Acta Math. Univ. Comenianae
Vol. LXII, 1(1993), pp. 117–121

ENTROPY–MINIMALITY

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In this note, we introduce a dynamical property of continuous maps, which we call **entropy-minimality**, lying between minimality and topological transitivity. We pay special attention to maps of the interval, showing that topological transitivity implies entropy-minimality for piecewise monotone maps but not for maps of the interval in general.

Let $f\colon X \to X$ be a continuous self-map of a compact metric space. Recall that f is **minimal** if the only nonempty, closed, f-invariant subset of X is X itself, and f is **topologically transitive** if the only closed, f-invariant subset of X with nonempty interior is X itself. We say that f is **entropy-minimal** if the only nonempty, closed, f-invariant subset Y of X such that $\operatorname{ent}(f|Y) = \operatorname{ent}(f)$ is $Y = X$. (Here $\operatorname{ent}(\cdot)$ denotes topological entropy [**AKM**].)

Clearly every minimal map is entropy-minimal. The converse is false. Any topologically transitive, piecewise monotone map of the interval provides a counterexample (see Theorem 2 below), as does any infinite, topologically transitive shift of finite type.

Theorem 1. *Every entropy-minimal map is topologically transitive.*

Proof. Let $f\colon X \to X$ be an entropy-minimal map. Let $\Omega = \Omega(f)$ denote the nonwandering set of f, defined by $x \in \Omega$ if and only if for every open set U containing x, there exists $n \geq 1$ such that $f^n(U) \cap U \neq \emptyset$. Ω is nonempty, closed, f-invariant, and [**W**, Corollary 8.6.1(iii)] $\operatorname{ent}(f) = \operatorname{ent}(f|\Omega)$. Therefore $\Omega = X$. By [**GH**, Theorem 7.21], $\Omega(f^n) = X$ for every $n \geq 1$.

We use the following equivalent formulation of topological transitivity: f is topologically transitive if and only if for every nonempty open set U, $\operatorname{cl} \cup_{n\geq 1} f^n(U) = X$. For ease of notation, if E is a subset of X, we write E^* in place of $\operatorname{cl} \cup_{n\geq 1} f^n(E)$. If f is not topologically transitive, there exists a nonempty open set U such that $U^* \neq X$. Let $V = X - U^*$. Since $X = U^* \cup V^*$, we have $\operatorname{ent}(f) = \max\{\operatorname{ent}(f|U^*), \operatorname{ent}(f|V^*)\}$ [**AKM**, Theorem 4]. Since $U^* \neq X$, we have $\operatorname{ent}(f|U^*) < \operatorname{ent}(f)$. Therefore $\operatorname{ent}(f|V^*) = \operatorname{ent}(f)$ and hence $V^* = X$.

Received May 28, 1992.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 58F08, 26A18; Secondary 54H20.

From the equivalent formulation of topological transitivity, there exists $n \geq 1$ such that $f^n(V) \cap U \neq \emptyset$. Let W be a nonempty open subset of V such that $f^n(W) \subseteq U$. Then $f^{kn}(W) \subseteq U^*$ for every $k \geq 1$. Since $U^* \cap V = \emptyset$, we have $f^{kn}(W) \cap W \subseteq f^{kn}(W) \cap V = \emptyset$ for every $k \geq 1$. But then no point of W is in $\Omega(f^n)$. □

We now turn to the question: when does topological transitivity imply entropy-minimality? Recall that an f-invariant, Borel probability measure μ on X is called a **measure of maximal entropy** if $\mathrm{ent}_\mu(f) = \mathrm{ent}(f)$. Here $\mathrm{ent}_\mu(f)$ denotes the measure-theoretic entropy [**W**] of the system (X, f, μ).

Theorem 2. *Every topologically transitive, piecewise monotone map of the interval is entropy-minimal.*

Proof. Let $f\colon [a,b] \to [a,b]$ be such a map. By [**P**, Corollary 3], f is topologically conjugate to a piecewise linear map, each of whose linear pieces has slope $\pm\beta$, where $\mathrm{ent}(f) = \log\beta$. Without loss of generality, we may assume that f itself has this property and hence satisfies the hypotheses of [**H**]. By [**H**, Theorem 8], f has a unique measure μ of maximal entropy and μ is positive on nonempty open sets.

Let $a = a_0 < \cdots < a_n = b$, where the intervals $[a_{i-1}, a_i]$ are maximal with respect to "f is monotone on J", and let $A = \{a_1, \ldots, a_{n-1}\}$. For $x \in [a,b] - \cup_{j\geq 0} f^{-j}(A)$, define $\varphi(x) \in \prod_0^\infty \{1, \ldots, n\}$ by $[\varphi(x)]_j = i$ if and only if $f^j(x) \in [a_{i-1}, a_i]$. The map φ^{-1} is uniformly continuous on $\varphi([a,b] - \cup_{j\geq 0} f^{-j}(A))$ and so extends to a continuous map ψ from $\Sigma = \mathrm{cl}\ \varphi(x \in [a,b] - \cup_{j\geq 0} f^{-j}(A))$ onto $[a,b]$. Then $\#\psi^{-1}(x) = 1$ or 2 for every $x \in [a,b]$ and $\psi \circ \sigma = f \circ \psi$, where σ is the shift on Σ.

Let X be a closed, f-invariant subset of $[a,b]$ and let $\Sigma' = \psi^{-1}(X)$. Then [**W**, Theorems 8.2, 8.7(v)] $\sigma|\Sigma'$ has a (not necessarily unique) measure ν' of maximal entropy. Let ν be the measure defined on X by $\nu(E) = \nu'(\psi^{-1}(E))$. Then $\mathrm{ent}(f|X) = \mathrm{ent}(\sigma|\Sigma') = \mathrm{ent}_{\nu'}(\sigma|\Sigma') = \mathrm{ent}_\nu(f|X)$, the first and last equalities because finite-to-one factor maps preserve topological entropy [**B**, Theorem 17], [**NP**, Corollary to Lemma 1]. Extend ν to all of $[a,b]$ by defining $\nu([a,b] - X) = 0$. If $X \neq [a,b]$, then $\nu \neq \mu$, and so $\mathrm{ent}_\nu(f|X) < \mathrm{ent}_\mu(f) = \mathrm{ent}(f)$. □

The proof above contains the easy proof of the following statement: **if a shift has a unique measure of maximal entropy, then the restriction of the shift to the support of this measure is entropy-minimal and has the same entropy as the original shift.** The converse is false: consider any minimal shift with entropy zero which has more than one invariant measure. See, for example, [**O**].

Below is an example which shows that Theorem 2 need not hold if the map is not piecewise monotone. Our example is a modified version of the map constructed by M. Barge and J. Martin [**BM**, Example 3]. It is defined on $[0,1]$ and has the property that for every $\varepsilon > 0$, there is a closed, f-invariant set $X_\varepsilon \subseteq [0, \varepsilon]$ such that $\operatorname{ent}(f|X_\varepsilon) = \operatorname{ent}(f)$. B. Gurevich and A. Zargaryan [**GZ**] used a similar construction to produce a map of the interval with no entropy-maximizing measure.

Example. Let (a_n) be a doubly infinite increasing sequence such that $\lim_{n\to-\infty} a_n = 0$ and $\lim_{n\to\infty} a_n = 1$. Let $f\colon [0,1] \to [0,1]$ be a map such that $f(0) = 0, f(1) = 1$, and for all n, $f(a_n) = (a_n)$ and f maps $[a_n, a_{n+1}]$ piecewise linearly onto $[a_{n-1}, a_{n+2}]$ with three linear pieces, as in Figure 1.

Figure 1.

As in [**BM**], it is easy to show that f is topologically transitive. We show that $\operatorname{ent}(f) = \log 5$ and that f is not entropy-minimal.

For $k = 2, 3, \dots$, let

$$X_k = \{x \in [0,1] : f^i(x) \in [a_{-k}, a_k] \quad \text{for } i = 0, 1, \dots\}.$$

Then $\operatorname{ent}(f) \geq \limsup_{k\to\infty} \operatorname{ent}(f|X_k)$, and $\operatorname{ent}(f|X_k) = \operatorname{ent}(f_k)$, where $f_k\colon [0,1] \to [0,1]$ is defined by

$$f_k(x) = \begin{cases} a_{-k}, & \text{if } f(x) \leq a_{-k}; \\ f(x), & \text{if } a_{-k} \leq f(x) \leq a_k; \\ a_k, & \text{if } f(x) \geq a_k. \end{cases}$$

Since $f_k \to f$ and entropy is C^0 lower semicontinuous [**M**, Theorem 2], $\operatorname{ent}(f) \leq \liminf_{k\to\infty} \operatorname{ent}(f_k)$. It follows that $\operatorname{ent}(f) = \lim_{k\to\infty} \operatorname{ent}(f_k)$.

Now $f_k = f$ on $[a_{-k+1}, a_{k-1}]$, and on $[a_{-k}, a_{-k+1}]$ and $[a_{k-1}, a_k]$, the graphs of f_k are as in Figure 2.

Figure 2.

By [**ALM**, Theorem 4.4.5], $\operatorname{ent}(f_k)$ is the logarithm of the spectral radius, denoted $\rho(\cdot)$, of the $(2k+1)\times(2k+1)$ matrix $B_k = (b_{i,j})$, indexed by $\{-k, \dots, k\}$ and defined by

$$\begin{aligned} b_{i,i} &= 1\,, \\ b_{i,i-1} &= b_{i,i+1} = 2\,, \\ b_{i,j} &= 0 \quad \text{otherwise.} \end{aligned}$$

We show that $5 - \frac{4}{k+1} \le \rho(B_k) \le 5$, from which it follows that $\operatorname{ent}(f) = \log 5$. We use the fact from Perron-Frobenius theory (see, for example, [**S**]) that for any irreducible nonnegative matrix B and any positive vector $\mathbf{v} = (v_i)$,

$$\min_i \frac{(B\mathbf{v})_i}{v_i} \le \rho(B) \le \max_i \frac{(B\mathbf{v})_i}{v_i}\,.$$

It is clear that B_k is irreducible. Setting $v_i = 1$ gives $\rho(B_k) \le 5$. To prove the other inequality, set

$$v_i = \begin{cases} k+1+i, & i \le 0\,; \\ k+1-i, & i \ge 0\,. \end{cases}$$

Then

$$\frac{(B\mathbf{v})_i}{v_i} = \begin{cases} 5, & i \ne 0; \\ 5 - \frac{4}{k+1}\,, & i = 0\,. \end{cases}$$

To show that f is not entropy-minimal, let

$$X = \{x \in [0,1] : f^i(x) \leq a_0 \quad \text{for } i = 0, 1, \dots\}.$$

As above, $\operatorname{ent}(f|X) = \log 5$.

Replacing a_0 by a_{-m} in the definition of X yields the statement that the entropy of f is concentrated on arbitrarily small closed intervals containing 0.

Acknowledgement. The authors thank A. Blokh and F. Hofbauer for useful conversations.

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