

On a sum involving powers of reciprocals of an arithmetical progression

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Submitted 25 September 2007; Accepted 9 December 2007

Abstract

Our purpose is to establish the following result: Let a and d be co-prime integers and $a, a + d, a + 2d, \dots, a + (k - 1)d$ ($k \geq 2$) be an arithmetical progression. Then for all integers $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$ the rational number $1/a^{\alpha_0} + 1/(a + d)^{\alpha_1} + \dots + 1/(a + (k - 1)d)^{\alpha_{k-1}}$ is never an integer. This result extends theorems of Taisinger (1915) and Kürschák (1918), and also generalizes a result of Erdős (1932).

Keywords: Harmonic sums, arithmetical progression, greatest prime factor.

In 1915, Taisinger proved that the harmonic number $H_n := 1 + \frac{1}{2} + \dots + \frac{1}{n}$ is never an integer except for H_1 . The more general result that the sum of reciprocals of consecutive terms, not necessarily starting with 1, is never an integer was proved by Kürschák in 1918 [3, p.157]. In 1932, Erdős proved that the sum of reciprocals of any integers in arithmetical progression is never a reciprocal and then an integer [2]. Our purpose is to give some extensions of the cited results.

Let n be a positive integer and p be a prime number. We define the p -valuation of n as the unique positive integer $v_p(n)$ satisfying $n = u \cdot p^{v_p(n)}$ with $\gcd(u, p) = 1$.

Our idea relies on the fundamental inequality about the valuation of a sum of two positive integers. Let n and m be integers. It is well known that $v_p(n + m) \geq \min\{v_p(n), v_p(m)\}$, with a remarkable implication that if $v_p(n) > v_p(m)$ then $v_p(n + m) = v_p(m)$.

The following Theorem is the key assertion behind all the results of this paper.

Theorem 1.1. *Let n_1, n_2, \dots, n_k be positive integers. Assume that there exists a prime P such that $v_P(n_{j_P})$ is maximal (non zero) for a unique $j_P \in \{1, 2, \dots, k\}$. Then*

$$\frac{1}{n_1} + \frac{1}{n_2} + \dots + \frac{1}{n_k}$$

is never an integer.

In fact this result is well-known and simple consequence of elementary properties of valuations (see [1]). However, for the convenience of the reader we give the proof of this statement.

Proof. Let us suppose that $N := \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k}$ is an integer. By setting $R := n_1 n_2 \cdots n_k / P^v$, where $v = 1 + \sum_{j \neq j_P} v_P(n_j)$, one has

$$RN - \sum_{j \neq j_P} \frac{R}{n_j} = \frac{R}{n_{j_P}}.$$

Each term of the left hand side is an integer, while the right hand side is not. It is contradiction, so the statement is proved. \square

We get the following as a simple and immediate consequence.

Corollary 1.2. Let n_1, n_2, \dots, n_k be positive integers. Assume that there exists a prime P such that $P \mid n_i$ for some i , and $P \nmid n_j$ when $j \neq i$. Then

$$\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k}$$

is never an integer.

The first main result of our paper is an extension of Taisinger's Theorem.

Theorem 1.3. Let n be an integer ≥ 2 and $\alpha_2, \dots, \alpha_n$ be positive integers. Then

$$1 + \frac{1}{2^{\alpha_2}} + \cdots + \frac{1}{n^{\alpha_n}}$$

is never an integer.

Proof. Let P be the greatest prime number $\leq n$. By Bertrand's postulate we have $n < 2P$. Thus P is coprime to all $k \in \{1, 2, \dots, n\} \setminus \{P\}$. The theorem follows then from Corollary 1.2. \square

To study the case of an arithmetical progression, we give the following result which is an immediate consequence of a theorem of Shorey and Tijdeman [4].

Theorem 1.4. Let a, d and k be positive integers, satisfying $\gcd(a, d) = 1$, $k \geq 2$.

By setting $\Delta = \prod_{j=1}^k (a + (j-1)d)$ and $P := \max_{p \mid \Delta} p$, the greatest prime factor of Δ , then for $d > 1$, we have $P \geq k$.

Now we are able to establish an extension of Erdős theorem, then of Kürschák's Theorem.

Theorem 1.5. *Let a, d and k be positive integers satisfying $k \geq 2$, and $a, a + d, a + 2d, \dots, a + (k - 1)d$ be an arithmetical progression. Then for all positive integers $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$ the rational number*

$$\frac{1}{a^{\alpha_0}} + \frac{1}{(a + d)^{\alpha_1}} + \dots + \frac{1}{(a + (k - 1)d)^{\alpha_{k-1}}}$$

is never an integer.

Proof. Let $\delta := \gcd(a, d)$. Consider the arithmetical progression $(a' + jd')$, $j = 0, \dots, k - 1$, where $a' = a/\delta$ and $d' = d/\delta$. For this progression, let P the prime given by Theorem 1.4. If $P \nmid \delta$, we conclude by using Corollary 1.2. Otherwise, we have

$$\frac{1}{a^{\alpha_0}} + \frac{1}{(a + d)^{\alpha_1}} + \dots + \frac{1}{(a + (k - 1)d)^{\alpha_{k-1}}} < \frac{k}{P} \leq 1.$$

□

Acknowledgement. The authors are grateful to the referee and would like to thank him/her for comments and several suggestions which improved the quality of this paper.

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