REMARKS ON UNIFORM DENSITY OF SETS OF INTEGERS

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Dedicated to the memory of Professor Péter Kiss

Abstract. The concept of the uniform density is introduced in papers [1], [2]. Some properties of this concept are studied in this paper. It is proved here that the uniform density has the Darboux property.

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Introduction

Let $A \subseteq N = \{1, 2, 3, \ldots\}$ and $m, n \in N$, $m < n$. Denote by $A(m, n)$ the cardinality of the set $A \cap [m, n]$. The numbers

$$\underline{d}(A) = \varliminf_{n\to\infty} \frac{A(1,n)}{n}, \qquad \bar{d}(A) = \varlimsup_{n\to\infty} \frac{A(1,n)}{n}$$

are called the lower and the upper asymptotic density of the set A. If there exists

$$d(A) = \lim_{n\to\infty} \frac{A(1,n)}{n}$$

then it is called the asymptotic density of A.

According to [1], [2] we set

$$\alpha_s = \min_{t\geq 0} A(t+1, t+s), \qquad \alpha^s = \max_{t\geq 0} A(t+1, t+s).$$

Then there exist

$$\underline{u}(A) = \lim_{s\to\infty} \frac{\alpha_s}{s}, \qquad \bar{u}(A) = \lim_{s\to\infty} \frac{\alpha^s}{s}$$

and they are called the lower and the upper uniform density of A, respectively.

It is obvious that for every $A \subseteq N$

$$\underline{u}(A) \leq \underline{d}(A) \leq \bar{d}(A) \leq \bar{u}(A).$$

Hence if $u(A)$ exists then $d(A)$ exists as well and $u(A) = d(A)$. The converse is not true. For example put

$$A = \bigcup_{k=1}^{\infty} \left\{10^k + 1, 10^k + 2, \dots, 10^k + k\right\}.$$

Then $d(A) = 0$, but $\underline{u}(A) = 0$, $\bar{u}(A) = 1$.

Note that the numbers α_s and α^s can be replaced by the numbers β_s and β^s, respectively, where

$$\beta_s = \varliminf_{t\to\infty} A(t+1, t+s), \quad \beta^s = \varlimsup_{t\to\infty} A(t+1, t+s)$$

(cf. [1], [2]).

In this paper we introduce some elementary remarks, observations on the concept of the uniform density and prove that this density has the Darboux property.

1. Uniform density $u(A)$ and $\lim\limits_{s\to\infty} \frac{A(t+1,t+s)}{s}$ (uniformly with respect to $t \geq 0$)

We introduce the following observation.

Theorem 1.1. *If there exists*

$$\lim_{s\to\infty} \frac{A(t+1, t+s)}{s} = L \tag{1}$$

uniformly with respect to $t \geq 0$, then there exists $u(A)$ and $u(A) = L$.

Proof. Let $\varepsilon > 0$. By the assumption there exists an $s_0 = s_0(\varepsilon) \in N$ such that for each $s > s_0$ and each $t \geq 0$ we have

$$(L - \varepsilon)s < A(t+1, t+s) < (L+\varepsilon)s.$$

By the definition of the numbers β_s, β^s we get from this for $s > s_0$

$$L - \varepsilon \leq \frac{\beta_s}{s} \leq \frac{\beta^s}{s} \leq L + \varepsilon.$$

If $s \to \infty$ we get

$$L - \varepsilon \leq \underline{u}(A) \leq \bar{u}(A) \leq L + \varepsilon.$$

Since $\varepsilon > 0$ is an arbitrary positive number, we get $u(A) = L$.

The foregoing theorem can be conversed.

Theorem 1.2. *If there exists $u(A)$ then*

$$\lim_{s \to \infty} \frac{A(t+1, t+s)}{s} = u(A)$$

uniformly with respect to $t \geq 0$.

Proof. Put $u(A) = L$. Since

$$L = \lim_{p \to \infty} \frac{\alpha_p}{p} = \lim_{p \to \infty} \frac{\alpha^p}{p}$$

for every $\varepsilon > 0$, there exists a p_0 such that for each $p > p_0$ we have

$$(L - \varepsilon)p < \alpha_p \leq \alpha^p < (L + \varepsilon)p.$$

So we get

$$(L - \varepsilon)p < \min_{t \geq 0} A(t+1, t+p) \leq \max_{t \geq 0} A(t+1, t+p) < (L + \varepsilon)p.$$

By the definition of $A(t+1, t+p)$ we get from this

$$\left| \frac{A(t+1, t+p)}{p} - L \right| \leq \varepsilon$$

for each $p > p_0$ and each $t \geq 0$. Hence

$$\lim_{p \to \infty} \frac{A(t+1, t+p)}{p} = L \quad (= u(A))$$

uniformly with respect to $t \geq 0$.

2. Uniform density and almost convergence

The concept of almost convergence was introduced in [5] (see also [10], p. 60). A sequence $(x_n)_1^\infty$ of real numbers almost converges to L if

$$\lim_{p \to \infty} \frac{x_{n+1} + x_{n+2} + \cdots + x_{n+p}}{p} = L$$

uniformly with respect to $n \geq 0$. If $(x_n)_1^\infty$ almost converges to L, we write
$$F - \lim x_n = L.$$

One can conjecture that there is a relationship between the uniform density of a set $A \subseteq N$ and the characteristic function χ_A of this set ($\chi_A(n) = 1$ if $n \in A$, $\chi_A(n) = 0$ if $n \in N \backslash A$).

Theorem 2.1. *Let $A \subseteq N$. Then $u(A) = v$ if and only if $F - \lim \chi_A(n) = v$.*

Proof. Let $t \geq 0$, $s \in N$. By the definition of the sequence $(\chi_A(n))_1^\infty$ we see that

$$\frac{A(t+1, t+s)}{s} = \frac{\chi_A(t+1) + \chi_A(t+2) + \cdots + \chi_A(t+s) - t}{s}.$$

The assertion follows from this equality by Theorem 1.1 and 1.2.

3. Another way for defining the uniform density of sets

If $A = \{a_1 < a_2 < \cdots < a_n < \cdots\} \subseteq N$ is an infinite set then it is well–known that
$$\underline{d}(A) = \varliminf_{n\to\infty} \frac{n}{a_n}, \quad \bar{d}(A) = \varlimsup_{n\to\infty} \frac{n}{a_n}$$
and
$$d(A) = \lim_{n\to\infty} \frac{n}{a_n}$$
(if $d(A)$ exists) (cf. [8], p. 247). A similar result can be stated also for the uniform density.

Theorem 3.1. *Let $A = \{a_1 < a_2 < \cdots < a_n < \cdots\} \subseteq N$ be an infinite set. Then $u(A) = L$ if and only if*

$$\lim_{p\to\infty} \frac{p}{a_{k+p} - a_{k+1}} = L \tag{2}$$

uniformly with respect to $k \geq 0$.

Proof. 1. Let $u(A) = L$. Consider that for $p \geq 2$

$$\frac{p}{a_{k+p} - a_{k+1}} = \frac{A(a_{k+1}, a_{k+p})}{a_{k+p} - a_{k+1}}.$$

By Theorem 1.2 (see (1)) the right-hand side converges by $p \to \infty$ (uniformly with respect to $k \geq 0$) to $u(A) = L$. Hence (2) holds.

2. Suppose that (2) holds (uniformly with respect to $k \geq 0$). By Theorem 1.1 it suffices to prove that
$$\lim_{p\to\infty} \frac{A(t+1, t+p)}{p} = L$$

uniformly with respect to $t \geq 0$.

We shall show it. Suppose in the first place that $t \geq a_1$. Then there exist $k, s \in N$ such that

$$a_k < t+1 \leq a_{k+1} < \cdots < a_{k+s} \leq t+p < a_{k+s+1}.$$

Then $A(t+1, t+p)$ equals to s and so

$$\frac{A(t+1,t+p)}{p} = \frac{s}{p}.$$

Further on the basis of choice of the numbers k, s we get

$$a_{k+s} - a_{k+1} \leq p-1 < a_{k+s+1} - a_k.$$

Therefore

$$\frac{s}{a_{k+s+1} - a_k + 1} < \frac{A(t+1,t+p)}{p} < \frac{s}{a_{k+s} - a_{k+1}}.$$

But $-a_k + 1 \leq -a_{k-1}$, so that

$$\frac{s}{a_{k+s+1} - a_k + 1} \geq \frac{s}{a_{k+s+1} - a_{k-1}} = \frac{s+3}{a_{k+s+1} - a_{k-1}} \frac{s}{s+3}$$
$$= \frac{s+3}{a_{k+s+1} - a_{k-1}} \left(1 - \frac{3}{s+3}\right).$$

So we get wholly

$$\frac{s+3}{a_{k+s+1} - a_{k-1}} \left(1 - \frac{3}{s+3}\right) < \frac{A(t+1,t+p)}{p} < \frac{s}{a_{k+s} - a_{k+1}}. \tag{3}$$

Let $\gamma > 0$. Then by assumption (see (2)) there exists a v_0 such that for each $v > v_0$ we have

$$-\gamma < \frac{v}{a_{k+v} - a_{k+1}} - L < \gamma \tag{4}$$

for all $k \geq 0$.

Using (4) we get from (3)

$$\frac{s+3}{a_{k+s+1} - a_{k-1}} - L - \frac{3}{a_{k+s+1} - a_{k-1}} < \frac{A(t+1,t+p)}{p} - L < \frac{s}{a_{k+s} - a_{k+1}} - L. \tag{5}$$

Let $s > v_0$. Then by (4) the right-hand side of (5) is less than γ. On the left–hand side we get

$$\frac{s+3}{a_{k+s+1} - a_{k-1}} - L > -\gamma.$$

Further

$$\frac{-3}{a_{k+s+1}-a_{k-1}} \geq \frac{-3}{s+2},$$

since

$$a_{k+s+1}-a_{k-1} = (a_k - a_{k-1}) + (a_{k+1}-a_k) + \cdots + (a_{k+s+1}-a_{k+s})$$

and each summand on the right-hand side is ≥ 1.

Hence for every $t \geq a_1$ we get from (5) ($s > v_0$)

$$-\gamma - \frac{3}{s+2} < \frac{A(t+1,t+p)}{p} - L < \gamma \tag{6}$$

From this

$$\lim_{p\to\infty} \frac{A(t+1,t+p)}{p} = L$$

uniformly with respect to $t \geq a_1$.

It remains the case if $0 \leq t < a_1$. Since there is only a finite number of such t's, it suffices to show that for each fixed t, $0 \leq t < a_1$, we have

$$\lim_{p\to\infty} \frac{A(t+1,t+p)}{p} = L. \tag{7}$$

If t is fixed, $0 \leq t < a_1$ and p is sufficiently large we can determine a k such that $a_k \leq t+p < a_{k+1}$. Then

$$0 \leq t < a_1 < a_2 < \cdots < a_k \leq t+p < a_{k+1}$$

and

$$A(t+1,t+p) = A(t+1,a_1) + A(a_2,a_k). \tag{8}$$

From this

$$p < a_{k+1}, \quad p > a_k - a_1 \tag{8$'$}$$

and so from (8), (8′) we obtain

$$\begin{aligned} \frac{A(t+1,a_1)}{p} + \frac{A(a_2,a_{k+1})-1}{a_{k+1}} &\leq \frac{A(t+1,t+p)}{p} \\ &\leq \frac{A(t+1,a_1)}{p} + \frac{k-1}{a_k-a_1}. \end{aligned} \tag{9}$$

Obviously we have $A(t+1, a_1) \le a_1$ and so

$$\frac{A(t+1, a_1)}{p} = o(1) \quad (p \to \infty).$$

We arrange the left-hand side of (9). We get

$$\frac{A(a_2, a_{k+1}) - 1}{a_{k+1}} = -\frac{1}{a_{k+1}} + \frac{k}{a_{k+1} - a_2}\frac{a_{k+1} - a_2}{a_{k+1}} = o(1) + \frac{k}{a_{k+1} - a_2}$$

(if $p \to \infty$ then $k \to \infty$, as well).

Wholly we have

$$\frac{k}{a_{k+1} - a_2} + o(1) \le \frac{A(t+1, t+p)}{p} \le \frac{k-1}{a_k - a_1} + o(1).$$

If $p \to \infty$, then $k \to \infty$ and by assumption (cf (2)) the terms

$$\frac{k-1}{a_k - a_1} - L, \quad \frac{k}{a_{k+1} - a_2} - L$$

converge to zero. But then (9) yields

$$\lim_{p \to \infty} \frac{A(t+1, t+p)}{p} = L$$

uniformly with respect to $t \ge 0$. So $u(A) = L$.

The following theorem is a simple consequence of Theorem 3.1

Theorem 3.2. *Let $A = \{a_1 < a_2 < \cdots\} \subseteq N$ be a lacunary set, i.e.*

$$\lim_{n \to \infty} (a_{n+1} - a_n) = +\infty. \tag{10}$$

Then $u(A) = 0$.

Proof. Let $\varepsilon > 0$. Choose $M \in N$ such that $M^{-1} < \varepsilon$. By the assumption there exists an n_0 such that for each $n > n_0$ we get $a_{n+1} - a_n > M$.

Let $k > n_0$, $s \in N$, $s > 1$. Then

$$a_{k+s} - a_{k+1} = (a_{k+2} - a_{k+1}) + (a_{k+3} - a_{k+2}) + \cdots + (a_{k+s} - a_{k+s-1}) > (s-1)M$$

and so

$$\frac{s}{a_{k+s} - a_{k+1}} < \frac{s}{(s-1)M} < 2\varepsilon.$$

Hence for each $k > n_0$ and $s \geq 2$ we have

$$\frac{s}{a_{k+s} - a_{k+1}} < 2\varepsilon.$$

If $0 \leq k \leq n_0$, k is fixed, then

$$\lim_{s\to\infty} \frac{s}{a_{k+s} - a_{k+1}} = 0, \tag{11}$$

since, for sufficiently large s

$$\begin{aligned} a_{k+s} - a_{k+1} &= [(a_{k+2} - a_{k+1}) + \cdots + (a_{n_0+1} - a_{n_0})] \\ &+ [(a_{n_0+2} - a_{n_0+1}) + \cdots + (a_{k+s} - a_{k+s-1})] > M(k + s - n_0 - 1) \\ &\geq M(s - (n_0 + 1)). \end{aligned}$$

There exists only a finite number of k's with $0 \leq k \leq n_0$, so we see that (11) holds uniformly with respect to k, $0 \leq k \leq n_0$. So we get wholly

$$\lim_{s\to\infty} \frac{s}{a_{k+s} - a_{k+1}} = 0$$

uniformly with respect to $k \geq 0$. So according to Theorem 3.1, $u(A) = 0$.

Remark. The assumption (10) in Theorem 3.2 cannot be replaced by the weaker assumption

$$\overline{\lim_{n\to\infty}} (a_{n+1} - a_n) = +\infty. \tag{10'}$$

This can be shown by the following example:

$$A = \bigcup_{k=1}^{\infty} \{k! + 1, k! + 2, \ldots, k! + k\} = \{a_1 < a_2 < \cdots < a_n < \cdots\}.$$

Here we have $\underline{u}(A) = 0$, $\bar{u}(A) = 1$ and (10′) is satisfied.

Example 3.1 Let $\alpha \in R$, $\alpha > 1$. Put $a_k = [k\alpha]$, $(k = 1, 2, \ldots)$, where $[v]$ denotes the integer part of v. We show that the uniform density of the set A is $\frac{1}{\alpha}$. This follows from Theorem 3.1, since

$$\lim_{p\to\infty} \frac{p}{a_{k+p} - a_{k+1}} = \frac{1}{\alpha}$$

uniformly with respect to $k \geq 0$. This uniform convergence can be shown by a simple calculation which gives the estimates ($p \geq 2$)

$$\frac{p}{(p-1)\alpha + 1} \leq \frac{p}{a_{k+p} - a_{k+1}} \leq \frac{p}{(p-1)\alpha - 1}.$$

4. Darboux property of the uniform density

For every $A \subseteq N$ having the uniform density the number $u(A)$ belongs to $[0,1]$. The natural question arises whether also conversely for every $t \in [0,1]$ there is a set $A \subseteq N$ such that $u(A) = t$. The answer to this question is positive.

Theorem 4.1.
If $t \in [0,1]$ then there is a set $A \subseteq N$ with $u(A) = t$.

Proof. We can already suppose that $0 < t < 1$. Construct the set

$$A = \left\{ \left[\frac{1}{t}\right], \left[\frac{2}{t}\right], \ldots, \left[\frac{k}{t}\right], \ldots \right\} = \{a_1 < a_2 < \cdots\}.$$

Put $a_k = [\frac{k}{t}]$ $(k = 1, 2, \ldots)$ and set in Example 3.1 $\alpha = \frac{1}{t} > 1$. So we get

$$\lim_{p \to \infty} \frac{p}{a_{k+p} - a_{k+1}} = \frac{1}{\alpha} = t$$

uniformly with respect to $k \geq 0$. The assertion follows by Theorem 3.1.

Let v be a non-negative set function defined on a class $S \subseteq 2^N$. The function v is said to have the Darboux property provided that if $v(A) > 0$ for $A \in S$ and $0 < t < v(A)$, then there is a set $B \subseteq A$, $B \in S$ such that $v(B) = t$ (cf. [6], [7], [9]).

Theorem 4.2. *The uniform density has the Darboux property.*

Proof. Let $u(A) = \delta > 0$,

$$A = \{a_1 < a_2 < \cdots < a_k < \cdots\}$$

and $0 < t < \delta$. Construct the set

$$B = \{b_1 < b_2 < \cdots < b_k < \cdots\}$$

in such a way that we set

$$b_k = a_{[k\frac{\delta}{t}]} \quad (k = 1, 2, \ldots).$$

Put $n_k = [k\frac{\delta}{t}]$ $(k = 1, 2, \dots)$. Then $n_1 < n_2 < \dots < n_k < \dots$,

$$B = \{a_{n_1} < a_{n_2} < \dots < a_{n_k} < \dots\}, \quad B \subseteq A.$$

We prove that $u(B) = t$.

By Theorem 3.1 it suffices to show that

$$\lim_{p \to \infty} \frac{p}{b_{m+p} - b_{m+1}} = t \tag{12}$$

uniformly with respect to $m \geq 0$.

We have $(p > 1)$

$$\frac{p}{b_{m+p} - b_{m+1}} = \frac{p}{a_{n_{m+p}} - a_{n_{m+1}}}.$$

By a simple arrangement we get

$$\frac{p}{b_{m+p} - b_{m+1}} = \frac{n_{m+p} - n_{m+1} + 1}{a_{n_{m+p}} - a_{n_{m+1}}} \frac{p}{n_{m+p} - n_{m+1} + 1}. \tag{13}$$

A simple estimation gives

$$(p-1)\frac{\delta}{t} - 1 < n_{m+p} - n_{m+1} < (p-1)\frac{\delta}{t} + 1.$$

Using this in (13) we get

$$\lim_{p \to \infty} \frac{p}{n_{m+p} - n_{m+1} + 1} = \frac{t}{\delta} \tag{14}$$

uniformly with respect to $m \geq 0$.

Further by assumption

$$\lim_{p \to \infty} \frac{p}{a_{s+p} - a_{s+1}} = \delta$$

uniformly with respect to $s \geq 0$ (Theorem 3.1).

So we get

$$\lim_{p \to \infty} \frac{n_{m+p} - n_{m+1} + 1}{a_{n_{m+p}} - a_{n_{m+1}}} = \delta \tag{15}$$

uniformly with respect to $m \geq 0$ since the sequence

$$\left(\frac{n_{m+p} - n_{m+1} + 1}{a_{n_{m+p}} - a_{n_{m+1}}} \right)_{p=2}^{\infty}$$

is a subsequence of the sequence

$$\left(\frac{p}{a_{s+p} - a_{s+1}}\right)_{p=1}^{\infty}.$$

By (13), (14), (15) we get (12) uniformly with respect to $m \geq 0$.

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