^3} + \frac{\alpha^2-5\alpha-4}{\ell^4} + O_\alpha\left(\frac{1}{\ell^5}\right) \qquad (\text{as } \ell \to \infty).$$

Thus, for x sufficiently large, the inequality (12) is valid for $\alpha \geq 1$, and its reverse is valid for $\alpha < 1$.

References

[1] M. Hassani, On an inequality of Ramanujan concerning prime counting function, The Ramanujan Journal, 28(2012) 435–442.

[2] B. C. Berndt, Ramanujan's Notebook (Part IV), Springer-Verlag, 1994.