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SUBMANIFOLD GEOMETRY AND HESSIANS ON THE PSEUDORIEMANNIAN MANIFOLD OF METRICS

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Abstract. Submanifolds of the manifold of metrics $\mathcal{M}$ appear in several contexts in differential geometry such as in the theory of Einstein metrics, the Yamabe problem and Teichmüller theory. Using the natural family of pseudometrics G^c on the manifold of metrics from [**GMN92**], I have tried to describe the pseudo-riemannian geometry of the relevant submanifolds of $\mathcal{M}$. They will be described as maximal integral submanifolds. Submanifold charts and formulas for the second fundamental forms and the induced connections will be given. In the conformal class, which is geodesically closed, also the geodesic distance is studied.

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Submanifolds of the manifold of metrics $\mathcal{M}$ appear in several contexts in differential geometry such as in the theory of Einstein metrics, the Yamabe problem and Teichmüller theory. Using the natural family of pseudometrics G^c on the manifold of metrics from [**GMN92**], I have tried to describe the pseudo-riemannian geometry of the relevant submanifolds of $\mathcal{M}$. They will be described as maximal integral submanifolds. Submanifold charts and formulas for the second fundamental forms and the induced connections will be given. In the conformal class, which is geodesically closed, also the geodesic distance is studied. For that I have frequently made use of the fact that the classical submanifold equations of Riemannian geometry hold also on manifolds modeled over convenient spaces in the sense of [**KrM92**] — as long as the submanifold admits a projection onto its tangent bundle (for the straight forward proof see [**Neu92**]).

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All functionals which will be studied are given by integration over the base manifold. For non compact manifolds they have distributional densities as their analogs. This is explained in [**Neu92**]. Consider, for example, the total scalar curvature, which is not defined as a function on non compact manifolds. But it

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has a natural well defined candidate for its derivative along a curve c such that for all t, $c(t) = c(0)$ outside some compact subset of the manifold. This fits exactly to the topology on $\mathcal{M}$.

The geodesic equation for the submanifold of metrics with constant volume and for the conformal class with constant volume will be computed, but I could not solve it. Even worse seems to be this problem for the Teichmüller space. There I could not even derive the geodesic equation.

It is shown which metric corresponds to the metric of Teichmüller theory on the manifold of all almost complex structures over a Riemannian surface. For the moduli space this was already done before [**FiT82**].

For the total scalar curvature I have proceeded as follows: On the whole manifold of metrics the ellipticity of the arising partial differential operator on $\mathcal{M}$ is studied; and in the conformal class light like and degenerate directions of the Hessian are studied. For the latter problem full characterization seems to be possible only in dimension 4.

I will follow the notation of [**GMN92**].

1. Submanifold Geometry on $\mathcal{M}$

1.1. Let M be a smooth and finite dimensional manifold without boundary. Denote by $\mathcal{M}$ the set of metrics on M. Let S^2M denote the vector bundle of symmetric $(0,2)$-tensors on M. $\mathcal{M}$ is a cone in the vector space of symmetric $(0,2)$–tensor fields $\mathcal{C}^\infty(S^2M)$. It is open therein iff M is compact. At any rate $\mathcal{M}$ is a smooth manifold modeled on $\mathcal{C}^\infty_c(S^2M)$, the space of symmetric $(0,2)$-tensor fields with compact support.

1.2. For $c \in \mathbb{R}$, $c \neq 0$ there is a non degenerate bilinear form G^c on $\mathcal{M}$ defined by

$$G^c_g = \int_M \big(\operatorname{Tr}(H_0K_0) + c\operatorname{Tr}(H)\operatorname{Tr}(K)\big)\operatorname{vol}(g)$$

with the $(1,1)$-tensor fields $H = g^{-1}h, K = g^{-1}k$ and where H_0, K_0 denotes the traceless parts of these tensor fields. Write G instead of $G^{\frac{1}{n}}$. If $c > 0$ then G^c defines a (weak) metric, for $c < 0$ a (weak) pseudometric with pointwise signature (number of negative eigenvalues -1: As the signature depends continuously on c, it is constant on $c > 0$ and $c > 0$ respectively. If $c > 0$ this follows from $\operatorname{Tr}(H_0K_0) + \frac{1}{n}\operatorname{Tr}(HK) = \operatorname{Tr}(HK)$. For $c < 0$ consider $c = \frac{1}{n} - 1$; then the integrand of G^c is $\operatorname{Tr}(H_0K_0) + (\frac{1}{n} - 1)\operatorname{Tr}(HK) = \operatorname{Tr}(HK) - \operatorname{Tr}(H)\operatorname{Tr}(K)$; note that on the symmetric forms with zeros at the diagonal this is positive definite and on the space of diagonal matrices it has signature 1 and they are orthogonal to each other.

If $a, b \in \mathbb{R} \setminus \{0\}$, then

$$G^b(h,k) = G^a(h - \frac{a-b}{an}\operatorname{Tr}(g^{-1}h)g, k). \tag{1.2.1}$$

1.3. In [**GMN92**] it was shown that the geodesic in $(\mathcal{M}, G^c)$ starting at g^0 in direction h is given by

$$\operatorname{Exp}^c_{g^0}(th) = g^0 e^{(a(t)Id + b(t)H_0)},$$

where $a(t) = a_{c,H}(t)$ and $b(t) = b_{c,H}(t)$ in $C^\infty(M, \mathbb{R})$ are defined as follows:

$$a(t) = \frac{2}{n} ln((1 + \frac{t}{4}\operatorname{Tr}(H))^2 + t^2 \frac{c^{-1}}{16}\operatorname{Tr}(H_0^2))$$

$$b(t) = \begin{cases} \frac{4}{\sqrt{c^{-1}\operatorname{Tr}(H_0^2)}} \arctan(\frac{t\sqrt{c^{-1}\operatorname{Tr}(H_0^2)}}{4 + t\operatorname{Tr}(H)}) & \text{for } c^{-1}\operatorname{Tr}(H_0^2) > 0 \\ \frac{4}{\sqrt{-c^{-1}\operatorname{Tr}(H_0^2)}} \operatorname{Artanh}(\frac{t\sqrt{-c^{-1}\operatorname{Tr}(H_0^2)}}{4 + t\operatorname{Tr}(H)}) & \text{for } c^{-1}\operatorname{Tr}(H_0^2) < 0 \\ \frac{t}{1 + \frac{t}{4}\operatorname{Tr}(H)} & \text{for } \operatorname{Tr}(H_0^2) = 0 \end{cases}$$

Here arctan is taken to have values in $(-\frac{\pi}{2}, \frac{\pi}{2})$ for the points of the basis manifold, where $\operatorname{Tr}(H) \geq 0$. Else we define

$$\arctan(\frac{t\sqrt{c^{-1}\operatorname{Tr}(H_0^2)}}{4 + t\operatorname{Tr}(H)}) = \begin{cases} \arctan \text{ in } [0, \frac{\pi}{2}) & \text{for } t \in [0, -\frac{4}{\operatorname{Tr}(H)}) \\ \frac{\pi}{2} & \text{for } t = -\frac{4}{\operatorname{Tr}(H)} \\ \arctan \text{ in } (\frac{\pi}{2}, \pi) & \text{for } t \in (-\frac{4}{\operatorname{Tr}(H)}, \infty). \end{cases}$$

1.4. Submanifolds of $\mathcal{M}$. For each $g \in M$ we have the decomposition of $T_g\mathcal{M} = \mathcal{C}^\infty(S^2M)$

$$T_g\mathcal{M} = \mathbb{R}g \oplus C_0^g(M, \mathbb{R})g \oplus \mathcal{C}^\infty(S^2_gM) \tag{DEC}$$

where $C_0^g(M,\mathbb{R}) = \{f \in C^\infty(M,\mathbb{R})\colon \int_M f \operatorname{vol}(g) = 0\}$ and $\mathcal{C}^\infty(S^2_gM) = \{h \in \mathcal{C}^\infty(S^2M)\colon \operatorname{Tr}_g(h) = 0\}$. This decomposition is G^c-orthogonal for all $c \neq 0$: Obviously $G^c\big(C_0^g(M,\mathbb{R})g, \mathcal{C}^\infty(S^2_gM)\big) = 0$ and if $r \in \mathbb{R}, f \in C_0^g(M,\mathbb{R})$ then $G^c(rg, fg) = cn^2 \cdot r\int_M f \operatorname{vol}(g) = 0$.
The corresponding sections of orthogonal projections $\pi_i \in \mathcal{C}^\infty(\operatorname{End}(T\mathcal{M}))$ are

$$\begin{aligned} &\pi_1(g)\colon \mathcal{C}^\infty(S^2M) \twoheadrightarrow \mathbb{R}g, && h \mapsto \frac{1}{n}\big(\frac{1}{\operatorname{Vol}(g)}\int_M \operatorname{Tr}_g(h)\operatorname{vol}(g)\big)g \\ &\pi_2(g)\colon \mathcal{C}^\infty(S^2M) \twoheadrightarrow C_0^g(M,\mathbb{R})g, && h \mapsto \frac{1}{n}\big(\operatorname{Tr}_g(h) - \frac{1}{\operatorname{Vol}(g)}\int_M \operatorname{Tr}_g(h)\operatorname{vol}(g)\big)g \\ &\pi_3(g)\colon \mathcal{C}^\infty(S^2M) \twoheadrightarrow \mathcal{C}^\infty(S^2_gM), && h \mapsto h - \frac{1}{n}\operatorname{Tr}_g(h)g. \end{aligned}$$

and

$$\pi_T(g) = \pi_1 \oplus \pi_2 \colon \mathcal{C}^\infty(S^2M) \to C^\infty(M,\mathbb{R})g$$

(DEC) defines several smooth distributions of $T\mathcal{M}$ given by sums of π_i's. In each case maximal integral manifolds through each $g \in M$ exist and are as follows:

Distribution:	Maximal integral submanifold through g:
$\pi_1(T\mathcal{M})$	$\mathbb{R}^+ g$
$\pi_2(T\mathcal{M})$	$\mathrm{Conf}^g_0 = \{fg \colon f > 0, \int_M f\,\mathrm{vol}(g) = 0\}$
$\pi_3(T\mathcal{M})$	$\mathcal{M}_{\mathrm{vol}(g)} = \{b \in \mathcal{M} \colon \mathrm{vol}(b) = \mathrm{vol}(g)\}$
$\pi_1(T\mathcal{M}) \oplus \pi_2(T\mathcal{M})$	$\mathrm{Conf}^g = \{fg \colon f \in C^\infty(M,\mathbb{R}), f > 0\}$
$\pi_2(T\mathcal{M}) \oplus \pi_3(T\mathcal{M})$	$\mathcal{M}_{\mathrm{Vol}(g)} = \{b \in \mathcal{M} \colon \mathrm{Vol}(b) = \mathrm{Vol}(g)\}$

1.5. Submanifold charts. There are obvious charts

$$\mathbb{R}^+ \cdot g \ni rg \to r \in R^+$$
$$\mathrm{Conf}^g \ni pg \to p \in \{f \in C^\infty(M,\mathbb{R}) \colon f > 0\}$$
$$\mathrm{Conf}^g_0 \ni pg \to p \in \{f \in C^\infty(M,\mathbb{R}) \colon f > 0, \int_M f\,\mathrm{vol}(g) = 0\}$$

For $g_0 \in \mathcal{M}_{\mathrm{vol}(g)}$ consider the map $\phi^{g_0} \colon g \mapsto \ln(g_0^{-1}g) \in \mathcal{C}^\infty(\mathrm{End}(TM))$ on a neighborhood $U(g_0) \in \mathcal{M}$ of g_0, where it is a homeomorphism. Its inverse is $l_{g_0} \circ \mathrm{Exp}$, l denoting left multiplication. The chart changes are given by mappings of the form $E \to \ln(g_1^{-1}g_0\,\mathrm{Exp}(E)) \in \mathcal{C}^\infty(\mathrm{End}(TM))$ which are diffeomorphisms of open sets in $\mathcal{C}^\infty(\mathrm{End}(TM))$. This chart is adapted to $\mathcal{M}_{\mathrm{vol}(g)}$, since for $g_0 \in \mathcal{M}_{\mathrm{vol}(g)}$, $g \in U(g_0)$ is $\mathrm{vol}(g) = \mathrm{vol}(g_0)$ iff $\det(g) = \det(g_0)$, i.e. iff $\det(g_0^{-1}g) \equiv 1$, i.e. iff $\det(\mathrm{Exp}(\phi^{g_0}(g))) = 1$, i.e. iff $e^{\mathrm{Tr}(\phi^{g_0}(g))} = 1$, i.e. iff $\mathrm{Tr}(\phi^{g_0}(g)) = 0$. Thus

$$\phi(\mathcal{M}_{\mathrm{vol}(g)} \cap U(g_0)) = \phi(U(g_0)) \cap \{E \in \mathcal{C}^\infty(\mathrm{End}(TM)) \colon \mathrm{Tr}(E) = 0\}.$$

For $g_0 \in \mathcal{M}_{\mathrm{Vol}(g)}$ consider the map $\psi^{g_0} \colon g \mapsto (\frac{g}{\mathrm{Vol}(g)^{\frac{2}{n}}}, \mathrm{Vol}(g) - 1)$, which determines a global diffeomorphism $\mathcal{M} \to \mathcal{M}_{\mathrm{Vol}(g_0)} \times (-1,\infty)$. Obviously, $\psi^{g_0}(\mathcal{M}_{\mathrm{Vol}(g_0)}) = \mathcal{M}_1 \times \{0\}$.

1.6. Calculation of second fundamental forms. In the sequel I will not use charts adapted to the submanifolds, since these charts are not easy to handle with.

1.7. $\omega^c_{\mathcal{M}_{\mathrm{Vol}(g)}}$. In this case the orthogonal projection on the normal bundle is

$$\pi_{N\mathcal{M}_{\mathrm{Vol}(g)}}(\nabla^c \tilde{k}(g)) = \frac{1}{n\,\mathrm{Vol}(g)} \int_M \Big(\mathrm{Tr}(g^{-1} d\tilde{k}(g).h) - \mathrm{Tr}(g^{-1}\Gamma^c_g(h,k)) \Big)\,\mathrm{vol}(g)$$

The extension $\tilde{k}$ satisfies $\int_M (g^{-1}\tilde{k}) \operatorname{vol}(\tilde{g}) = 0$ for $\tilde{g} \in \mathcal{M}_{\operatorname{Vol}(g)}$. Thus

$$\int_M \operatorname{Tr}(g^{-1} d\tilde{k}(g).h) \operatorname{vol}(g) = \int_M \Big(\operatorname{Tr}(HK) - \frac{1}{2} \operatorname{Tr}(K) \operatorname{Tr}(H) \Big) \operatorname{vol}(g) g.$$

For the Christoffel symbol one obtains

$$\begin{aligned}\int_M & - \operatorname{Tr}(g^{-1}\Gamma_g^c(h,k)) \operatorname{vol}(g) \\ &= \int_M \big(-\frac{4c+1}{4c} \operatorname{Tr}(HK) + \frac{cn+1}{4cn} \operatorname{Tr}(H) \operatorname{Tr}(K) \big) \operatorname{vol}(g).\end{aligned}$$

Therefore

$$\omega^c_{\mathcal{M}_{\operatorname{Vol}(g)}}(h,k) = \frac{1}{n \operatorname{Vol}(g)} \int_M \Big(\frac{1}{4c} \operatorname{Tr}(HK) + \frac{1-4cn}{4cn} \operatorname{Tr}(H) \operatorname{Tr}(K) \Big) \operatorname{vol}(g) g.$$

In particular $\mathcal{M}_{\operatorname{Vol}(g)}$ is geodesically closed iff $n=1$ and $c=2$. For $n=1$:

$$\omega^c_{\mathcal{M}_{\operatorname{Vol}(g)}}(h,k) = \frac{1}{n \operatorname{Vol}(g)} \int_M \frac{2-c}{4c} hk \operatorname{vol}(g) g.$$

1.8. $\omega^c_{\mathcal{M}_{\operatorname{vol}(g)}}$**.** Since $\pi_{N\mathcal{M}_{\operatorname{vol}(g)}} = \pi_1 \oplus \pi_2 = \pi_T$ the second fundamental form is $\frac{1}{n} \operatorname{Tr}(\nabla_h \tilde{k}) g$. For $\tilde{g} \in \mathcal{M}_{\operatorname{vol}(g)}$ is $\operatorname{Tr}(\tilde{g}^{-1}\tilde{k}) = 0$, hence $\operatorname{Tr}(g^{-1} d\tilde{k}(g).h) = \operatorname{Tr}(HK)$. On the other hand

$$\begin{aligned}\operatorname{Tr}(g^{-1}\Gamma_g^c(h,k)) = \Big(& \operatorname{Tr}(HK) - \frac{1}{2} \operatorname{Tr}(H) \operatorname{Tr}(K) \\ & + \frac{1}{4c} \operatorname{Tr}(HK) + \frac{cn-1}{4cn} \operatorname{Tr}(H) \operatorname{Tr}(K) \Big) g\end{aligned}$$

As $\operatorname{Tr}(H) = \operatorname{Tr}(K) = 0$,

$$- \operatorname{Tr}(g^{-1}\Gamma_g^c(h,k)) = \frac{1-4c}{4c} \operatorname{Tr}(HK) g.$$

Finally,

$$\omega^c(h,k) = \frac{1}{4cn} \operatorname{Tr}(HK).$$

Therefore, $\mathcal{M}_{\operatorname{vol}(g)}$ is never geodesically closed.

1.9. Conf^g**.** It has followed implicitly from 1.7 that Conf^g is geodesically closed. A direct proof using the geodesics on $\mathcal{M}$ will be given in 6.1.

1.10. $\omega^c_{\mathrm{Conf}^g_0}$**.** Since $\pi_{N\,\mathrm{Conf}^g_0} = \pi_1 \oplus \pi_3$ the second fundamental form is
$(*)$

$$\pi_{\mathrm{Conf}^g_0}(\nabla_h \tilde{k})(g) = \left(\frac{1}{n\,\mathrm{Vol}(g)} \int_M \mathrm{Tr}(g^{-1}\nabla_h\tilde{k})\,\mathrm{vol}(g)\right) g + \left(\nabla_h\tilde{k} - \frac{1}{n}\,\mathrm{Tr}(g^{-1}\nabla_h\tilde{k})g\right)$$

Here $h = f_1 g, k = f_2 g$ for some $f_1, f_2 \in C^g_0(M,\mathbb{R})$. If $\tilde{g} \in \mathrm{Conf}^g_0$ is

$$\int_M \mathrm{Tr}(\tilde{g}^{-1}\tilde{k})\,\mathrm{vol}(\tilde{g}) = 0.$$

Therefore for the first summand in $(*)$ one obtains the terms

$$\begin{aligned}\int_M \mathrm{Tr}(g^{-1}d\tilde{k}(g).h)\,\mathrm{vol}(g) &= \int_M \left(\mathrm{Tr}(HK) - \frac{1}{2}\,\mathrm{Tr}(H)\,\mathrm{Tr}(K)\right)\mathrm{vol}(g)\\ &= \frac{n(n-2)}{2}\int_M f_1 f_2\,\mathrm{vol}(g)\end{aligned}$$

and (see the calculation in 1.7)

$$\int_M -\Gamma^c_g(h,k)\,\mathrm{vol}(g) = \int_M \frac{-3cn - n + 1}{4c} f_1 f_2\,\mathrm{vol}(g).$$

The second summand of $(*)$ vanishes (which reflects the fact that Conf^g is geodesically closed):

$$d\tilde{k}(g).h - \frac{1}{n}\,\mathrm{Tr}(g^{-1}d\tilde{k}(g).h) - \Gamma^c_g(h,k)_0.$$

At first note that $\tilde{k} - \frac{1}{n}\,\mathrm{Tr}(g^{-1}\tilde{k})g = 0$ along Conf^g_0 and therefore

$$\begin{aligned}d\tilde{k}(g).h - \frac{1}{n}\,\mathrm{Tr}(g^{-1}d\tilde{k}(g).h) &= -\frac{1}{n}\,\mathrm{Tr}(HK)g + \frac{1}{n}\,\mathrm{Tr}(H)k\\ &= -f_1 f_2 g + f_1 f_2 g = 0.\end{aligned}$$

and finally from the expression of the Christoffel symbol it follows easily that

$$\Gamma^c_g(h,k)_0 = \Gamma^c_g(f_1 g, f_2 g)_0 = 0.$$

Therefore

$$\omega^c(h,k) = \frac{cn - n^2 - n + 1}{4cn\,\mathrm{Vol}(g)} \int_M f_1 f_2\,\mathrm{vol}(g) g.$$

In particular Conf^g_0 is geodesically closed iff $c = \frac{n^2+n-1}{n}$. Note that for such a c the metric G^c is positive definite.

1.11. Covariant derivative on $\mathcal{M}_{\mathrm{Vol}(g)}$. Put $V = \mathrm{Vol}(g)$ and let $\widetilde{\nabla}^c = \pi_3 \circ \nabla^c$ be the Christoffel symbol on $\mathcal{M}_V$. Then for a vector field k along $\mathcal{M}_V$, the covariant derivative in direction $h \in T_g\mathcal{M}_V$ is given by

$$\begin{aligned}\widetilde{\nabla}^c_h k(g) =& \nabla^c_h k - \omega(h, k(g))\\ =& dk(g).h - \frac{1}{2}hg^{-1}k - \frac{1}{2}kg^{-1}h + \frac{1}{4}\operatorname{Tr}(H)k + \frac{1}{4}\operatorname{Tr}(K)h\\ &- \frac{1}{4cn}\operatorname{Tr}(HK)g + \frac{cn-1}{4cn^2}(\operatorname{Tr}(H)\operatorname{Tr}(K))g\\ &- \frac{1}{nV}\int_M \left(\frac{1}{4c}\operatorname{Tr}(HK) + \frac{1-4cn}{4cn}\operatorname{Tr}(H)\operatorname{Tr}(K)\right)\operatorname{vol}(g).g\end{aligned}$$

For $c = \frac{1}{n}$:

$$\begin{aligned}\widetilde{\nabla}^c_h k(g) =& dk(g).h - \frac{1}{2}hg^{-1}k - \frac{1}{2}kg^{-1}h\\ &+ \frac{1}{4}\operatorname{Tr}(H)k + \frac{1}{4}\operatorname{Tr}(K)h - \frac{1}{4}\operatorname{Tr}(HK)g\\ &- \frac{1}{nV}\int_M \left(\frac{n}{4}\operatorname{Tr}(HK) - \frac{3}{4}\operatorname{Tr}(H)\operatorname{Tr}(K)\right)\operatorname{vol}(g).g\end{aligned}$$

1.12. Geodesics on $\mathcal{M}_{\mathrm{Vol}(g)}$. Thus the geodesic equation is

$$\begin{aligned}g_{tt} =& g_t g^{-1} g_t - \frac{1}{2}\operatorname{Tr}(g^{-1}g_t) + \frac{1}{4}\operatorname{Tr}(g^{-1}g_t g^{-1}g_t)g\\ &+ \frac{1}{nV}\int_M \left(\frac{n}{4}\operatorname{Tr}(g^{-1}g_t g^{-1}g_t) - \frac{3}{4}\operatorname{Tr}^2(g^{-1}g_t)\right)\operatorname{vol}(g).g\end{aligned}$$

The substitution $J = g^{-1}g_t$ does not eliminate g, since then

$$\begin{aligned}J_t =& -\frac{1}{2}\operatorname{Tr}(J)J + \frac{1}{4}\operatorname{Tr}(J^2)\operatorname{Id}\\ &+ \frac{1}{nV}\int_M \left(\frac{n}{4}\operatorname{Tr}(J^2) - \frac{3}{4}\operatorname{Tr}^2(J)\right)\operatorname{vol}(g).\operatorname{Id}\end{aligned}$$

Nevertheless $J_0' = -\frac{1}{2}\operatorname{Tr}(J)J_0$. Assume that the solution is of the form $g(t) = g^0\operatorname{Exp}(a(t)\operatorname{Id} + b(t)H_0)$. Then the volume element along the geodesic is

$$\operatorname{vol}(g) = \operatorname{vol}(g^0)e^{\frac{n^2}{2}a(t)}$$

and the geodesic equation is

$$\begin{cases} a'' = -\dfrac{n}{4}a'^2 - \dfrac{1}{2V}n\displaystyle\int_M a'^2 e^{\frac{n^2}{2}a}\operatorname{vol}(g^0)\\ \qquad + \dfrac{\operatorname{Tr}(H_0^2)}{4}\big(b'^2 + \dfrac{1}{V}\displaystyle\int_M b'^2 e^{\frac{n^2}{2}a}\operatorname{vol}(g^0)\big)\\ b'' = -\frac{1}{2}na'b'.\end{cases}$$

Up to now I have not been able to solve it; even not if M is the 1-sphere or the torus for the geodesic starting at the standard metric. In these cases I have tried to use Fourier transformation. But the Fourier transform of $e^{(a)}$ has a very complicated dependence on the Fourier coefficients of a.

2. Application to Teichmüller Theory

2.1. Riemannian Surfaces. In the following five paragraphs I will review some basic results about Riemannian surfaces. For proofs see e.g. [**FiT82**]: Let M be a compact, oriented manifold of dimension 2. Denote by $\mathcal{C}$ the set of complex structures and by $\mathcal{A}$ the set of almost complex structures that respect the orientation of M. Each complex structure yields an almost complex structure by realification. $\mathcal{A}$ is a smooth manifold with tangent space

$$T_J\mathcal{A} = \{L \in \mathcal{C}^\infty(T^1_1(M))\colon\, J\circ L + L\circ J = 0\}.$$

In dimension 2 the Nijenhuis tensor

$$N(J)(X,Y) = 2\Big([JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]\Big)$$

of every almost complex structure J vanishes: If $X_x \neq 0$, then $\{X_x, JX_x\}$ is a basis of T_xM; $N(J)$ is antisymmetric and $N(X_x, JX_x) = 0$.

Thus by the Newlander-Nirenberg theorem the sets $\mathcal{C}$ and $\mathcal{A}$ are the same. The diffeomorphism group acts on $\mathcal{C}$ by pullback of charts, on $J \in \mathcal{A}$ for $\phi \in \operatorname{Diff}(M)$ by the pullback $\phi^*(J) = (T\phi^{-1})\circ(J\circ\phi)\circ T\phi$. The identification is $\operatorname{Diff}(M)$-invariant. Hence one has the correspondences $\mathcal{C}/\operatorname{Diff}(M) \leftrightarrow \mathcal{A}/\operatorname{Diff}(M)$ and $\mathcal{C}/\operatorname{Diff}_0(M) \leftrightarrow \mathcal{A}/\operatorname{Diff}_0(M)$ for the connected component of the identity $\operatorname{Diff}_0(M)$.

$\mathcal{R} = \mathcal{C}/\operatorname{Diff}(M)$ is called the Riemannian space of moduli of M and $\mathcal{T} = \mathcal{C}/\operatorname{Diff}_0(M)$ is called the Teichmüller space of M.

The modular group $\Gamma = \operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ is discrete and $\mathcal{T}/\Gamma = \mathcal{R}$. For $\operatorname{genus}(M) \geq 2$ $\mathcal{T}$ is a cell of dimension $6(\operatorname{genus}(M) - 1)$, in particular it delivers a stratification of $\mathcal{R}$.

2.2. Let $\mathcal{P} = \{f \in C^\infty(M,\mathbb{R})\colon\, f > 0\}$ be the cone of strictly positive functions. $\mathcal{P}$ acts on $\mathcal{M}$ by multiplication. Define the mapping

$$\psi\colon \mathcal{M} \to \mathcal{A}, \quad \psi(g) = -c^1_1(g^{-1}\otimes \operatorname{vol}(g)).$$

As $\operatorname{vol}(fg) = f^{\frac{\dim(M)}{2}}\operatorname{vol}(g) = f\operatorname{vol}(g)$ for $f \in \mathcal{P}$, ψ is invariant under the action of $\mathcal{P}$ on $\mathcal{M}$. The mapping $\widehat{\psi}\colon \mathcal{M}/\mathcal{P} \to \mathcal{A}$ induced by ψ is a diffeomorphism. Denote by $\mathcal{M}_{-1}$ the set of metrics g with scalar curvature $\operatorname{scal}(g) \equiv -1$ (As the total volume is always positive, the notation does not clash with the notation $\mathcal{M}_{\mathrm{Vol}}$). Then $\pi_{\mathcal{M}/\mathcal{P}} \restriction \mathcal{M}_{-1}\colon \mathcal{M}_{-1} \to \mathcal{M}/\mathcal{P}$ is a diffeomorphism; the proof of that is based on the existence of isothermal coordinates. Thus $\psi \restriction \mathcal{M}_{-1} = \widehat{\psi}\circ\pi_{\mathcal{M}/\mathcal{P}}\colon \mathcal{M}_{-1} \to \mathcal{A}$ is a diffeomorphism. $\widehat{\psi}\circ\pi_{\mathcal{M}/\mathcal{P}}$ is moreover $\operatorname{Diff}(M)$-invariant and thus induces a diffeomorphism of the moduli spaces $\mathcal{M}_{-1}/\operatorname{Diff}(M) \cong \mathcal{R}$, $\mathcal{M}_{-1}/\operatorname{Diff}_0(M) \cong \mathcal{A}$.

2.3. The Weil-Peterson metric. There is a metric on $\mathcal{A}$ given by

$$H_J(L_1, L_2) = \int_M \operatorname{Tr}(L_1 \circ L_2^*) \operatorname{vol}(\psi^{-1}(J))$$

where L^* is the $\psi^{-1}(J)$-adjoint of L_2: $\psi^{-1}(J)(L_2(X), Y) = \psi^{-1}(J)(X, L_2^*(Y))$ for all $X, Y \in \mathfrak{X}(M)$. In normal coordinates for $\psi^{-1}(J)$ one sees that the adjoint $L \mapsto L^*$ on $T_J\mathcal{A}$ is the identity. H is $\operatorname{Diff}(M)$-invariant and factors to a metric $\hat{H}$ on $\mathcal{T}$ and $\mathcal{R}$. $(\mathcal{T}, \hat{H})$ is Kähler and is called the Weil-Peterson metric. $\mathcal{A} \twoheadrightarrow T$ is a Riemannian submersion.

2.4. Tangent space of $\mathcal{M}_{-1}$. The variation of the scalar curvature is

$$d\operatorname{scal}(g).h = -g(h, \operatorname{Ric}(g)) - \delta^g\delta^g h + \Delta^g(\operatorname{Tr}_g h).$$

In $\dim(M) = 2$ one has $\operatorname{Ric}(g) = \frac{1}{2}\operatorname{scal}(g).g$ and thus

$$T_g\mathcal{M}_{-1} = \{h \in \mathcal{C}^\infty(S^2M) \colon \delta^g\delta^g h + \Delta^g(\operatorname{Tr}_g h) = \frac{1}{2}\operatorname{Tr}_g(h)\}.$$

Remember that $\mathcal{C}^\infty(S^2M) = \ker(\delta^g) \oplus \operatorname{im}(\delta^{g*})$, thus also

$$\mathcal{C}^\infty(S^2_gM) = (\ker(\delta^g) \cap \ker \operatorname{Tr}_g) \oplus (\operatorname{im}(\delta^{g*}) \cap \ker \operatorname{Tr}_g)$$

is a G_g-orthogonal decomposition. Let $\pi_{TD}(g) \colon \mathcal{C}^\infty(S^2M) \twoheadrightarrow \ker(\delta^g) \cap \ker \operatorname{Tr}_g$ be the projection. If $h \in T_g\mathcal{M}_{-1} \cap \ker(\delta^g)$, then multiplication with $\operatorname{Tr}_g h$ and partial integration yields

$$\begin{aligned} 0 =& \langle \operatorname{Tr} 2_g h, \operatorname{Tr}_g h\rangle_g - \langle \operatorname{Tr}_g h, \Delta^g(\operatorname{Tr}_g h)\rangle_g \\ =& \| \operatorname{Tr}_g h\|_g^2 + \|df\|_g^2 = 0, \end{aligned}$$

thus $\operatorname{Tr}_g h = 0$ and $T_g(\mathcal{M}_{-1}) \cap \ker(\delta^g) = \ker(\delta^g) \cap \ker \operatorname{Tr}_g$. If $h \in \operatorname{im}(\delta^{g*})$, then h is given by $h = \frac{d}{dt}\big|_0 (\phi_t *)g$ for some curve of diffeomorphisms ϕ_t. Since $\operatorname{scal}(\phi_t^* g) = \phi_t^*(\operatorname{scal}(g)) = \operatorname{scal}(g) \circ \phi_t$, $d\operatorname{scal}(g).h = \frac{d}{dt}\big|_0 \operatorname{scal}(\phi_t^* g) = 0$ and $T_g\mathcal{M}_{-1} \cap \operatorname{im}(\delta^{g*}) = \operatorname{im}(\delta^{g*})$. Summing up one gets

$$T_g\mathcal{M}_{-1} = (\ker(\delta^g) \cap \ker \operatorname{Tr}_g) \oplus \operatorname{im}(\delta^{g*}). \tag{2.4.1}$$

From this one realizes, that

$$T_g(\mathcal{M}_{-1}/\operatorname{Diff}(M)) = \ker(\delta^g) \cap \ker \operatorname{Tr}_g .$$

2.5. Geometry of $\mathcal{M}_{-1}$. The problem is that the tangent space is not easy to describe and so is the geodesic equation. As $\mathcal{M}_{\mathrm{vol}(g)}$ is diffeomorphic to $\mathcal{M}_{-1}$, one might suspect that the geometry of $\mathcal{M}_{-1}$, which is well-known, is of some help. But it is easy to see that this diffeomorphism is not an isometry. And even worse, $\mathcal{M}_{\mathrm{vol}(g)}$ is not $\mathrm{Diff}(M)$-invariant.
On the other hand, the map $\psi\colon \mathcal{M}_{-1} \to \mathcal{A}$ is not an isometry either. The pullback metric on $\mathcal{M}_{-1}$ is

$$\psi^*(H)_g(h,k) = \int_M \mathrm{Tr}(d\psi(g).h \circ d\psi(g).k)\,\mathrm{vol}(g).$$

Put $I(h,k) = \mathrm{Tr}(d\psi(g).h \circ d\psi(g).k)$. The derivative of ψ is given by

$$d\psi(g).h = c_1^1(g^{-1}hg^{-1} \otimes \mathrm{vol}(g)) + \frac{1}{2}\,\mathrm{Tr}_g(h)\psi(g).$$

Therefore, the integrand $I(h,k)$ is given by

$$\begin{aligned}
I(h,k) = &\,\mathrm{Tr}\left(c_1^1\big(g^{-1}hg^{-1} \otimes \mathrm{vol}(g)\big) \circ c_1^1\big(g^{-1}kg^{-1} \otimes \mathrm{vol}(g)\big)\right)\\
&+ \frac{1}{2}\,\mathrm{Tr}_g(k)\,\mathrm{Tr}\left(c_1^1\big(g^{-1}hg^{-1} \otimes \mathrm{vol}(g) \circ \psi(g)\big)\right)\\
&+ \frac{1}{2}\,\mathrm{Tr}_g(h)\,\mathrm{Tr}\left(c_1^1\big(g^{-1}kg^{-1} \otimes \mathrm{vol}(g) \circ \psi(g)\big)\right)\\
&+ \frac{1}{4}\,\mathrm{Tr}_g(h)\,\mathrm{Tr}_g(k)\,\mathrm{Tr}(\psi(g) \circ \psi(g)).
\end{aligned}$$

As $\mathrm{Tr}(L_1 \circ L_2) = \mathrm{Tr}(L_1 \circ L_2^*)$ (see 2.3) and

$$\mathrm{vol}(g)_{jl}g^{lr}\,\mathrm{vol}(g)_{nr} = -\,\mathrm{vol}(g)_{jl}g^{lr}\,\mathrm{vol}(g)_{rn} = g_{jn},$$

one has

$$\begin{aligned}
&\mathrm{Tr}\left(c_1^1\big(g^{-1}hg^{-1} \otimes \mathrm{vol}(g)\big) \circ c_1^1\big(g^{-1}kg^{-1} \otimes \mathrm{vol}(g)\big)\right)\\
=&h^{il}\,\mathrm{vol}(g)_{lm}k^{mn}\,\mathrm{vol}(g)_{ni}\\
=&h^{ij}\,\mathrm{vol}(g)_{jl}k^{mn}\,\mathrm{vol}(g)_{nr}g^{lr}g_{im} = \mathrm{Tr}(g^{-1}hg^{-1}k).
\end{aligned}$$

Furthermore, as $\mathrm{vol}(g)_{mn}g^{nr}\,\mathrm{vol}(g)_{ri} = -g_{mi}$,

$$\mathrm{Tr}\big(c_1^1(g^{-1}hg^{-1} \otimes \mathrm{vol}(g) \circ \psi(g))\big) = h^{ij}\,\mathrm{vol}(g)_{mn}g^{nr}\,\mathrm{vol}(g)_{ri} = -\,\mathrm{Tr}_g(h).$$

Finally, $\mathrm{Tr}(\psi(g) \circ \psi(g)) = \mathrm{Tr}(-\,\mathrm{Id}) = -2$. Summing up one obtains

$$I = g(h,k) - \frac{3}{2}\,\mathrm{Tr}_g(h)\,\mathrm{Tr}_g(k). \tag{2.5.1}$$

As $\mathrm{Tr}(HK) + a\,\mathrm{Tr}(H)\,\mathrm{Tr}(K) = \mathrm{Tr}(H_0K_0) + \frac{an+1}{n}\,\mathrm{Tr}(H)\,\mathrm{Tr}(K)$ for $a \in \mathbb{R}$, we have just proved the following lemma.

2.6. Proposition. *The metric on $\mathcal{M}_{-1}$ that is induced by $\psi\colon \mathcal{M}_{-1} \to \mathcal{A}$ is given by $\psi^*(H) = G^{-1} \restriction \mathcal{M}_{-1}$ (see 1.2).*

$\mathcal{M}_{-1}$ is not geodesically closed in $\mathcal{M}$. Denote by $\hat{G}$ the metric on $\mathcal{M}_{-1}/\operatorname{Diff}_0(M)$ induced by $G = G^{\frac{1}{n}}$.

2.7. Corollary. *(comp. [**FiT82**]) The diffeomorphism of moduli spaces*

$$\psi \circ \pi_{\mathcal{M}/\mathcal{P}}\colon \mathcal{M}_{-1}/\operatorname{Diff}_0(M) \to \mathcal{A}/\operatorname{Diff}_0(M)$$

induced by $g \mapsto -c_1^1(g^{-1} \otimes \operatorname{vol}(g))$ is an isometry between $(\mathcal{M}_{-1}/\operatorname{Diff}_0(M), \hat{G})$ and $(\mathcal{A}/\operatorname{Diff}_0(M), \hat{H})$.

Proof. Follows from 2.5.1 and the direct sum decomposition (2.4.1) of $T\mathcal{M}_{-1}$ in 2.4. □

2. Some Tensor Calculus

In the sequel the relation between contractions of ∇^2, the second order Ricci identity and the Lichnerowitz Laplacian is studied. This will give an alternative formula for the derivation of the Ricci curvature and allows to compare to already known formulas (e.g. [**Bes88**]).

3.1. Notation. Denote by $T^{*p}M$ the vector bundle of all $(0,p)$-tensors and by S^pM (resp. Λ^pM) the vector bundle of all symmetric (resp. antisymmetric) $(p,0)$-tensors on M. Then for the Whitney sum $S^2M \oplus \Lambda^2M = T^2M$. For the smooth section write $\mathcal{C}^\infty(S^2M) = \mathcal{C}^\infty(S^2M)$ and $\Omega^2(M) = \mathcal{C}^\infty(\Lambda^2M)$ Let g be a metric on M. The induced pseudo-metric on the vector bundle T^pM is given by

$$g(X_1 \otimes \cdots \otimes X_p, Y_1 \otimes \cdots \otimes Y_p) = g(X_1, Y_1) \cdots g(X_p, Y_p)$$

and will be denoted also by g. Note that on T^2M there is also the pseudo-metric $\tilde{g}$ given by $\tilde{g}(h,k) = \operatorname{Tr}(g^{-1}hg^{-1}k)$. Then $\tilde{g} \restriction S^2M = g$, $\tilde{g} \restriction \Lambda^2M = -g$. Let $\nabla\colon \mathcal{C}^\infty(TM) \to \mathcal{C}^\infty(TM) \otimes \mathcal{C}^\infty(TM)$ be the Levi-Civita connection of g. Its extension to T^pM will be denoted also by ∇:

$$\nabla\colon \mathcal{C}^\infty(S^pM) \to \Omega^1M \oplus \mathcal{C}^\infty(S^pM)$$
$$\nabla\colon \Omega^pM \to \Omega^1M \oplus \Omega^pM$$

Let : $\mathcal{C}^\infty(T^pM) \to \mathcal{C}^\infty(S^pM)$ be the symmetrisation

$$\mathbf{sym}(T)(X_1, \dots, X_p) = \frac{1}{p!} \sum_{\tau \in S(p)} T(X_{\sigma(1)}, \dots, X_{\sigma(p)})$$

and $\mathbf{alt}\colon \mathcal{C}^\infty(T^pM) \to \Omega^p(M)$ be the antisymmetrisation

$$\mathbf{alt}(T)(X_1,\dots,X_p) = \frac{1}{p!}\sum_{\tau\in S(p)} \operatorname{sign}(\tau) T(X_{\sigma(1)},\dots,X_{\sigma(p)})$$

Covariant derivatives of tensors in $\mathcal{C}^\infty(S^2M)$ and $\Omega^p(M)$ are symmetric (resp. antisymmetric) in all factors but the first. Define the symmetric (resp. antisymmetric) covariant derivative by

$$\begin{aligned} &\delta^*\colon \mathcal{C}^\infty(T^pM) \to \mathcal{C}^\infty(S^pM), &&\delta^* = \mathbf{sym}\circ\nabla \\ &d\colon \mathcal{C}^\infty(T^pM) \to \Omega^p(M), &&d = \mathbf{alt}\circ\nabla. \end{aligned}$$

d is an extension of the usual exterior differential on $\Omega^p(M)$.

3.2. The $(3,1)$ curvature tensor $\operatorname{Riem}(g)$ acts on $\mathcal{C}^\infty(S^2M)$ by

$$\operatorname{Riem}(g).h = c^{12}_{12}\big(g\otimes c^1_1(h\otimes \operatorname{Riem}(g))\big) = c^{123}_{123}(g\otimes h\otimes \operatorname{Riem}(g)).$$

In a local basis $\{\partial_i\}$ of $\mathcal{C}^\infty(TM)$ this is the symmetric $(2,0)$-tensor

$$(X,Y)\mapsto \sum_{ij} g^{ij} h(\operatorname{Riem}(g)(X,\partial_i)Y,\partial_j)$$

3.3. For a metric g and for each $p\geq 1$ there is the non degenerate pairing of convenient vector spaces $\tilde G_g\colon \mathcal{C}^\infty(T^pM)\otimes \mathcal{C}^\infty_c(T^pM)$, $(h,k)\mapsto \int_M g(h,k)\operatorname{vol}(g)$. If the metric g is somehow fixed, I will write $\tilde G$ instead of $\tilde G_g$. For $p=0$ and $p=1$ the common notation is $\tilde G(.,.) = \langle .,.\rangle_g$.

3.4. Contractions of ∇^2. If the metric g is fixed, then contractions of $T\in \mathcal{C}^\infty(T^pM)$

$$c^{1\dots 2l}_{i_1\dots i_{2l}}(\underbrace{g^{-1}\otimes\cdots\otimes g^{-1}}_{l\text{ times}}\otimes T)$$

will often be abbreviated by $c_{i_1\dots i_{2l}}(T)$. For $h\in\mathcal{C}^\infty(S^2M)$ put

$$\begin{aligned} M(h) &= \mathbf{sym}(c_{14}\nabla^2 h) = \mathbf{sym}(c_{13}\nabla^2) \\ \Delta(h) &= -c_{12}\nabla^2(h) = (\delta\circ\nabla)(h) \end{aligned}$$

Note that not even on $\mathcal{C}^\infty(S^2M)$ or on $\Omega^1(M)$, the operators $\Delta = \delta\circ\nabla$ and $\delta\circ\delta^*$ coincide.
The remaining two traces are

$$\begin{aligned} \operatorname{Hess}(\operatorname{Tr}_g h) &= \nabla d(\operatorname{Tr}_g h) = \nabla(c_{23}\nabla h) = c_{34}(\nabla^2 h) \\ (\delta^*\circ\delta)(h) &= -\mathbf{sym}(\nabla(c_{12}(\nabla h))) = -\mathbf{sym}\circ c_{23}(\nabla^2 h) = -\mathbf{sym}\circ c_{24}(\nabla^2 h) \end{aligned}$$

The full contractions are

$$\begin{aligned} \operatorname{Tr}_g(M(h)) &= -\operatorname{Tr}_g((\delta^*\circ\delta)(h)) = \delta\delta h \\ \operatorname{Tr}_g(\Delta h) &= -\operatorname{Tr}_g(\operatorname{Hess}(\operatorname{Tr}_g h)) = \Delta(\operatorname{Tr}_g h) \end{aligned}$$

Furthermore there is the identity $G_g((\delta\delta h)g, h) = G_g(h, \operatorname{Hess}(\operatorname{Tr}_g(h)))$.

3.5. Let D be a connection on a vector bundle E over M. We will use the curvature R^D of (E, D), the 2-form on M with values in $E^* \otimes E$ with the following sign:

$$R^D_{X,Y}(s) = D_{[X,Y]}s - [D_X, D_Y]s$$

for $s \in \mathcal{C}^\infty(E)$. Let furthermore ∇ be a connection on M with torsion T. The second derivative $D^2 s \in \mathcal{C}^\infty(T^*M \otimes T^*M \otimes E)$ of a section $s \in \mathcal{C}^\infty(E)$ is defined by

$$D^2_{X,Y}s = D_X(D_Y s) - D_{\nabla_X Y}s.$$

On natural bundles (such as (T^pM, ∇)) this coincides with the composition: $\nabla^2 h = \nabla(\nabla h)$. We will make use of the second order Ricci Identity

$$\text{(RId)} \qquad D^2_{X,Y}s - D^2_{Y,X} = -R^D_{X,Y}s - D_{T(X,Y)}.$$

3.6. For $h \in \mathcal{C}^\infty(S^2M)$ the second order Ricci Identity becomes

$$\begin{aligned}(\nabla^2 h)(X, Y, A, B) &= (\nabla^2 h)(Y, X, A, B) - (\operatorname{Riem}(g)(X, Y)h)(A, B)\\ &= (\nabla^2 h)(Y, X, A, B)\\ &\quad + h(\operatorname{Riem}(g)(X, Y)A, B) + h(A, \operatorname{Riem}(g)(X, Y)B)\end{aligned}$$

For the traces of $\nabla^2 h$ it follows that

$$c_{14}(\nabla^2 h)(X, Y) = c_{24}(\nabla^2 h)(X, Y) - (\operatorname{Riem}(g).h)(X, Y) + \operatorname{Ric}(g)g^{-1}h.$$

Thus there is the following relation between $M(h)$ and $\delta^*\delta h$:

$$\text{(3.6.1)} \qquad 2M(h) = -2\delta^*\delta h - 2\operatorname{Riem}(g).h + \operatorname{Ric}(g)g^{-1}h + hg^{-1}\operatorname{Ric}(g).$$

This equality will be used later to obtain alternative formulas for the Hessian of the total scalar curvature.

3.7. This formula suggests the definition of the Lichnerowitz Laplacian (see [**Aub82**])

$$\Delta_L : \mathcal{C}^\infty(T^pM) \to \mathcal{C}^\infty(T^pM), \quad \Delta_L = \Delta + \Gamma = \delta \circ \nabla + \Gamma,$$

where

$$\begin{aligned}(\Gamma T)_{i_1\dots i_p} &= \sum_{i=1}^{p} g^{ij}\operatorname{Ric}(g)_{i_k i} T_{i_1\dots j\dots i_p}\\ &\quad - \sum_{k\neq l} g^{ij} g^{mn}\operatorname{Riem}(g)_{i_k i i_l m} T_{i_1\dots j\dots n\dots i_p}\end{aligned}$$

In particular, for the restriction $\Delta_L\colon \mathcal{C}^\infty(S^2M) \to \mathcal{C}^\infty(S^2M)$ note that the action of $\operatorname{Riem}(g)$ on $\mathcal{C}^\infty(S^2M)$ is given in coordinates by $\operatorname{Riem}(g).h = h^{kl}\operatorname{Riem}(g)_{iklj}$ and thus by the symmetries of h, of the Riemann curvature and of the Ricci curvature one computes that

$$\begin{aligned}(\Gamma h)_{ij} &= \operatorname{Ric}(g)_{ik}h^k{}_j + \operatorname{Ric}(g)_{jk}h_i{}^k - \operatorname{Riem}(g)_{ikjl}h^{kl} - \operatorname{Riem}(g)_{jkil}h^{kl}\\ &= \big(\operatorname{Ric}(g)g^{-1}h + hg^{-1}\operatorname{Ric}(g) - 2\operatorname{Riem}(g).h\big)_{ij}.\end{aligned}$$

This is exactly the part of equality (3.6) that does not contain derivatives of h. Therefore, one obtains the equality

$$2M(h) + \Delta(h) = \Delta_L - 2\delta^*\delta h. \tag{3.7.1}$$

It will be used later to compare results of [**Bes88**] with our calculations.

4. The Hessian of the Total Scalar Curvature on $(\mathcal{M}, G^c)$

The Hessian will be given in the form $\operatorname{Hess}^c(h,h) = G^c_g(L^c h, h)$ for some partial differential operator L^c. This operator is elliptic only for $(\dim(M) = 1, c = \frac{1}{3})$ and its principal symbol depends on c.

4.1. At first I will present the calculation via the covariant derivative. Let $\tilde\zeta\colon \mathcal{M} \to \mathcal{M} \times \mathcal{C}^\infty(S^2M)$, $\tilde\zeta(g) = (g, \zeta(g)$ with $\zeta(g) = \operatorname{grad}^c \operatorname{Scal}(g)$. The differential of $\tilde\zeta$ at $g \in \mathcal{M}$ in direction $h \in \mathcal{C}^\infty(S^2M) = T_g\mathcal{M}$ is

$$T_g\tilde\zeta(h) = (g, \operatorname{grad}^c\operatorname{Scal}(g), h, d\zeta(g).h) \in \mathcal{T}^2M = \mathcal{M} \times (\times^3 \S^2 M).$$

4.2. A simple calculation yields $\zeta(g) = a_1 \operatorname{scal}(g)g - \operatorname{Ric}(g)$ with $a_1(c) = \frac{cn-1}{cn^2} + \frac{1}{2cn} = \frac{2cn-2+n}{2cn^2}$ $(a(\frac{1}{n}) = \frac{1}{2})$. Thus,

$$\begin{aligned}d\zeta(g).h &= a_1\big((d\operatorname{scal}(g).h)g + \operatorname{scal}(g)h\big) - d\operatorname{Ric}(g).h\\ &= a_1\big(-(\operatorname{Ric}(g), h) + \Delta^g(\operatorname{Tr}_g h)g + (\delta\delta h)g + \operatorname{scal}(g)h\big)\\ &\quad + \frac{1}{2}\operatorname{Hess}(\operatorname{Tr}_g h) - \frac{1}{2}\Delta^g h - M(h).\end{aligned}$$

4.3. For the Christoffel symbols one computes that

$$\begin{aligned}\Gamma^c_g(\zeta(g), h) &= a_1\operatorname{scal}(g)h - \frac{1}{2}\operatorname{Ric}(g)g^{-1}h - \frac{1}{2}hg^{-1}\operatorname{Ric}(g)\\ &\quad - \frac{1}{4}\big(\operatorname{Tr}_g(a_1\operatorname{scal}(g) - \operatorname{Ric}(g))\big)h - \frac{1}{4}(\operatorname{Tr}_g h)\big(a_1\operatorname{scal}(g)g - \operatorname{Ric}(g)\big)\\ &\quad + \frac{1}{4cn}(\operatorname{Tr}_g(a_1\operatorname{scal}(g) - \operatorname{Ric}(g), h)_g\\ &\quad + \frac{cn-1}{4cn^2}\operatorname{Tr}_g h\operatorname{Tr}_g\operatorname{Tr}_g(a_1\operatorname{scal}(g) - \operatorname{Ric}(g))g\end{aligned}$$

Put $a_2(c) = -\frac{1}{4}\frac{cn-1}{cn^2}$ ($a_2(\frac{1}{n}) = 0$) and $a_3(c) = a_1(c) - \frac{na_1}{4} + \frac{1}{4} = \frac{1}{8cn^2}(cn - n^2 - 8 + 8cn)$ ($a_3(\frac{1}{n}) = \frac{6-n}{8}$). Thus

$$\begin{aligned}\Gamma^c_g(\zeta(g), h) &= a_3 \operatorname{scal}(g)h - \frac{1}{2}\operatorname{Ric}(g)g^{-1}h - \frac{1}{2}hg^{-1}\operatorname{Ric}(g)\\ &\quad - \frac{1}{4cn}(\operatorname{Ric}(g), h)_g g + \frac{1}{4}(\operatorname{Tr}_g h)\operatorname{Ric}(g) + a_2 \operatorname{scal}(g)(\operatorname{Tr}_g h)g.\end{aligned}$$

4.4. Put $a_4(c) = -a_1 + \frac{1}{4cn} = \frac{1}{4cn^2}(-4cn + 4 - n)$ ($a_4(\frac{1}{n}) = -\frac{1}{4}$) and note that $a_1 - a_3 = \frac{n-2}{8cn}$. From the difference of the terms in 4.2 and in 4.3 one obtains

$$\begin{aligned}\operatorname{Hess}^c \operatorname{Scal}(g)(h,h) = G^c\Big(&a_1\big(\Delta^g(\operatorname{Tr}_g h)g + (\delta\delta h)\big)g\\ &+ \frac{1}{2}\operatorname{Hess}(\operatorname{Tr}_g h) - \frac{1}{2}\Delta^g h - M(h)\\ &+ \frac{1}{2}\operatorname{Ric}(g)g^{-1}h + \frac{1}{2}hg^{-1}\operatorname{Ric}(g) - \frac{1}{4}(\operatorname{Tr}_g h)\operatorname{Ric}(g)\\ &+ a_4(\operatorname{Ric}(g), h)_g g + \frac{n-2}{8cn}\operatorname{scal}(g)h - a_2(\operatorname{Tr}_g h)\operatorname{scal}(g)g, h\Big).\end{aligned}$$

By equality 3.6.1 one obtains the alternative formula

$$\begin{aligned}\operatorname{Hess}^c \operatorname{Scal}(g)(h,h) = G^c\Big(&a_1\big(\Delta^g(\operatorname{Tr}_g h) + (\delta\delta h)\big)g\\ &+ \frac{1}{2}\operatorname{Hess}(\operatorname{Tr}_g h) - \frac{1}{2}\Delta^g h + \delta^*\delta h\\ &+ \operatorname{Riem}(g).h - \frac{1}{4}(\operatorname{Tr}_g h)\operatorname{Ric}(g)\\ &+ a_4(\operatorname{Ric}(g), h)g + \frac{n-2}{8cn}\operatorname{scal}(g)h\\ &- a_2(\operatorname{Tr}_g h)\operatorname{scal}(g)g, h\Big).\end{aligned} \tag{4.4.1}$$

For $c = \frac{1}{n}$ use the formula $G((\delta\delta h)g, h) = G(h, \operatorname{Hess}(\operatorname{Tr}_g h))$.

$$\begin{aligned}\operatorname{Hess}\operatorname{Scal}(g)(h,h) =G\Big(&\frac{1}{2}\Delta^g(\operatorname{Tr}_g h)g + (\delta\delta h)g\\ &- \frac{1}{2}\Delta^g h - M(h)\\ &+ \frac{1}{2}\operatorname{Ric}(g)g^{-1}h + \frac{1}{2}hg^{-1}\operatorname{Ric}(g)\\ &- \frac{1}{2}(\operatorname{Tr}_g h)\operatorname{Ric}(g) + \frac{n-2}{8}\operatorname{scal}(g)h, h\Big).\end{aligned}$$

Or, alternatively, by 3.6.1

$$\begin{aligned}\operatorname{Hess}\operatorname{Scal}(g)(h,h) =G\Big(&\frac{1}{2}\Delta^g(\operatorname{Tr}_g h)g + (\delta\delta h)g\\ &- \frac{1}{2}\Delta^g h + \delta^*\delta h\\ &+ \operatorname{Riem}(g).h - \frac{1}{2}(\operatorname{Tr}_g h)\operatorname{Ric}(g) + \frac{n-2}{8}\operatorname{scal}(g)h, h\Big).\end{aligned}$$

4.5. Calculation via geodesics. The geodesics on $(\mathcal{M}, G)$ are of the form $g(t) = g_0 e^{(a(t)\,\mathrm{Id} + b(t) H_0)}$ with $H = g_0^{-1}$. Thus

$$g(t)^{-1} g'(t) = g(t)\big(a'(t)\,\mathrm{Id} + b'(t) H_0\big), g'(0) = h$$

The functions $a(t)$ and $b(t)$ satisfy:

$$\begin{aligned} &a(0) = 0, && a'(0) = \frac{\mathrm{Tr}_g h}{n}, && a''(0) = -\frac{1}{2n}(\mathrm{Tr}\, H)^2 + \frac{1}{4}\mathrm{Tr}(H^2) \\ &b(0) = 0, && b'(0) = 1, && b''(0) = -\frac{\mathrm{Tr}\, H}{2}. \end{aligned}$$

Put $\zeta(t) = \operatorname{grad}\mathrm{Scal}(g(t)) = \frac{\mathrm{scal}(g(t))}{2} g(t) - \mathrm{Ric}(g(t))$. $(\mathrm{Hess}\,\mathrm{Scal})(g_0)(h,h) = \frac{\partial^2}{\partial t^2}\Big|_0 (\mathrm{Scal} \circ g)(t)$.

$$\begin{aligned} \alpha(t) \colon &= \frac{d}{dt'}|_t (\mathrm{Scal}(g(t')) = dS(g(t)).g'(t) \\ &= G_{g(t)}(\zeta(t), g'(t)) \\ \frac{d}{dt}\Big|_0 \alpha &= \frac{d}{dt}\Big|_0 \int_M \beta(t)\,\mathrm{vol}(g(t)) \end{aligned}$$

with $\beta(t) = \mathrm{Tr}\big(g(t)^{-1}\zeta(t) g(t)^{-1} g'(t)\big)$. Putting into the special form of $\zeta(g)$ yields

$$\begin{aligned} \beta(t) &= \frac{n-2}{2} a'(t)\,\mathrm{scal}(g(t)) - b'(t)\,\mathrm{Tr}(H_0 g(t)^{-1}\,\mathrm{Ric}(g)) \\ \beta(0) &= \mathrm{Tr}(H)\frac{\mathrm{scal}(g_0)}{2} - (\mathrm{Ric}(g_0), h)_{g_0}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{d}{dt}\Big|_0 \beta(t) = \frac{1}{2}\mathrm{Tr}(H) d\,\mathrm{scal}(g_0).h + \frac{n}{2} a''(t)\,\mathrm{scal}(g_0) \\ - \mathrm{Tr}((a''(t)\,\mathrm{Id} + b''(t) H_0) g_0^{-1}\,\mathrm{Ric}(g_0)) - \mathrm{Tr}(H^2 g_0^{-1}\,\mathrm{Ric}(g_0)) \\ - \mathrm{Tr}(H g_0^{-1} d\,\mathrm{Ric}(g_0).h). \end{aligned}$$

Put the terms in appropriate forms:

$$\begin{aligned} &\frac{n}{2} a''(t)\,\mathrm{scal}(g) = (-\frac{\mathrm{scal}(g)}{4}(\mathrm{Tr}\, H) g_0 + \frac{n}{8}\,\mathrm{scal}(g) h, h)_{g_0} \\ &\mathrm{Tr}((a''(t)\,\mathrm{Id} + b''(t) H_0) g_0^{-1}\,\mathrm{Ric}(g_0)) = (\frac{1}{4}\,\mathrm{scal}(g) h - \frac{\mathrm{Tr}\, H}{2}\,\mathrm{Ric}(g_0), h)_{g_0} \\ &\mathrm{Tr}(H^2 g_0^{-1}\,\mathrm{Ric}(g_0)) = (h g_0^{-1}\,\mathrm{Ric}(g_0), h)_{g_0}. \end{aligned}$$

Using the formulas for the derivation of the scalar curvature and the Ricci curvature one computes easily

$$\begin{aligned}\frac{d}{dt}\Big|_0 \beta = \big(&\frac{1}{2}(\Delta^{g_0} \operatorname{Tr} H)g_0 + \frac{1}{2}(\delta^{g_0}\delta^{g_0} h)g_0 - \frac{\operatorname{Tr} H}{2}\operatorname{Ric}(g_0)\\ &+ \frac{n-2}{8}\operatorname{scal}(g_0)h + \frac{1}{2}\operatorname{Ric}(g_0)g_0^{-1}h + \frac{1}{2}hg_0^{-1}\operatorname{Ric}(g_0)\\ &- \frac{\operatorname{scal}(g)}{4}(\operatorname{Tr} H)g_0 + \frac{\operatorname{Tr} H}{2}\operatorname{Ric}(g_0)\\ &+ \frac{1}{2}\operatorname{Hess}(\operatorname{Tr} H) - \frac{1}{2}\Delta^{g_0} h - M(h), h\big)_{g_0}\end{aligned}$$

The terms $\frac{\operatorname{Tr} H}{2}\operatorname{Ric}(g)$ cancel and one has

$$\begin{aligned}(\operatorname{Hess}\operatorname{Scal})(g_0)(h,h) &= \frac{d}{dt}\Big|_0 \alpha(t)\\ &= \int_M \beta(0) d\operatorname{vol}(g).h + \int_M \beta'(0)\operatorname{vol}(g_0)\\ &= G_{g_0}(\frac{\operatorname{scal}(g)}{4}(\operatorname{Tr} H)g_0 - \frac{1}{2}(\operatorname{Tr} H)\operatorname{Ric}(g_0), h)\\ &+ \int_M \beta'(0)\operatorname{vol}(g_0).\end{aligned}$$

Summing up one obtains

$$\begin{aligned}(\operatorname{Hess}\operatorname{Scal})(g_0)(h,h) = G_{g_0}\Big(&\frac{1}{2}(\Delta^{g_0}\operatorname{Tr} H)g_0 + (\delta^{g_0}\delta^{g_0} h)g_0\\ &- \frac{1}{2}\Delta^{g_0} h - M(h)\\ &+ \frac{1}{2}\operatorname{Ric}(g_0)g_0^{-1}h + \frac{1}{2}hg_0^{-1}\operatorname{Ric}(g_0)\\ &+ \frac{n-2}{8}\operatorname{scal}(g_0)h - \frac{1}{2}(\operatorname{Tr} H)\operatorname{Ric}(g_0), h\Big).\end{aligned}$$

This is exactly the same expression for the Hessian as in 4.4.

4.6. Principal symbol. The formula of the Hessian computed in 4.4 is of the form

$$\operatorname{Hess}^c \operatorname{Scal}(g)(h,h) = G^c(L_g^c h, h)$$

for some second order partial differential operator L_g^c. By (4.4.1), the principal part — i.e. the part of highest order — of L_g^c is given by

$$h \mapsto a_1(c)\big(\Delta^g(\operatorname{Tr}_g h) + \delta\delta h\big)g - \frac{1}{2}\Delta^g h + \frac{1}{2}\operatorname{Hess}\operatorname{Tr}_g h + \delta^*\delta h$$

and depends on c.

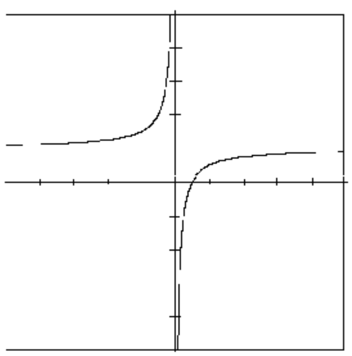
The principal symbol of a differential operator $P\colon \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(F)$ (E, F vector finite dimensional real vector bundles over M) of order m at $\xi \in T_x^*M$ is given by $\sigma_\xi(P).h = \lim_{t\to\infty} t^{-m}e^{-t\phi}P(e^{t\phi}h)$ where $\phi\colon M \to \mathbb{R}$, $\phi(x) = 0$ and $d\phi(x) = \xi$. The principal symbol of ∇^2 is $\sigma_\xi(\nabla^2).h = \xi\otimes\xi\otimes h$. Thus

$$\begin{aligned}
&\sigma_\xi(\Delta).h = -\|\xi\|^2 h,\\
&\sigma_\xi(\delta^*\delta).h = -\mathbf{sym}\circ c_{13}(\xi\otimes\xi\otimes h),\\
&\sigma_\xi(\operatorname{Hess}\operatorname{Tr}_g).h = -(\operatorname{Tr}_g h)\xi\otimes\xi,\\
&\sigma_\xi(\Delta\operatorname{Tr}_g).h = -\|\xi\|^2\operatorname{Tr}_g h,\\
&\sigma_\xi(\delta\delta).h = -h(g^{-1}(\xi), g^{-1}(\xi))
\end{aligned}$$

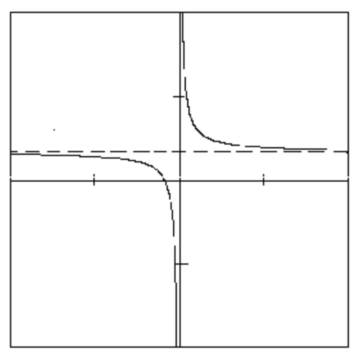
and the principal symbol of L_g^c is

$$\begin{aligned}
\sigma_\xi(L_g^c).h = {}&-a_1(c)\big(\|\xi\|^2\operatorname{Tr}_g(h) + h(g^{-1}(\xi), g^{-1}(\xi))\big)g\\
&+\frac{1}{2}\|\xi\|^2 h - \frac{1}{2}(\operatorname{Tr}_g h)\xi\otimes\xi - \frac{1}{2}\xi\otimes h(g^{-1}(\xi),.) - \frac{1}{2}h(g^{-1}(\xi),.)\otimes\xi
\end{aligned}$$

L_g^c depends on c via $a_1(c)$. For $\dim(M) = n = 2$, $a_1(c) \equiv \frac{1}{2}$ is constant. In all other dimensions $\dim(M) = n$, the image of $\mathbb{R}\setminus\{0\} \ni c \mapsto a_1(c)$ is $\mathbb{R}\setminus\{\frac{1}{n}\}$:



$n = 1$, $c \mapsto a_1(c)$



$n > 2$, $c \mapsto a_1(c)$

4.7. Theorem. *If* $\dim(M) = 1$. *then* L_g^c *is elliptic iff* $c \neq \frac{1}{3}$. *If* $\dim(M) = 2, 3$ *or* 4 *then for each* $c \in \mathbb{R}\setminus\{0\}$*, the operator* L_g^c *is not elliptic.*

Proof. If $\dim(M) = 1$, then in a g-normal basis at x

$$\sigma_{\xi_x}(L_g^c).h = (-2a_1(c) + 1)\xi_x^2 h.$$

Thus L_g^c is elliptic iff $c \neq \frac{1}{3}$.

Now assume $\dim(M) = 2$. Here $a_1(c) \equiv \frac{1}{2}$. Let $x \in M$ be an arbitrary point and chose a normal chart of M centered at x for the metric g. Let $(\xi_1, \xi_2) = \xi \in T_x^*M$ be such that $\xi_1^2 + \xi_2^2 = 1$. The symmetric $(2,0)$-tensors at x are identified with $\mathbb{R}^3$ by $h \mapsto (h_{11}, h_{12}, h_{22})$. With this identification the principal symbol at ξ is given by the matrix

$$A_2(\xi_1, \xi_2) = \begin{pmatrix} -\frac{1}{2} & -\xi_1\xi_2 & -1 \\ 0 & 0 & -2\xi_1\xi_2 \\ -\frac{1}{2} - \xi_1^2 & -\xi_1\xi_2 & -2\xi_2^2 \end{pmatrix}$$

The determinant is $\det(A_2(\xi_1, \xi_2)) = 2\xi_1^4(\xi_1^2 - 1)$. Thus at $\xi = (1, 0)$ $\xi = (0, 1)$, $A_2(\xi_1, \xi_2)$ is singular.

For $n = \dim(M) > 2$, $c \mapsto a_1(c)$ maps $\mathbb{R} \setminus \{0\}$ onto $\mathbb{R} \setminus \{\frac{1}{n}\}$. Put $\xi_3 = \cdots = \xi_n = 0$, $\|\xi\| = 1$. Then I_n, the subspace of the space of symmetric matrices defined by

$$I_n = \{h \in \mathbb{R}^{(n\times n)} \colon h^T = h, h_{ij} = 0 \text{ if } (i,j) \notin \{1,2\} \times \{1,2\} \text{ and } i \neq j \ \},$$

is invariant under the local expression of principal symbol. E.g. for $n = 3$:

$$\begin{aligned} \sigma_\xi(L_g^c).h = & -a_1\big((1 + \xi_1^2)h_{11} + (1 + \xi_2^2)h_{22} + h_{33} + 2h_{12}\xi_1\xi_2\big) \operatorname{Id} \\ & + \frac{1}{2}h - \frac{1}{2}(h_{11} + h_{22} + h_{33}) \begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & 0 \\ \xi_1\xi_2 & \xi_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ & - \mathbf{sym} \begin{pmatrix} h_{11}\xi_1^2 + h_{12}\xi_1\xi_2, & h_{11}\xi_1\xi_2 + h_{12}\xi_2^2, & 0 \\ h_{12}\xi_2^2 + h_{22}\xi_1\xi_2, & h_{12}\xi_1\xi_2 + h_{22}\xi_2^2, & 0 \\ h_{13}\xi_1^2 + h_{23}\xi_1\xi_2, & h_{13}\xi_1\xi_2 + h_{23}\xi_2^2, & 0 \end{pmatrix} \end{aligned}$$

With the identification $(h_{11}, h_{12}, h_{22}, h_{33}) \mapsto \mathbb{R}^4$ the principal symbol acts on $\mathbb{R}^4$ by the matrix

$$A_3 = \begin{pmatrix} -a_1 + (\frac{1}{2} - a_1)\xi_1^2 & -\xi_1\xi_2 & -a_1(1 + \xi_1^2) - \frac{1}{2}\xi_2^2 & -a_1(1 + \xi_1^2) \\ (-2a_1 + 1)\xi_1\xi_2 & 0 & -(2a_1 + 1)\xi_1\xi_2 & -2a_1\xi_1\xi_2 \\ -a_1 - (a_1 + \frac{1}{2})\xi_1^2 & -\xi_1\xi_2 & -a_1(1 + \xi_2^2) + \frac{1}{2} - \frac{3}{2}\xi_2^2 & -a_1(1 + \xi_2^2) \\ -(a_1 + \frac{1}{2}\xi_1^2) & -\frac{1}{2}\xi_1\xi_2 & -(a_1 + \frac{1}{2}\xi_2^2) & \frac{1}{2} - a_1 \end{pmatrix}$$

One computes that $\det(A_3(a_1, \xi)) = \frac{(1+2a_1)\xi_1^2(\xi_1^2-1)(1-2a_1+4a_1\xi_1^2)}{4}$. Thus at $\xi = (1, 0, 0)$ and $\xi = (0, 1, 0)$, $\sigma_\xi(L_g^c)$ maps I_3 on a proper subspace of I_3. Hence at $\xi = (1, 0, 0)$ and $\xi = (0, 1, 0)$, $\sigma_\xi(L_g^c) \colon \mathcal{S}^2(T_x^*M) \to \mathcal{S}^2(T_x^*M)$ is not an isomorphism.

For $n = 4$, the subspace $I_4 = \{h \in \mathbb{R}^{(4\times 4)} \colon h^t + h, h_{13} = h_{23} = h_{14} = h_{24} = h_{34} = 0\}$ is invariant and with the identification $(h_{11}, h_{12}, h_{22}, h_{33}, h_{44}) \mapsto \mathbb{R}^5$, the relevant determinant is

$$\det(A_4(a_1, \xi)) = \frac{(1 + 2a_1)\xi_1^2(\xi_1^2 - 1)(1 - 2a_1 + 4a_1\xi_1^2)}{8}$$

and $A_4(a_1, \xi)$ is singular for $\xi = (1, 0, 0, 0)$. $\square$

4.8. Conjecture. *For $n \geq 3$, the relevant determinant is*

$$\det(A_n(a_1,\xi)) = \frac{(1+2a_1)\xi_1^2(\xi_1^2-1)(1-2a_1+4a_1\xi_1^2)}{2^{n-1}}.$$

Thus at $\xi = (1,0,\dots,0)$, $\sigma_\xi(L_g^c)$ is not invertible, and L_g^c is not elliptic.

5. The Hessian on Submanifolds

In 5.1 a method for computing the Hessian on splitting submanifolds is presented. In particular, it allows to compute the Hessian on critical points without any knowledge of the Christoffel symbols of the metric.

This method is applied to $\mathcal{M}_{\mathrm{vol}(g)}$ and $\mathcal{M}_{\mathrm{Vol}(g)}$. For $\mathcal{M}_{\mathrm{vol}(g)}$ a second proof is given using geodesics.

5.1. Method. Consider a section of projections π which is some sum of sections π_1, π_2, π_3 from 1.4. Let $F\colon \mathcal{M} \to \mathbb{R}$ be a smooth function the gradient $\operatorname{grad}^c F$ of which is smooth. Consider an integral submanifold $\mathcal{S}$ of $\mathcal{M}$ corresponding to π, the restriction $\tilde{G} = G^c \restriction \mathcal{S}$ with covariant derivative $\tilde{\nabla} = \pi \circ \nabla$, and the restriction $f = F \restriction \mathcal{S}$. Then $\tilde{\zeta} = (\mathrm{Id}, \zeta) = (\mathrm{Id}, \pi \circ \operatorname{grad}^c F)$ is an extension of $\operatorname{grad} f$, the (thus smooth) gradient of f. This observation allows to compute the Hessian of f using the global chart of $\mathcal{M}$. For $g \in \mathcal{S}, h \in T_g\mathcal{S}$ by the orthogonality of the projection

$$\begin{aligned}(\operatorname{Hess} f)(g)(h,h) =& \tilde{G}_g(\tilde{\nabla}_h \operatorname{grad}^c f(g), h) = G_g^c(\pi((\nabla_h\tilde{\zeta})(g)), h)\\ =& G_g^c((\nabla_h\tilde{\zeta})(g), h) =\\ =& G_g\big(\pi\big(d\zeta(g).h - \Gamma_g^c(\zeta(g),h)\big), h\big)\end{aligned}$$

In particular this method allows to derive the Hessian *at critical points* of such submanifolds only out of the gradient and without explicit knowledge of the Christoffel symbols. Nevertheless, in the literature often strange calculations are applied (e.g. in [**Bes88,** Proposition 4.55]).

5.2. The Hessian of Scal **on $\mathcal{M}_{\mathrm{vol}(g)}$.** As $G^c \restriction \mathcal{M}_{\mathrm{vol}(g)} = G \restriction \mathcal{M}_{\mathrm{vol}(g)}$ for all $c \neq 0$ I will consider $c = \frac{1}{n}$. Let $g \in \mathcal{M}_{\mathrm{vol}(g)}$ and $h \in T_g\mathcal{M}_{\mathrm{vol}(g)} = \mathcal{C}^\infty(S_g^2 M)$. Then $\zeta(g) = \pi_3((\operatorname{grad}\mathrm{Scal})(g)) = -\operatorname{Ric}_0(g)$. The necessary terms are very easy to compute:

$$d\zeta(g).h = -d\operatorname{Ric}_0(g).h = -d\operatorname{Ric}(g).h - \frac{1}{n}(h, \operatorname{Ric}(g))_g g + \frac{1}{n}\operatorname{scal}(g)h$$

and

$$\pi_3(\Gamma_g(\zeta(g),h)) = -(\frac{1}{2}\operatorname{Ric}(g)_0 g^{-1}h + \frac{1}{2}hg^{-1}\operatorname{Ric}(g)_0)_0.$$

Note that for $H = g^{-1}h$

$$\operatorname{Tr}(g^{-1}\operatorname{Ric}(g)_0 H^2) = \operatorname{Tr}(g^{-1}\operatorname{Ric}(g) H^2) - \frac{1}{n}\operatorname{scal}(g)\operatorname{Tr}(H^2)$$

With the notation $\widetilde{\mathrm{Scal}} = \mathrm{Scal} \restriction \mathcal{M}_{\mathrm{vol}(g)}$ one obtains

$$\begin{aligned}(\mathrm{Hess}^c\,\widetilde{\mathrm{Scal}})(g)(h,h) =& G_g(\pi_3(d\zeta(g).h - \Gamma_g(\zeta(g),h)),h) = \\ =& G_g(-\frac{1}{2}\Delta^g h - M(h) + \frac{1}{2}\operatorname{Ric}(g)g^{-1}h + \frac{1}{2}hg^{-1}\operatorname{Ric}(g),h)\end{aligned}$$

5.3. Calculation via geodesics. In [**FrG89**] the geodesics on $\mathcal{M}_{\mathrm{vol}(g)}$ are described. They are of the form $g(t) = g_0 e^{tH}$ for $H = g_0^{-1}h$. Thus

$$g'(t) = g_0 e^{tH} H = g(t)H, \quad g''(t) = g(t)H^2.$$

Now using that $\operatorname{Tr} H = 0$ one computes

$$\begin{aligned}\frac{d}{ds}\Big|_t \mathrm{Scal}(g(s)) &= dS(g(t))g'(t) = G_{g(t)}(\operatorname{grad} S(g), g'(t)) = \\ &= \int_M \operatorname{Tr}\big(g(t)^{-1}\big(\frac{1}{2}\operatorname{scal}(g(t))g(t) - \operatorname{Ric}(g)\big)H\big)\operatorname{vol}(g(t)) = \\ &= -\int_M \operatorname{Tr}(g(t)^{-1}\operatorname{Ric}(g(t))H)\operatorname{vol}(g).\end{aligned}$$

Therefore,

$$\frac{d^2}{dt^2}\Big|_0 S(g(t)) = \int_M \big(\operatorname{Tr}(Hg_0^{-1}\operatorname{Ric}(g_0)H) - \operatorname{Tr}(g_0^{-1}(d\operatorname{Ric}(g_0).h)H)\big)\operatorname{vol}(g).$$

As $\operatorname{Tr} H = 0$, the derivation of Ric in direction h is given by $d\operatorname{Ric}(g_0).h = \frac{1}{2}\Delta^{g_0}h + M(h)$. Thus

$$(\mathrm{Hess}(\widetilde{\mathrm{Scal}}))(g)(h,h) = G_{g_0}(-\frac{1}{2}\Delta^{g_0}h - M(h) + \frac{1}{2}\operatorname{Ric}(g_0)g_0^{-1}h + \frac{1}{2}hg_0^{-1}\operatorname{Ric}(g_0),h),$$

which is exactly the same expression as in 5.2.

5.4. The Hessian of Scal **on** $\mathcal{M}_{\mathrm{Vol}(g)}$**.** As the problems of the geodesics on $\mathcal{M}_{\mathrm{Vol}(g)}$ seems to be unsolved up to now (see 1.12), so far only the method from 5.1 applies. Put $\widetilde{\mathrm{Scal}} = \mathrm{Scal} \restriction \mathcal{M}_{\mathrm{Vol}(g)}$ and

$$\begin{aligned}\zeta(g) =& (\pi_2 \oplus \pi_3)(\operatorname{grad}\mathrm{Scal}) \\ =& \frac{n-2}{2n}(\operatorname{scal}(g) - \frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)})g - \operatorname{Ric}(g)_0 \\ =& \frac{1}{2}(\operatorname{scal}(g) - \frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)})g + \frac{1}{n}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)}g - \operatorname{Ric}(g).\end{aligned}$$

It was already proved that $d\,\mathrm{Scal}(g).h = G_g(\frac{1}{2}\,\mathrm{scal}(g)g - \mathrm{Ric}(g), h)$. For $h \in T_g\mathcal{M}_{\mathrm{vol}(g)}$, $d\,\mathrm{Vol}(g).h = 0$

$$\begin{aligned} d\zeta(g).h =& \frac{1}{2}\big(-(\mathrm{Ric}(g),h)_g g + \Delta^g(\mathrm{Tr}_g h)g + (\delta^g\delta^g h)g\big) + \frac{1}{2}\,\mathrm{scal}(g)h \\ &+ \frac{1}{2}\,\mathrm{Hess}(\mathrm{Tr}_g h) - \frac{1}{2}\Delta^g h - M(h) \\ &- \frac{n-2}{2n}\frac{1}{\mathrm{Vol}(g)}G_g(\frac{1}{2}\,\mathrm{scal}(g)g - \mathrm{Ric}(g),h)g - \frac{n-2}{2n}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)}h \end{aligned}$$

and $\mathrm{Tr}_g(\zeta(g)) = \frac{n-2}{2}(\mathrm{scal}(g) - \frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)})$.

$$\begin{aligned} \Gamma_g(\zeta(g),h) =& \frac{1}{2}(\mathrm{scal}(g) - \frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)})g + \frac{1}{n}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)}g - \frac{1}{2}\,\mathrm{Ric}(g)g^{-1}h - \frac{1}{2}hg^{-1}\,\mathrm{Ric}(g) \\ &- \frac{1}{4}(h,\mathrm{Ric}(g))_g g - \frac{n-2}{8}(\mathrm{scal}(g) - \frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)})h + \frac{1}{4}(\mathrm{Tr}_g h)\,\mathrm{Ric}_g \end{aligned}$$

Since $G_g((\delta^g\delta^g h)g, h) = G_g(\mathrm{Hess}\,\mathrm{Tr}_g h, h)$,

$$\begin{aligned} (\mathrm{Hess}\,\widetilde{\mathrm{Scal}})(g)(h,h) =& G_g\bigg(\frac{1}{2}(\Delta^g(\mathrm{Tr}_g h)g + 2\,\mathrm{Hess}(\mathrm{Tr}_g h) - \Delta^g h) - M(h) \\ &+ \frac{1}{2}\,\mathrm{Ric}(g)g^{-1}h + \frac{1}{2}hg^{-1}\,\mathrm{Ric}(g) \\ &- \frac{1}{2}(\mathrm{Ric}(g),h)g + \frac{1}{2}(\mathrm{scal}(g) - \frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)})h \\ &- \frac{n-2}{2n}G_g(\frac{1}{2}\,\mathrm{scal}(g)g - \mathrm{Ric}(g),h)g \\ &- \frac{1}{2}(\mathrm{scal}(g) - \frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)})g + \frac{n+2}{8}(\mathrm{scal}(g) - \frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)})h, h\bigg) \end{aligned}$$

If g is critical then $\mathrm{Ric} = \frac{1}{n}\,\mathrm{scal}(g)g$ and $G(\frac{1}{2}\,\mathrm{scal}(g)g - \mathrm{Ric}(g),h) = 0$.
Then in particular $\frac{1}{2}(\mathrm{Ric}(g),h)g = \frac{1}{2n}\,\mathrm{scal}(g)(\mathrm{Tr}_g h)g$ and thus

$$\begin{aligned} (\mathrm{Hess}\,\widetilde{\mathrm{Scal}})(g)(h,h) =& G_g\bigg(\frac{1}{2}(\Delta^g(\mathrm{Tr}_g h)g + 2\,\mathrm{Hess}(\mathrm{Tr}_g h) - \Delta^g h) - M(h) \\ &+ \frac{1}{2}\,\mathrm{Ric}(g)g^{-1}h + \frac{1}{2}hg^{-1}\,\mathrm{Ric}(g) \\ &- \frac{1}{2n}\,\mathrm{scal}(g)(\mathrm{Tr}_g h)g, h\bigg) \\ =& G_g\bigg(\frac{1}{2}(\Delta^g(\mathrm{Tr}_g h)g + 2\,\mathrm{Hess}(\mathrm{Tr}_g h) - \Delta^g h) + \delta^*\delta h \\ &+ \mathrm{Riem}(g).h - \frac{1}{2n}\,\mathrm{scal}(g)(\mathrm{Tr}_g h)g, h\bigg) \end{aligned}$$

where I have used equality 3.6.1 to obtain the last identity. Note the slight mistake in the term in [**Bes88,** Proposition 4.55].

The principal part of the partial differential operator $\tilde{L}_g$ defined by

$$G_g(\tilde{L}_g h, h) = (\operatorname{Hess}\widetilde{\operatorname{Scal}})(g)(h,h)$$

coincides with the principal part of the operator $L_g^{\frac{1}{n}}$ for the Hessian of the total scalar curvature on $\mathcal{M}$.

6. The Conformal Class

If a metric g is fixed and as long as no confusion arises, I will write $\langle,\rangle$ instead of $\langle,\rangle_g$ for the inner product on $C_c^\infty(M,\mathbb{R})$, $\Omega_c^1(M)$ and $\mathfrak{X}_c(M)$.

6.1. Geodesics. Let M be a compact finite dimensional manifold. From 1.3 it follows that the geodesic in $\mathcal{M}$ starting at $\tilde{g} \in \operatorname{Conf}^g$ in direction $h = f\tilde{g}$ for some $f \in C^\infty(M,\mathbb{R})$ is

$$c(t) = \tilde{g}(1 + \frac{nt}{4}f)^{\frac{4}{n}}$$

since then $H_0 = 0$ and $a(t) = (1 + \frac{nt}{4}f)^{\frac{4}{n}}$. Thus Conf^g is geodesically closed in $\mathcal{M}$.

Conf^g is not geodesically complete, but any two metrics in Conf^g can be joined by a unique geodesic segment in Conf^g. For existence assume $\tilde{g} = \psi g$ with $\psi \in C^\infty(M,\mathbb{R})$, $\psi > 0$ and put $f = \frac{4}{n}(\psi^{\frac{n}{4}} - 1)$, the exponent being defined as $\psi > 0$. Then for $0 \le t \le 1$ holds $(1-t) + t\psi^{\frac{n}{4}} > 0$ and thus

$$c(t) = (1 + \frac{nt}{4}f)^{\frac{4}{n}}g = ((1-t) + t\psi^{\frac{n}{4}})g$$

is a required geodesic segment.

On the other hand assume without restriction that $c(0) = g$, $c(1) = \psi g = \tilde{g}$. Then $c(1) = (1 + \frac{n}{4}f)^{\frac{4}{n}} = \psi g$ which determines f uniquely.

6.2. Length of geodesics. Let $c : [0,1] \to \operatorname{Conf}^g$ be a geodesic segment $c(t) = (1 + \frac{nt}{4}f)^{\frac{4}{n}}g$. Thus $c'(t) = c(t)\frac{f}{1+\frac{nt}{4}f}$, $\operatorname{vol}(c(t)) = (1 + \frac{nt}{4}f)^2 \operatorname{vol}(g)$ and

$$\begin{aligned} G_{c(t)}(c'(t), c'(t)) &= n \int_M \frac{f^2}{(1+\frac{nt}{4}f)^2}(1 + \frac{nt}{4}f)^2 \operatorname{vol}(g) \\ &= n \int_M f^2 \operatorname{vol}(g) \end{aligned}$$

This yields the arclength of c as

$$\int_0^1 \sqrt{G_{c(t)}(c'(t), c'(t))}dt = \sqrt{n \int_M f^2 \operatorname{vol}(g)}.$$

6.3. The geodesic distance $d(g,\psi g)$ between two metrics g and ψg is

$$\begin{aligned} d(g,\psi g) &= \sqrt{n\int_M \big(\frac{4}{n}(\psi^{\frac{n}{4}}-1)\big)^2 \operatorname{vol}(g)} \\ &= \frac{4}{\sqrt{n}}\sqrt{\operatorname{Vol}(g)+\int_M (\psi^{\frac{n}{2}}-2\psi^{\frac{n}{4}})\operatorname{vol}(g)} \\ &= \frac{4}{\sqrt{n}}\sqrt{\operatorname{Vol}(g)+\operatorname{Vol}(\psi g)-2\operatorname{Vol}(\sqrt{\psi}g)}. \end{aligned}$$

In general, for $\phi,\psi\in C^\infty(M,\mathbb{R})$, $\phi,\psi>0$:

$$\begin{aligned} d(\phi g,\psi g) &= d(\phi g,(\psi\phi^{-1})\phi g) \\ &= \frac{4}{\sqrt{n}}\sqrt{\operatorname{Vol}(\phi g)+\operatorname{Vol}(\psi g)-2\operatorname{Vol}(\sqrt{\phi\psi}g)} \\ &= \frac{4}{\sqrt{n}}\sqrt{\int_M \big(\phi^{\frac{n}{4}}-\psi^{\frac{n}{4}}\big)^2\operatorname{vol}(g)}. \end{aligned}$$

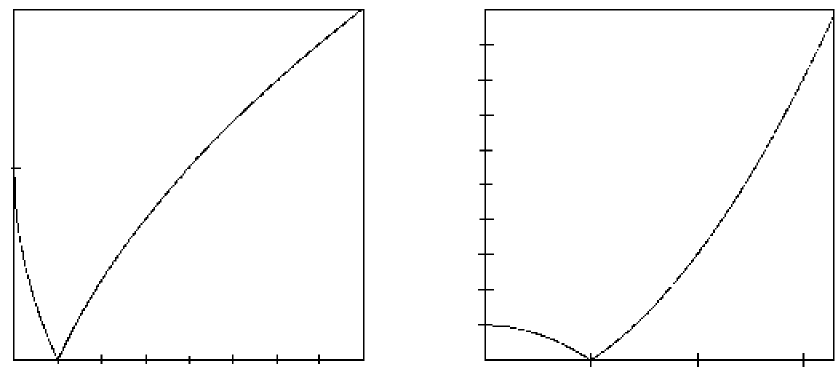
6.4. Proposition. *Denote the diagonal in* $\operatorname{Conf}^g\times\operatorname{Conf}^g$ *by* $\operatorname{Diag}(\operatorname{Conf}^g)$. *The geodesic distance is given by*

$$d(\phi g,\psi g) = \frac{4}{\sqrt{n}}\sqrt{\int_M \big(\phi^{\frac{n}{4}}-\psi^{\frac{n}{4}}\big)^2\operatorname{vol}(g)},$$

and it is smooth exactly on $\operatorname{Conf}^g\times\operatorname{Conf}^g\setminus\operatorname{Diag}(\operatorname{Conf}^g)$.
In particular, for $\psi\equiv r\in\mathbb{R}$*:*

$$d(g,rg) = \frac{4}{\sqrt{n}}\,\big|(r^{\frac{n}{4}}-1)\big|\,\sqrt{\operatorname{Vol}(g)}.$$

The mapping $\mathbb{R}^+\ni r\mapsto d(g,rg)$ is — depending on the dimension of the manifold $\dim(M)$ — of the form with the singularity at $r=1$. If $r\to 0$ then $d(g,rg)\to\frac{4}{\sqrt{n}}\operatorname{Vol}(g)<\infty$. If $r\to\infty$ then $d(g,rg)\to\infty$. For $\dim(M)=4$ the mapping $\mathbb{R}^+\ni +r\mapsto d(g,rg)$ is a straight line crushed at $r=1$, similar to the standard geodesic distance on $\mathbb{R}^+$.



$r \mapsto d(rg, g)$ if $\dim(M) < 4$ $r \mapsto d(rg, g)$ if $\dim(M) > 4$

6.5. The functional $\tilde{N}$. Put $\tilde{N} = N \restriction \mathrm{Conf}^g$. The gradient of $\tilde{N}$ is

$$\operatorname{grad} \tilde{N}(\tilde{g}) = \frac{n-2}{2n} \operatorname{Vol}(\tilde{g})^{\frac{2-n}{n}} (\operatorname{scal}(\tilde{g}) - \frac{\operatorname{Scal}(\tilde{g})}{\operatorname{Vol}(\tilde{g})})\tilde{g}$$

and $\operatorname{grad}(\tilde{g}) = 0$ iff $\operatorname{scal}(\tilde{g}) = \frac{\operatorname{Scal}(\tilde{g})}{\operatorname{Vol}(\tilde{g})}$, in particular iff $\operatorname{scal}(\tilde{g})$ is constant. By a result proved by Schoen [**Sch84**] in each conformal class is a metric of constant scalar curvature, i.e. a critical point of $\tilde{N}$. Einstein metrics are unique in their conformal class with constant total volume.

A conformal class with Einstein metric admits a characteristic function as follows: Assign to each metric $\tilde{g}$ the arclength of the unique geodesic segment from $\tilde{g}$ to the Einstein metric with the same total volume.

6.6. Assume $\dim(M) = n \geq 2$. In the parameterization $\tilde{g} = f^{\frac{4}{n-2}} g$ ($f \in C^\infty(M, \mathbb{R})$, $f > 0$) one may compute that

$$\text{(TS)} \qquad \begin{aligned} &4\frac{n-1}{n-2}\Delta^g f + \operatorname{scal}(g) f = \operatorname{scal}(\tilde{g}) f^{\frac{n+2}{n-2}} \\ &\operatorname{vol}(\tilde{g}) = f^{\frac{2n}{n-2}} \operatorname{vol}(g) \end{aligned}$$

Thus

$$\tilde{N}(\tilde{g}) = \frac{\int_M \left(4\frac{n-1}{n-2}(\Delta^g f) f + \operatorname{scal}(g) f^2\right) \operatorname{vol}(g)}{\left(\int_M f^{\frac{2n}{n-2}} \operatorname{vol}(g)\right)^{\frac{n-2}{n}}}$$

and $\tilde{N}$ turns out to be the Yamabe-functional in the case $q = \frac{2n}{n-2}$ of [**Aub82,** p. 126].

6.7. $(n \geq 2)$. Let $\tilde{g} = f^{\frac{4}{n-2}}$. The transformation formula for the scalar curvature (TS) from 6.6 suggests the partial differential operator

$$L^g = -\Delta^g - \frac{n-2}{4(n-1)} \operatorname{scal}(g),$$

which is called the conformal Laplacian ([**Sch84**], [**ScY88**] — our sign coincides with the sign there), and which satisfies for all $\psi \in C^\infty(M, \mathbb{R})$ the identity

$$L^g(f\psi) = f^{\frac{n+2}{n-2}} L^{\tilde{g}}(\psi)$$

One may express $\tilde{N}(g)$ with the conformal Laplacian:

$$\tilde{N}(\tilde{g}) = -4\frac{n-1}{n-2} \frac{\langle L^g(f), f\rangle_g}{\left(\int_M f^{\frac{2n}{n-2}} \operatorname{vol}(g)\right)^{\frac{n-2}{n}}}.$$

6.8. The Hessian. The Hessian of $\tilde{N}$ is by definition

$$\operatorname{Hess} \tilde{N}(g)(h,h) = \frac{d}{dt}\bigg|_0 G_{c(t)}(\operatorname{grad} \tilde{N}(c(t)), c'(t))$$

for the geodesic $c(t)$ with $c(0) = g$, $c'(0) = fg = h$. A geodesic c satisfies

$$c'(t) = \frac{f}{1 + \frac{nt}{4}} c(t), \quad c''(0) = \frac{4-n}{4} f^2 g.$$

Thus

$$\begin{aligned} &G_{c(t)}(\operatorname{grad} \tilde{N}(c(t)), c'(t)) = \\ =&\frac{n-2}{2} \operatorname{Vol}(c(t))^{\frac{2-n}{n}} \int_M \big(\operatorname{scal}(c(t)) - \frac{\operatorname{Scal}(c(t))}{\operatorname{Vol}(c(t))}\big) \frac{f}{1+\frac{nt}{4}} \operatorname{vol}(c(t)). \end{aligned}$$

And an easy calculation using the identities $\delta^g(\delta^g(h)) = -\Delta^g f$ and $(\operatorname{Ric}(g), h)_g = f \operatorname{scal}(g)$ yields

$$\begin{aligned} \operatorname{Hess} \tilde{N}(g)(h,h) =& \frac{n-2}{2} \operatorname{Vol}(g)^{\frac{2-n}{n}} \times \\ &\times \Big(\int_M \big((n-1)\Delta f - \operatorname{scal}(g) f\big) f \operatorname{vol}(g) \\ &\quad + \frac{1}{\operatorname{Vol}(g)} \int_M f \operatorname{scal}(g) \operatorname{vol}(g) \int_M f \operatorname{vol}(g) \\ &\quad + (1-n) \int_M f \operatorname{vol}(g) \int_M \big(\operatorname{scal}(g) - \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}\big) f \operatorname{vol}(g) \\ &\quad + \frac{n}{4} \int_M \big(\operatorname{scal}(g) - \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}\big) f^2 \operatorname{vol}(g) \Big). \end{aligned}$$

6.9. Symmetrisation. Put $\alpha = \frac{n-2}{n}\operatorname{Vol}(g)^{\frac{2-n}{n}}$. Then $\alpha > 0$. In the sequel assume $c, c_1, c_2 \in \mathbb{R}$ and $f, f_1, f_2 \in C_0^g(M, \mathbb{R})$.

For the symmetrisation of the Hessian one computes using $\int_M \Delta f \operatorname{vol}(g) = 0$ and $\int_M \operatorname{scal}(g)\operatorname{vol}(g) = \operatorname{Scal}(g)$ that

$$\operatorname{Hess}\tilde{N}(g)(c_1 g, c_2 g) = 0 \tag{6.9.1}$$

for all $c_1, c_2 \in \mathbb{R}$ and that

$$\frac{1}{\alpha}\operatorname{Hess}\tilde{N}(g)(cg, fg) = -\frac{n}{4}c\int_M \operatorname{scal}(g) f \operatorname{vol}(g) \tag{6.9.2}$$

for all $c \in \mathbb{R}$ and $f \in C_0^g(M, \mathbb{R})$. Moreover one sees that

$$\begin{aligned}\frac{1}{\alpha}\operatorname{Hess}\tilde{N}(g)(f_1 g, f_2 g) &= \int_M \big((n-1)\Delta f_1 - \operatorname{scal}(g) f_1\big) f_2 \operatorname{vol}(g)\\ &\quad + \frac{n}{4}\int_M \big(\operatorname{scal}(g) - \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}\big) f_1 f_2 \operatorname{vol}(g).\end{aligned}$$

since these terms are already symmetric.
Summing up one gets for $\tilde{f}_i \in C^\infty(M, \mathbb{R})$ $(i = 1, 2)$ and $c_i = \pi_T(\tilde{f}_i) \in \mathbb{R}$, $f_i = \pi_3(\tilde{f}_i) \in C^\infty(M, \mathbb{R})$

$$\begin{aligned}&\frac{1}{\alpha}\operatorname{Hess}\tilde{N}(g)(\tilde{f}_1 g, \tilde{f}_2 g) =\\ &= -\frac{n}{4}\big(c_1\int_M \operatorname{scal}(g) f_2 \operatorname{vol}(g) + c_2\int_M \operatorname{scal}(g) f_1 \operatorname{vol}(g)\big)\\ &+ \int_M \big((n-1)\Delta f_1 - \operatorname{scal}(g) f_1\big) f_2 \operatorname{vol}(g) + \frac{n}{4}\int_M (\operatorname{scal}(g) - \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}) f_1 f_2 \operatorname{vol}(g).\end{aligned}$$

6.10. The symmetric form $H(g)$, the quadratic form $Q(g)$ and the PDO P^g. For $g \in \mathcal{M}$ define the symmetric form $H(g)$ on $C^\infty(M, \mathbb{R})$

$$H(g)(\tilde{f}_1, \tilde{f}_2) = \frac{1}{\alpha(n-1)}\operatorname{Hess}\tilde{N}(g)(\tilde{f}_1 g, \tilde{f}_2 g)$$

and the associated quadratic form $Q(g)$ on $C^\infty(M, \mathbb{R})$

$$Q(g)(\tilde{f}) = H(g)(\tilde{f}, \tilde{f})$$

Define the elliptic partial differential operator $P^g : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$

$$P^g = \Delta^g + \frac{n-4}{4(n-1)}\operatorname{scal}(g) - \frac{n}{4(n-1)}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}.$$

In $C^\infty(M,\mathbb{R})$ orthogonal complements are taken with respect to $\langle,\rangle_g$: e.g. $\mathrm{scal}(g)^\perp = \{f \in C^\infty(M,\mathbb{R})\colon \langle f, \mathrm{scal}(g)\rangle_g = 0\}$. One easily verifies that

$$P^g\colon\ \mathrm{scal}(g)^\perp \cap C_0^g(M,\mathbb{R}) \to C_0^g(M,\mathbb{R})$$

and that for $f_1, f_2 \in C_0^g(M,\mathbb{R})$ and $\tilde f_1, \tilde f_2 \in \mathrm{scal}(g)^\perp$

$$H(g)(f_1,f_2) = \langle P^g(f_1), f_2\rangle_g, \quad H(g)(\tilde f_1, \tilde f_2) = \langle P^g(\tilde f_1), \tilde f_2\rangle_g.$$

Furthermore, $P^g + L^g = \frac{1}{4(n-1)}\big(-2\,\mathrm{scal}(g) - n\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)}\big)$.

The subspace of degeneracy of $H(g)$ will be denoted by $D(g)$. The maximal strictly positive (or strictly negative) subspace will be denoted by $H(g)^+$ (resp. $H(g)^-$). They exist, since $H(g)$ is symmetric and P^g is elliptic, and thus there is a maximal set of smoooth functions in $C^\infty(M,\mathbb{R})$ that diagonalize $H(g)$. Furthermore,

$$C^\infty(M,\mathbb{R}) = H(g)^+ \oplus D(g) \oplus H(g)^-.$$

$f \in C^\infty(M,\mathbb{R})$ is called lightlike iff $H(g)(f,f) = 0$. If $D(g) = \{0\}$ then the set of lightlike vectors is called the lightcone.
From 6.9.1 and 6.9.2 follows that $\mathbb{R}$ is always lightlike for $H(g)$ and

$$H(g)(\mathbb{R}, C_0^g(M,\mathbb{R})) = 0 \text{ iff } g \text{ is a critical point of } \tilde N.$$

In particular $\mathbb{R}g \subset D(g)$ iff g critical.

6.11. Consider the trivial line bundle $\mathbb{R} \to C^\infty(M,\mathbb{R}) \to C_0^g(M,\mathbb{R})$.
(1) ($Q(g)$) For every $f \in C_0^g(M,\mathbb{R})$ with $\int_M f\,\mathrm{scal}(g)\,\mathrm{vol}(g) \neq 0$ exists exactly one $c \in \mathbb{R}$, in fact $c = \big(\int_M f\,\mathrm{scal}(g)\,\mathrm{vol}(g)\big)^{-1} Q(g)(f)$ such that $Q(g)(f+c) = 0$. In other words: Above each function in $C_0^g(M,\mathbb{R})$ which is not perpendicular to the scalar curvature $\mathrm{scal}(g)$ lies exactly one lightlike vector.
(2) For $f \in C_0^g(M,\mathbb{R}) \cap \mathrm{scal}(g)^\perp$ is $Q(g)(f) = 0$ iff $Q(f+c) = 0$ for all $c \in \mathbb{R}$. In other words: For $f \in C_0^g(M,\mathbb{R}) \cap \mathrm{scal}(g)^\perp$ either the whole fiber through f or no point in this fiber is lightlike.
If g is critical then $\mathrm{scal}(g)^\perp = C_0^g(M,\mathbb{R})$ and therefore only (2) applies.

6.12. Properties if g is critical. Some of them are well known — sometimes in disguise. If g is critical

$$H(g)(c_1 + f_1, c_2 + f_2) = \int_M \big(\Delta f_1 - \frac{\mathrm{scal}(g)}{n-1} f_1\big) f_2\,\mathrm{vol}(g)$$

and $P^g = \Delta^g - \frac{\mathrm{scal}(g)}{n-1}$.
Thus $\mathbb{R} \subset D(g)$ (in particular lightlike) and with $f \in C^\infty(M,\mathbb{R})$ the whole fiber through f is in $D(g)$. Consider therefore $\tilde H(g) = H(g) \restriction C_0^g(M,\mathbb{R})$ and

$\tilde{Q}(g) = Q(g) \restriction C_0^g(M, \mathbb{R})$.
Use the notation $\operatorname{Spec}^+(\Delta) = \operatorname{Spec}(\Delta \restriction C_0^g(M,\mathbb{R}))$ for the strictly positive spectrum of the Laplacian. For $\lambda \in \operatorname{Spec}^+$ denote by $\operatorname{Eig}_\lambda(\Delta^g)$ the eigenspace of $\Delta \restriction C_0^g(M,\mathbb{R})$ corresponding to λ:

$$\operatorname{Eig}_\lambda(\Delta^g) = \{f \in C_0^g(M,\mathbb{R})\colon \Delta^g f = \lambda f\}.$$

Take an complete orthonormal system $\{e_1\}_{i\geq 0}$ of eigenvectors of Δ. The $\{e_i\}_{i\geq 0}$ yield a diagonalization of $\tilde{H}(g)$: Denote by $\lambda_{n(i)}$ the eigenvalue corresponding to e_i. $\lambda_0 = 0$, $\lambda_n \leq \lambda_{n+1}$ and $\lambda_{n(i)} \to \infty$ (conf. [**Aub82**]). Then for $i \neq j$

$$\tilde{H}(g)(e_i, e_j) = \lambda_{n(i)}\langle e_i, e_j\rangle - \frac{\operatorname{scal}(g)}{n-1}\langle e_i, e_j\rangle = 0$$

and

$$\tilde{H}(g)(e_i, e_i) = \big(\lambda_{n(i)} - \frac{\operatorname{scal}(g)}{n-1}\big)\langle e_i, e_i\rangle.$$

Thus $\{\lambda_{n(i)} - \frac{\operatorname{scal}(g)}{n-1}\}_{i\geq 0}$ is the sequence of eigenvalues of P^g.

6.13. (1) $\tilde{Q}(g)$ has lightlike vectors iff $\lambda_1 \leq \frac{\operatorname{scal}(g)}{n-1}$. In particular, $\operatorname{scal}(g) \geq 0$.
(2) g is a minimum of $\tilde{N} \restriction \operatorname{Conf}_0^g$ iff $\lambda_1 \leq \frac{\operatorname{scal}(g)}{n-1}$.
g is a minimum of $\tilde{N}$ iff $\operatorname{Scal}(g) \leq 0$.
(3a) $\tilde{H}(g)$ is degenerate iff $\frac{\operatorname{scal}(g)}{n-1} \in \operatorname{Spec}^+(\Delta)$ and in this case $D(g)$ is the eigenspace to $\frac{\operatorname{scal}(g)}{n-1}$.
(3b) If g is Einstein then $\tilde{H}(g)$ is degenerate iff $\lambda_1 = \frac{\operatorname{scal}(g)}{n-1}$, which is by Obata's theorem equivalent to (M, g) being isometric to the standard sphere (comp. [**Bes82**, remark 4.65]).
The Lichnerowitz equality for the lowest non-trivial eigenvalue of the Laplacian says that $\lambda_1 \geq \frac{n}{n-1}\operatorname{Ric}_{min}$, where Ric_{min} is the smallest number r such that $\operatorname{Ric} - rg \geq 0$. As g is Einstein, $\operatorname{Ric}_{min} = \frac{\operatorname{scal}(g)}{n}$, and therefore $\lambda_1 \geq \frac{\operatorname{scal}(g)}{n-1}$. Thus the assertion follows from (3a).
(4) The degree of degeneracy is finite compare (comp. [**Bes82**, remark 4.65]). For the standard sphere the degree of degeneracy is 1.
This follows from (3a) and the fact that the multiplicity of the first eigenvalue of an elliptic operator is 1.
(5) The number $\dim(\tilde{H}(g)^-)$ of strictly negative eigenvalues counted with multiplicities is finite. The number $\dim(\tilde{H}(g)^+)$ of strictly positive eigenvalues counted with multiplicities is infinite.
This is consequence of the ellipticity and positivity of Δ.
Since $\lambda_i \to \infty$ and their multiplicities $m(\lambda_i)$ are finite, we are done. Explicitly,

$$\dim(\tilde{H}(g)^-) = \sum_{\lambda_i < \frac{\operatorname{scal}(g)}{n-1}} m(\lambda_i) < \infty.$$

From this and since $\dim D(g) < \infty$, $\dim C_0^g(M,\mathbb{R}) = \infty$, follows that

$$\dim(\tilde{H}(g)^+) = \sum_{\lambda_i > \frac{\mathrm{scal}(g)}{n-1}} m(\lambda_i) = \infty.$$

(6) $\tilde{N}$ does not admit a local maximum.
Consequence of $\dim(\tilde{H}(g)^+) = \infty$.

6.14. $Q(g)$ **for arbitrary** g**.** Recall the operator

$$P^g = \Delta^g + \frac{n-4}{4(n-1)}\,\mathrm{scal}(g) - \frac{n}{4(n-1)}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)}$$

from 6.10. We have three cases depending on the dimension of M.
$\dim M = 3$: The coefficient of $f\,\mathrm{scal}(g)$ is negative.
$\dim M = 4$: $P^g f = \Delta^g f - \frac{1}{3}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)} f$.
$\dim M \geq 5$: The coefficient of $f\,\mathrm{scal}(g)$ is positive.
Accordingly, one has the following

6.15. Proposition. *Assume that* $\tilde{Q}(g)$ *has non-trivial lightlike vectors:*
If $\dim M = 3$*, then not* $\mathrm{scal}(g) \leq 0$
If $\dim M = 4$*, then* $\lambda_1 \leq \frac{1}{3}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)}$
If $\dim M \geq 5$ *and* $\mathrm{scal}(g) \geq 0$*, then* $\lambda_1 \leq \frac{n}{4(n-1)}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)}$.

Proof. Assume $f \in C_0^g(M,\mathbb{R})$ to be lightlike, $f \not\equiv 0$.
λ_1 is given by (conf. [**Aub82,** Theorem 4.2])

$$\lambda_1 = \inf_{\substack{l \in C_0^g(M,\mathbb{R}) \\ l \not\equiv 0}} \frac{\langle \nabla l, \nabla l\rangle}{\langle l, l\rangle} \leq \frac{\langle \nabla f, \nabla f\rangle}{\langle f, f\rangle}.$$

If $\dim M = 3$ and $\mathrm{scal}(g) \leq 0$, then $-\langle \mathrm{scal}(g) f, f\rangle \geq 0$ and

$$\langle \nabla f, \nabla f\rangle - \frac{3}{8}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)}\langle f, f\rangle \leq 0.$$

Hence $\lambda_1 \leq \frac{3}{8}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)} \leq 0$, a contradiction.
Analogously, if $\dim M \geq 5$.
If $\dim M = 4$,

$$\langle \nabla f, \nabla f\rangle - \frac{1}{3}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)}\langle f, f\rangle = 0.$$

Hence $\lambda_1 \leq \frac{1}{3}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)} \leq 0$. $\square$

6.16. One might attempt to get independent results under the conditions $\operatorname{scal}(g) - \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} \geq 0$ and $\operatorname{scal}(g) - \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} \leq 0$. But then already $\operatorname{scal}(g) = \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}$: Assume in the first case $\operatorname{scal}(g) \restriction U > \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} \restriction U$ for some open $U \subset M$. Then $0 < \int_M (\operatorname{scal}(g) - \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}) \operatorname{vol}(g) = \operatorname{Scal}(g) - \operatorname{Scal}(g) = 0$, a contradiction. The other case is analogous.

6.17. *$H(g)$* **for arbitrary** *g***.** Let $\operatorname{scal}(g) = \sum_{i \geq 0} \operatorname{scal}(g)^i e_i$ be the expansion in an eigenbasis of Δ^g. Note that $\operatorname{scal}(g) = 0$ for all $i \geq 1$ iff g critical.
Denote by $\pi_1 \colon C^\infty(M,\mathbb{R}) \to \mathbb{R}$, $\pi_2 \colon \to C_0^g(M,\mathbb{R})$ the natural projections. In 6.9 it was shown that for $\tilde{f}_1, \tilde{f}_2 \in C^\infty(M,\mathbb{R})$

$$\begin{aligned} H(g)(\tilde{f}_1, \tilde{f}_2) =& \langle P^g \pi_2(\tilde{f}_1), \pi_2(\tilde{f}_2)\rangle \\ &- \frac{n}{4(n-1)}\big(\pi_1(\tilde{f}_1)\langle \operatorname{scal}(g), \pi_2(\tilde{f}_2)\rangle + \pi_1(\tilde{f}_2)\langle \operatorname{scal}(g), \pi_2(\tilde{f}_1)\rangle\big). \end{aligned}$$

Note that $H(g)(\tilde{f}_1, c_2) = -\frac{n}{4(n-1)} c_2 \langle \pi_2(\tilde{f}_1), \operatorname{scal}(g)\rangle$ for $c_2 \in \mathbb{R}$. Therefore a degenerate $\tilde{f}$ has a $C_0^g(M,\mathbb{R})$-component orthogonal to $\operatorname{scal}(g)$: $\int_M \pi_2(\tilde{f}) \operatorname{scal}(g) \operatorname{vol}(g) = 0$. In the Δ^g-eigenbasis the hyperplane in $C_0^g(M,\mathbb{R})$ which is orthogonal to $\operatorname{scal}(g)$ is spanned by functions of the form e_i for $\operatorname{scal}(g)^i = 0$ and $\big(\operatorname{scal}(g)^k e_l - \operatorname{scal}(g)^l e_k\big)$ for $\operatorname{scal}(g)^k \neq 0, \operatorname{scal}(g)^l \neq 0$.
Now $P^g : C_0^g(M,\mathbb{R}) \cap \operatorname{scal}(g)^\perp \to C_0^g(M,\mathbb{R})$. Therefore for $f \in C_0^g(M,\mathbb{R}) \cap \operatorname{scal}(g)^\perp$ one has $\langle P^g f, \tilde{f}_2\rangle_g = \langle P^g f, \pi_3(\tilde{f}_2)\rangle_g$. Since $\pi(\tilde{f}_2) = \frac{1}{\operatorname{Vol}(g)} \int_M \tilde{f}_2 \operatorname{vol}(g)$

$$\langle \operatorname{scal}(g), \tilde{f}_2\rangle_g = \langle \operatorname{scal}(g), \pi_2(\tilde{f}_2)\rangle_g + \langle \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}, \tilde{f}_2\rangle_g.$$

Thus $\tilde{f}$ is degenerate iff $\tilde{f}$ satisfies the system of equations

$$\text{(EQ)} \qquad \begin{aligned} &P^g \pi_2(\tilde{f}) = \pi_1(\tilde{f}) \frac{n}{4(n-1)}\big(\operatorname{scal}(g) - \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}\big) \\ &\langle \pi_2(\tilde{f}), \operatorname{scal}(g)\rangle_g = 0 \end{aligned}$$

If $\tilde{f} \in C^\infty(M,\mathbb{R})$ is a solution of (EQ) and $\dim M \neq 4$, then integration over M yields $(n-4)\langle \pi_2(\tilde{f}), \operatorname{scal}(g)\rangle = 0$. Thus for $\dim M \neq 4$ $\tilde{f} \in C^\infty(M,\mathbb{R})$ is degenerate iff it satisfies (EQ).

6.18. Case $\pi_1(\tilde{f}) = 0$**.** If $\dim M \neq 4$ then $f \in C_0^g(M,\mathbb{R})$ is degenerate iff $Pf = 0$.
If $\dim M = 4$, then $P^g f = \Delta^g f - \frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} f$.

6.19. Proposition. (1) *Assume* $\dim M \neq 4$ *and that* $H(g)$ *has a degenerate direction in* $C_0^g(M,\mathbb{R})$.
Then not $\operatorname{scal}(g) \leq \frac{n}{n-4}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}$.

(1a) *If* $\dim M = 3$*, then not* $\operatorname{scal}(g) \le -3\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}$*; In particular: if* $\operatorname{scal}(g) \le 0$*, then* $H(g)$ *does not admit a degenerate direction in* $C_0^g(M,\mathbb{R})$*.*
(2) *If* $\dim M = 4$*, then there is a degenerate direction in* $C_0^g(M,\mathbb{R})$ *iff* $\frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} \in \operatorname{Spec}^+(\Delta^g)$ *and at least one of the corresponding eigenvectors is orthogonal to* $\operatorname{scal}(g)$*. In this case the intersection of the degenerate subspace and* $C_0^g(M,\mathbb{R})$ *equals the intersection of the eigenspace to* $\frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}$ *and the orthogonal complement of* $\operatorname{scal}(g)$*:* $D(g) \cap C_0^g(M,\mathbb{R}) = \operatorname{Eig}_{\frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}}(\Delta^g) \cap \operatorname{scal}(g)^\perp$
In particular: If $\operatorname{Scal}(g) \le 0$*, then* $H(g)$ *does not admit a degenerate direction in* $C_0^g(M,\mathbb{R})$*.*

Proof. (1) Let $f \in C_0^g(M,\mathbb{R})$, $f \not\equiv 0$ be degenerate.
$0 = P^g f = \Delta^g f + \frac{n-4}{4(n-1)}\operatorname{scal}(g) f - \frac{n}{4(n-1)}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} f$. Δ^g is strictly positive on $C_0^g(M,\mathbb{R})$. Thus not $\operatorname{scal}(g) \le \frac{n}{n-4}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}$.
(1a) follows from (1), since then if $\operatorname{scal}(g) \le 0$ also $\operatorname{scal}(g) + 3\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} \le 0$.
(2) follows directly from the special form of P^g in this dimension. □

To illustrate the case $\dim M = 4$ expand $P^g f = 0$ in $\{e_i\}$: $\sum_{i\ge 1}(\lambda_i f^i - \frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} f^i)e_i = 0$, which is equivalent to $(\lambda_i - \frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)})f^i = 0$ for all $i \ge 1$. Hence, $f^i = 0$ for $\lambda_{n(i)} \ne \frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}$ and the remaining finitely many f^i's satisfy $\sum_{\lambda_{n(i)}=\frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}} \operatorname{scal}(g)^i f^i = 0$.

6.20. Case $\pi_1(\tilde f) \ne 0$**.** Without loss of generality assume $c = \frac{4(n-1)}{n}$ to obtain from (EQ) the equation for $f \in C_0^g(M,\mathbb{R})$:

(EQ') $$P^g f = \operatorname{scal}(g) - \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}.$$

This allows to reduce the problem to an equation in $C_0^g(M,\mathbb{R})$.

6.21. Proposition. *If* $\dim M \ne 4$ *then* $H(g)$ *has a degenerate direction outside* $C_0^g(M,\mathbb{R})$ *iff EQ' has a solution* f_1 *in* $C_0^g(M,\mathbb{R})$*.*
If f_1 *is such a solution then* $f + \frac{4(n-1)}{n}$ *points in a degenerate direction.*

6.22. If $\dim M = 4$ then $f + \frac{4(n-1)}{n}$ (with $f \in C_0^g(M,\mathbb{R})$) is a degenerate direction for $H(g)$ iff

(EQ(4)) $$\Delta f - \frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} f = \operatorname{scal}(G) - \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}$$
$$\text{and } \langle f, \operatorname{scal}(g)\rangle = 0$$

In the expansion in $\{e_i\}$ EQ(4) becomes:

(EQ(4)) $$f_0 = 0, \quad \sum_{i\ge 1}\big(\lambda_{n(i)} f^i.e_i - \frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} f^i.e_i\big) = \sum_{i\ge 1}\operatorname{scal}(g)^i.e_i.$$

thus for all $i \geq 1$: $\big(\lambda_{n(i)} - \frac{1}{3}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)}\big) f^i = \mathrm{scal}(g)^i$.
There are two cases:
(1) $\frac{1}{3}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)} \in \mathrm{Spec}^+(\Delta^g)$ and one of the corresponding coefficients $\mathrm{scal}(g)^i$ does not vanish. Then (EQ(4)) cannot be solved.
(2) $\frac{1}{3}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)} \notin \mathrm{Spec}^+(\Delta^g)$ or if $\frac{1}{3}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)} \in \mathrm{Spec}^+(\Delta^g)$ and all the corresponding coefficients $\mathrm{scal}(g)^i$ do vanish. Then (EQ(4)) can be solved by setting (for $i \geq 1$)

$$f^i = \begin{cases} \frac{\mathrm{scal}(g)^i}{\lambda_{n(i)} - \frac{1}{3}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)}} & \text{if } \lambda_{n(i)} - \frac{1}{3}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)} \neq 0 \\ \text{arbitrary} & \text{else.} \end{cases}$$

The thus defined sequence $(f^i)_{i>0)}$ satisfies $\frac{f^i}{\mathrm{scal}(g)^i} \to 0$. Therefore with $(\mathrm{scal}(g)^i)$ also (f^i) is a Schwartz sequence and $\sum_i f^i e_i \in C_0^g(M, \mathbb{R})$. In the expansion the condition $\int_M \mathrm{scal}(g) f \,\mathrm{vol}(g) = 0$ reads

$$\sum_i \mathrm{scal}(g)^i f^i = 0. \tag{6.22.1}$$

If $f \in C_0^g(M, \mathbb{R})$ is a solution of (EQ(4)) then this 6.22.1 becomes a condition only on the scalar curvature:

$$\sum_{\substack{i \geq 1 \\ \lambda_{n(i)} \neq \frac{1}{3}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)}}} \frac{(\mathrm{scal}(g)^i)^2}{\lambda_{n(i)} - \frac{1}{3}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)}} = 0 \tag{$*$}$$

since $\mathrm{scal}(g)^i f^i = 0$ if $\lambda_{n(i)} = \frac{1}{3}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)}$.

6.23. Lemma. *If $\dim M = 4$, $H(g)$ has a degenerate direction outside $C_0^g(M, \mathbb{R})$ iff the expansion $\mathrm{scal}(g) = \sum_{i \geq 0} \mathrm{scal}(g)^i e_i$ of the scalar curvature in the complete orthonormal system corresponding to Δ satisfies the equation ($*$) and either $\frac{1}{3}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)} \notin \mathrm{Spec}^+(\Delta^g)$ or $\frac{1}{3}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)} \in \mathrm{Spec}^+(\Delta^g)$ and all the corresponding coefficients of $\mathrm{scal}(g)$ vanish.*

6.24. Corollary. *If $\mathrm{Scal}(g) \leq 0$ then $H(g)$ has no degenerate directions outside of $C_0^g(M, \mathbb{R})$ unless $\mathrm{scal}(g) \equiv \frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)}$, i.e. unless g is critical.*

Proof. In either case $\frac{1}{3}\frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)} \notin \mathrm{Spec}^+$ and ($*$) becomes

$$\sum_{i \geq 1} c^i \big(\mathrm{scal}(g)^i\big)^2 = 0 \tag{$*$'}$$

for some strictly positive coefficients $c^i \in \mathbb{R}$. The left hand side of ($*$') is strictly positive unless $\mathrm{scal}(g) \equiv \frac{\mathrm{Scal}(g)}{\mathrm{Vol}(g)}$. □

Putting together 6.19, 6.23 and 6.24 one obtains for $\dim M = 4$ the following theorem.

6.25. Theorem. *Assume* $\dim M = 4$ *and* g *not critical.*
(1) *If* $\operatorname{Scal}(g) \le 0$*, then* $H(g)$ *is not degenerate.*
(2) *If* $\operatorname{Scal}(g) > 0$*, then the (possibly trivial) degenerate subspace* $D(g)$ *is obtained as follows: Assume the expansion* $\operatorname{scal}(g) = \sum_{i\ge 0} \operatorname{scal}(g)^i e_i$ *of the scalar curvature in the complete orthonormal system* $\{e_i\}$ *corresponding to* Δ^g *and put*

$$f = \sum_{\substack{i\ge 1\\ \lambda_{n(i)} \neq \frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}}} \frac{\operatorname{scal}(g)^i}{\lambda_{n(i)} - \frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}} e_i + 3 \quad \in C^\infty(M,\mathbb{R}) \setminus C_0^g(M,\mathbb{R}).$$

(2a) *If* $\frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} \in \operatorname{Spec}^+(\Delta)$ *and one of the corresponding coefficients* $\operatorname{scal}(g)^i$ *does not vanish, then* $D(g) = \operatorname{Eig}_{\frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}}(\Delta^g) \cap \operatorname{scal}(g)^\perp \cap C_0^g(M,\mathbb{R}) \subset C_0^g(M,\mathbb{R})$.
(2b) *If* $\frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} \in \operatorname{Spec}^+(\Delta)$*, all of the corresponding coefficients* $\operatorname{scal}(g)^i$ *vanish and* $(*)$ *is not satisfied, then* $D(g) = \operatorname{Eig}_{\frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}}(\Delta^g) \cap C_0^g(M,\mathbb{R}) \subset C_0^g(M,\mathbb{R})$.
(2c) *If* $\frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} \in \operatorname{Spec}^+(\Delta)$*, all of the corresponding coefficients* $\operatorname{scal}(g)^i$ *vanish and* $(*)$ *is satisfied, then* $D(g) = \operatorname{Eig}_{\frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}}(\Delta^g) \cap C_0^g(M,\mathbb{R}) \oplus \mathbb{R}f \not\subseteq C_0^g(M,\mathbb{R})$.
(2d) *If* $\frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} \notin \operatorname{Spec}^+(\Delta)$ *and* $(*)$ *is not satisfied, then* $D(g) = \{0\}$.
(2e) *If* $\frac{1}{3}\frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} \notin \operatorname{Spec}^+(\Delta)$ *and* $(*)$ *is satisfied, then* $D(g) = \mathbb{R}f$.

6.26. The conformal class with constant volume. Here the situation is less complicated. Put $\widehat{\operatorname{Scal}} = \operatorname{Scal} \restriction \operatorname{Conf}_0^g$. Then a computation like in section 5 yields

$$\begin{aligned}\operatorname{Hess}(\widehat{\operatorname{Scal}})(g)(fg,fg) =& \frac{n-2}{2}\int_M \big((n-1)\Delta f - \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} f \\ &+ \frac{n-4}{4}(\operatorname{scal}(g) - \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)})f\big) f \operatorname{vol}(g).\end{aligned}$$

If g is a critical point of $\widehat{\operatorname{Scal}}$, $\operatorname{scal}(g) - \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} = 0$ and thus

$$\operatorname{Hess}(\widehat{\operatorname{Scal}})(g)(fg,fg) = \frac{n-2}{2}\int_M \big((n-1)\Delta f - \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)} f\big) f \operatorname{vol}(g).$$

The critical points of $\widehat{\operatorname{Scal}}$ are exactly the critical points of $\widetilde{N}$ that lie in Conf_0^g. At critical points the classification of light like and degenerate directions of $\widehat{\operatorname{Scal}}$ yields the same results as for $\widetilde{N} \restriction C_0^g(M,\mathbb{R})\cdot g \times C_0^g(M,\mathbb{R})\cdot g$. At non critical points this classification is also similar, but much easier, since the constant direction is lacking. Note that for $f \in C_0^g(M,\mathbb{R})$

$$\operatorname{Hess}(\widehat{\operatorname{Scal}})(g)(fg,fg) = \operatorname{Hess}(\widetilde{N})(g)(fg,fg) - \int_M \big(\operatorname{scal}(g) - \frac{\operatorname{Scal}(g)}{\operatorname{Vol}(g)}\big) f^2 \operatorname{vol}(g).$$

In particular they do not coincide.

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