

# ON *M*-PROJECTIVE CURVATURE TENSOR OF A GENERALIZED SASAKIAN SPACE FORM

VENKATESHA AND B. SUMANGALA

ABSTRACT. In the present paper, we have studied *M*-projectively flat generalized Sasakian space form,  $\eta$ -Einstein generalized Sasakian space form and irrotational *M*-projective curvature tensor on a Sasakian space form.

#### 1. INTRODUCTION

A Riemannian manifold with constant sectional curvature C is known as real-space-form and its curvature tensor is given by

$$R(X,Y)Z = C\{g(Y,Z)X - g(X,Z)Y\}.$$

A Sasakian manifold  $(M, \phi, \xi, \eta, g)$  is said to be a Sasakian space form [3], if all the  $\phi$ -sectional curvatures  $K(X \wedge \phi X)$  are equal to a constant C, where  $K(X \wedge \phi X)$  denotes the sectional curvature of the section spanned by the unit vector field X, orthogonal to  $\xi$  and  $\phi X$ . In such a case, the

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Riemannian curvature tensor of M is given by

1.1)  

$$R(X,Y)Z = \frac{C+3}{4} \{g(Y,Z)X - g(X,Z)Y\} + \frac{C-1}{4} \{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + \frac{C-1}{4} \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}.$$

As a natural generalization of these manifolds, P. Alegre, D.E. Blair and A. Carriazo [3], [1] introduced the notion of generalized Sasakian space form.

Sasakian space form and Generalized Sasakian space form have been studied by several authors, viz., [3], [2], [6], [14], [10].

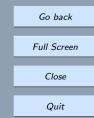
In 1971, G. P. Pokhariyal and R. S. Mishra [13] defined a tensor field  $W^*$  on a Riemannian manifold as

$$W^*(X,Y)Z = R(X,Y)Z - \frac{1}{4n}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]$$

(1.2)

Such a tensor field  $W^*$  is known as *M*-projective curvature tensor.

The properties of the *M*-projective curvature tensor in Sasakian and Kaehler manifold were studied by R. H. Ojha [11] [12]. He showed that it bridges the gap between the conformal curvature tensor, coharmonic curvature tensor and concircular curvature tensor. S. K. Chaubey and R. H. Ojha [8] studied the properties of the *M*-projective curvature tensor in Riemannian and Kenmotsu manifold. S. K. Chaubey [9] also studied the properties of *M*-projective curvature tensor



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in LP-Sasakian manifold. C. S. Bagewadi, E. Girish Kumar and Venkatesha [4] studied irrotational *D*-conformal curvature tensor in Kenmotsu and trans-Sasakian manifolds. C. S. Bagewadi, Venkatesha and N. S. Basavarajappa [5] proved that if pseudo projective curvature tensor in a LP-Sasakian manifold is irrotational, then the manifold is Einstein. Motivated by these ideas, in the present paper, we made an attempt to study the properties of *M*-projective curvature tensor in generalized Sasakian space form. The present paper is organized as follows.

In Section 2, we review some preliminary results. In Section 3, we study *M*-projectively flat generalized Sasakian space form and obtain necessary and sufficient conditions for a generalized Sasakian space form to be *M*-projectively flat. And in Section 4, we study  $\eta$ -Einstein generalized Sasakian space form satisfying  $W^*(\xi, X) \cdot R = 0$ . Finally in Section 5, we prove that *M*-projective curvature tensor in an  $\eta$ -Einstein generalized Sasakian space form is irrotational if and only if  $f_3 = \frac{3f_2}{(1-2n)}$ .

### 2. Preliminaries

An odd-dimensional Riemannian manifold (M, g) is called an almost contact manifold if there exists a (1, 1) tensor field  $\phi$ , a vector field  $\xi$  and a 1-form  $\eta$  on M, such that

(2.1) 
$$\phi^2(X) = -X + \eta(X)\xi,$$

(2.2) 
$$\eta(\phi X) = 0$$

(2.3) 
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

(2.4) 
$$\phi \xi = 0, \qquad \eta(\xi) = 0, \qquad g(X,\xi) = \eta(X),$$

).

for any vector fields X, Y on M.

If in addition,  $\xi$  is a Killing vector field, then M is said to be a K-contact manifold. It is well known that a contact metric manifold is a K-contact manifold if and only if

(2.5) 
$$(\nabla_X \xi) = -\phi(X)$$

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for any vector field X on M.

On the other hand, the almost contact metric structure on M is said to be normal if  $[\phi, \phi](X,Y) = -2d\eta(X,Y)\xi$  for any X, Y, where  $[\phi, \phi]$  denotes the Nijenhuis tensor of  $\phi$  given by

$$[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X,\phi Y] - \phi[\phi X,Y] - \phi[X,\phi Y]$$

A normal contact metric manifold is called a Sasakian manifold. It can be proved that Sasakian manifold is K-contact, and that an almost contact metric manifold is Sasakian if and only if

(2.6) 
$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X$$

Given an almost contact metric manifold  $(M, \phi, \xi, \eta, g)$ , we say that M is an generalized Sasakian space form if there exists three functions  $f_1$ ,  $f_2$  and  $f_3$  on M such that

2.7)  

$$R(X,Y)Z = f_{1}\{g(Y,Z)X - g(X,Z)Y\} + f_{2}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_{3}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}$$

for any vector fields X, Y, Z on M, where R denotes the curvature tensor of M. This kind of manifold appears as a natural generalization of the well-known Sasakian space form M(C), which can be obtained as particular cases of generalized Sasakian space form by taking  $f_1 = \frac{C+3}{4}$  and  $f_2 = f_3 = \frac{C-1}{4}$ .



Further in a (2n + 1)-dimensional generalized Sasakian space form, we have [1]

(2.8) 
$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi,$$

(2.9) 
$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y),$$

(2.10) 
$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3,$$

(2.11) 
$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y],$$

2.12) 
$$R(\xi, X)Y = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X]$$

(2.13) 
$$\eta(R(X,Y)Z) = (f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)],$$

(2.14) 
$$S(X,\xi) = 2n(f_1 - f_3)\eta(X).$$

## 3. M-projectively flat generalized Sasakian space form

For a (2n+1)-dimensional (n > 1) *M*-projectively flat generalized Sasakian space form, from (1.2), we have

(3.1) 
$$R(X,Y)Z = \frac{1}{4n} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY].$$

In view of (2.8) and (2.9), the equation (3.1) takes the form

$$R(X,Y)Z = \frac{1}{4n} [2(2nf_1 + 3f_2 - f_3)\{g(Y,Z)X - g(X,Z)Y\} - (3f_2 + (2n-1)f_3)\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi\}].$$



(3.2)



Using (2.7), the equation (3.2) reduces to

$$f_{1}\{g(Y,Z)X - g(X,Z)Y\} + f_{2}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_{3}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}$$

$$(3.3) = \frac{1}{4n}[2(2nf_{1} + 3f_{2} - f_{3})\{g(Y,Z)X - g(X,Z)Y\} - (3f_{2} + (2n-1)f_{3})\{\eta(Y)\eta(Z)X - \eta(X,Z)\eta(Y)\xi\}].$$

Replacing Z by  $\phi Z$  in (3.3), we obtain

$$(3.4) \begin{aligned} f_1\{g(Y,\phi Z)X - g(X,\phi Z)Y\} \\ &+ f_2\{g(X,\phi^2 Z)\phi Y - g(Y,\phi^2 Z)\phi X + 2g(X,\phi Y)\phi^2 Z\} \\ &+ f_3\{g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi\} \\ &= \frac{1}{4n}[2(2nf_1 + 3f_2 - f_3)\{g(Y,\phi Z)X - g(X,\phi Z)Y\} \\ &- (3f_2 + (2n-1)f_3)\{g(Y,\phi Z)\eta(X)\xi - g(X,\phi Z)\eta(Y)\xi\}]. \end{aligned}$$

Putting  $X = \xi$  in (3.4), we get

(3.5)

$$4nf_1g(Y,\phi Z)\xi - 4nf_3g(Y,\phi Z)\xi = [4nf_1 + 3f_2 - (1+2n)f_3]g(Y,\phi Z)\xi.$$

Simplifying (3.5), we get

(3.6)  $[(1-2n)f_3 - 3f_2]g(Y,\phi Z)\xi = 0.$ 





Since  $g(Y, \phi Z) \neq 0$ , it follows from (3.6) that

$$f_3 = \frac{3f_2}{(1-2n)}$$

Conversely, suppose that

(3.9)

$$f_3 = \frac{3f_2}{(1-2n)}$$

holds. Then in view of (2.7) and (2.9), we can write the equation (1.2) as

$$\dot{W}^{*}(X,Y,Z,W) = f_{2}\{g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W) + 2g(X,\phi Y)g(\phi Z,W)\} + f_{3}\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) + g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W) + g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\},$$

where 
$$\dot{W}^*(X, Y, Z, W) = g(W^*(X, Y)Z, W)$$
.  
Replacing X by  $\phi X$  and Y by  $\phi Y$  in (3.8), we get

$$\begin{split} \dot{W}^*(\phi X, \phi Y, Z, W) &= f_2 \{ g(\phi X, \phi Z) g(\phi^2 Y, W) - g(\phi Y, \phi Z) g(\phi^2 X, W) \\ &+ 2g(\phi X, \phi^2 Y) g(\phi Z, W) \} + f_3 \{ g(\phi Y, Z) g(\phi X, W) \\ &- g(\phi X, Z) g(\phi Y, W) \}. \end{split}$$



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Putting  $Y = W = e_i$  where  $\{e_i\}$ , is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over  $i \ (1 \le i \le 2n+1)$ , we get

(3.10)  
$$\sum_{i=1}^{2n+1} \dot{W}^*(\phi X, \phi e_i, Z, e_i) = f_2 \{ -g(\phi X, \phi Z)g(\phi e_i, \phi e_i) + g(\phi^2 Z, \phi^2 X) + 2g(\phi^2 X, \phi^2 Z) \} - f_3 g(\phi Z, \phi X).$$

Putting  $X = Z = e_i$ , where  $e_i$ , is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i  $(1 \le i \le 2n + 1)$ , we get after simplification that  $f_2 = 0$ . But then  $f_3 = 0$  by (3.7).

Therefore,

(3.11) 
$$R(X,Y)Z = f_1[g(Y,Z)X - g(X,Z)Y]$$

The above equation gives

(3.12) 
$$S(X,Y) = 2nf_1g(X,Y).$$

Hence in view of (1.2), we have  $W^*(X, Y)Z = 0$ . This leads us to state the following.

**Theorem 3.1.** A (2n+1)-dimensional (n > 1) generalized Sasakian space form is M-projectively flat if and only if  $f_3 = \frac{3f_2}{1-2n}$ .

But in [14], the author proved that if a (2n+1)-dimensional (n > 1) generalized Sasakian space form is Ricci semisymmetric, then  $f_3 = \frac{3f_2}{1-2n}$ . Hence we conclude the following.

**Corollary 3.1.** If a (2n + 1)-dimensional (n > 1) generalized Sasakian space form is Ricci semisymmetric, then it is M-projectively flat.



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4. An  $\eta$ -EINSTEIN GENERALIZED SASAKIAN SPACE FORM SATISFYING  $W^*(\xi, X)R = 0$ In view of (2.4), (2.8), (2.9) and (2.12), (1.2) becomes

(4.1) 
$$W^*(\xi, X)Y = \frac{1}{4n} [(1-2n)f_3 - 3f_2] \{g(X,Y)\xi - \eta(Y)X\}.$$

Now we have

(4.2) 
$$(W^*(\xi, X)R)(Y, Z)U = W^*(\xi, X)R(Y, Z)U - R(W^*(\xi, X)Y, Z)U - R(Y, W^*(\xi, X)Z)U - R(Y, Z)W^*(\xi, X)U.$$

But as we assume  $W^*(\xi, X)R = 0$ , (4.2) takes the form

(4.3) 
$$W^{*}(\xi, X)R(Y, Z)U - R(W^{*}(\xi, X)Y, Z)U - R(Y, W^{*}(\xi, X)Z)U - R(Y, Z)W^{*}(\xi, X)U = 0.$$

Using (2.4), (2.11), (2.12), (2.13) and (4.1) in (4.3), we get

$$\begin{aligned} &\frac{1}{4n} [(1-2n)f_3 - 3f_2] [\dot{R}(X, Y, Z, U)\xi + \eta(Y)R(X, Z)U \\ &+ \eta(Z)R(Y, X)U + \eta(U)R(Y, Z)X - (f_1 - f_3)\{g(Z, U)\eta(Y)X \\ &- g(Y, U)\eta(Z)X + g(X, Y)g(Z, U)\xi - g(X, Y)\eta(U)Z \\ &- g(X, Z)g(Y, U)\xi + g(X, Z)\eta(U)Y + g(X, U)\eta(Z)Y \\ &- g(X, U)\eta(Y)Z\}] = 0, \end{aligned}$$

where

(4.4)

(4.5)  $\dot{R}(X,Y,Z,U) = g(X,R(Y,Z)U).$ 



Taking inner product of (4.4) with respect to the Riemannian metric g and then using (2.4) and (2.13), we have

(4.6) 
$$\frac{1}{4n}[(1-2n)f_3 - 3f_2][\dot{R}(X, Y, Z, U) - (f_1 - f_3)\{g(X, Y)g(Z, U) - g(X, Z)g(Y, U)\}] = 0.$$

Then

$$f_3 = \frac{3f_2}{(1-2n)}$$

or

(4.7) 
$$\dot{R}(X,Y,Z,U) = (f_1 - f_3)\{g(X,Y)g(Z,U) - g(X,Z)g(Y,U)\}$$

Using (2.4) and (4.5) in (4.7), we get

(4.8) 
$$R(Y,Z)U = (f_1 - f_3)\{g(Z,U)Y - g(Y,U)Z\}$$

Contracting (4.8) with respect to the vector field Y, we find

(4.9) 
$$S(Z,U) = 2n(f_1 - f_3)g(Z,U)$$

Therefore,

(4.10) 
$$QZ = 2n(2n+1)(f_1 - f_3)Z.$$

Hence,

(4.11)

$$r = 2n(2n+1)(f_1 - f_3)$$
 and so  $f_3 = \frac{3f_2}{(1-2n)}$ 

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Thus, we state following theorem.

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**Theorem 4.1.** A (2n + 1)-dimensional (n > 1)  $\eta$ -Einstein generalized Sasakian space form satisfies the condition  $W^*(\xi, X)R = 0$  if and only if  $f_3 = \frac{3f_2}{(1-2n)}$ .

In the light of Theorems 3.1 and 4.1, we state next collorary.

**Corollary 4.1.** A (2n + 1)-dimensional (n > 1) generalized Sasakian space form satisfies the condition  $W^*(\xi, X)R = 0$  if and only if it is M-projectively flat.

### 5. The irrotational M-projective curvature tensor

**Definition 5.1.** The rotation (curl) of *M*-projective curvature tensor  $W^*$  on a Riemannian manifold is given by [1]

(5.1)  

$$\operatorname{Rot} W^* = (\nabla_U W^*)(X, Y)Z + (\nabla_X W^*)(U, Y)Z + (\nabla_Y W^*)(X, U)Z - (\nabla_Z W^*)(X, Y)U.$$

By virtue of second Bianchi identity, we have

$$(\nabla_U W^*)(X, Y)Z + (\nabla_X W^*)(U, Y)Z + (\nabla_Y W^*)(X, U)Z = 0.$$

Therefore, (5.1) becomes

(5.2) 
$$\operatorname{Rot} W^* = -(\nabla_Z W^*)(X, Y)U.$$

If the *M*-projective curvature tensor is irrotational, then  $\operatorname{curl} W^* = 0$ , and so by (5.2) we get

$$(\nabla_Z W^*)(X, Y)U = 0.$$

Thus,

(5.3)

$$(\nabla_Z W^*)(X, Y)U = W^*(\nabla_Z X, Y)U + W^*(X, \nabla_Z Y)U + W^*(X, Y)\nabla_Z U.$$



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Replacing  $U = \xi$  in (5.3), we have

(5.4) 
$$(\nabla_Z W^*)(X,Y)\xi = W^*(\nabla_Z X,Y)\xi + W^*(X,\nabla_Z Y)\xi + W^*(X,Y)\nabla_Z \xi.$$

Now, substituting  $Z = \xi$  in (1.2) and then using (2.4), (2.8), (2.11) and (2.14), we obtain

(5.5) 
$$(\nabla_Z W^*)(X,Y)\xi = k[\eta(Y)X - \eta(X)Y],$$

where

(5.6) 
$$k = \frac{1}{4n} [(1-2n)f_3 - 3f_2]$$

Using (5.5) in (5.4), we obtain

(5.7) 
$$W^*(X,Y)\phi Z = k[g(Z,\phi X)Y - g(Z,\phi Y)X].$$

Replacing Z by  $\phi Z$  in (5.7) and simplifying by using (2.1) and (2.3), we get

(5.8) 
$$W^*(X,Y)Z = k[g(Z,Y)X - g(Z,X)Y].$$

Also equations (1.2) and (5.8) give

(5.9) 
$$k[g(Z,Y)X - g(Z,X)Y] = R(X,Y)Z - \frac{1}{4n}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY].$$

Contracting the above equation with respect to the vector X and then using (5.6), we find (5.10)  $S(Y,Z) = 2n(f_1 - f_3)g(Y,Z),$ 

which gives

(5.11) 
$$r = 2n(2n+1)(f_1 - f_3)$$



In consequence of (1.2), (5.6), (5.8), (5.10) and (5.11) we can find

(5.12) 
$$R(X,Y)Z = -(f_1 - f_3)[g(Y,Z)X - g(X,Z)Y].$$

Therefore, we can state the following theorem.

**Theorem 5.1.** The *M*-projective curvature tensor in an  $\eta$ -Einstein generalized Sasakian space form is irrotational if and only if  $f_3 = \frac{3f_2}{(1-2n)}$ .

Theorem 4.1 together with Theorem 5.1 lead to the following corollaries.

**Corollary 5.1.** A (2n + 1)-dimensional (n > 1) generalized Sasakian space form satisfies the condition  $W^*(\xi, X)R = 0$  if and only if the M-projective curvature tensor is irrotational.

**Corollary 5.2.** A (2n+1)-dimensional (n > 1) generalized Sasakian space form is irrotational if and only if it is M-projectively flat.



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Venkatesha, Department of Mathematics, Kuvempu University, Shankaraghatta-577 451, Shimoga, Karnataka, India, *e-mail*: vensmath@gmail.com

B. Sumangala, Department of Mathematics, Kuvempu University, Shankaraghatta-577 451, Shimoga, Karnataka, India, *e-mail:* suma.srishaila@gmail.com



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