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TOPOLOGICAL REPRESENTATIONS OF QUASIORDERED SETS

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Abstract. We prove that for every infinite cardinal number α there exists a space X with $|X| = \alpha$, metrizable whenever $\alpha \geq \mathfrak{c}$, strongly paracompact whenever $\omega \leq \alpha \leq \mathfrak{c}$, such that every quasiordered set $(Q, \leq)$ with $|Q| \leq \alpha$ can be represented by closed subspaces of X in the sense that there exists a system $\{X_q | q \in Q\}$ of non-homeomorphic closed subspaces of X such that

$$q_1 \leq q_2 \text{ if and only if } X_{q_1} \text{ is homeomorphic to a subset of } X_{q_2}.$$

In fact, stronger results are proved here.

1. Introduction and the Main Results

Every class $\mathcal{M}$ of continuous maps, closed with respect to the composition and containing all homeomorphisms, determines a relation $\preceq$ on the class **Top** of all topological spaces by the rule

$$X \preceq Y \quad \text{if and only if there exists} \quad f : X \to Y \quad \text{in} \quad \mathcal{M}.$$

Clearly, the relation $\preceq$ is reflexive and transitive but not antisymmetric, i.e. it is a quasiorder on **Top**. We say that a quasiordered set $(Q, \leq)$ has an *$\mathcal{M}$-representation within a class $\mathbb{C}$ of topological spaces* if there exists a system $\{X_q | q \in Q\}$ of non-homeomorphic spaces in $\mathbb{C}$ such that, for every q_1, $q_2 \in Q$,

$$q_1 \leq q_2 \quad \text{if and only if} \quad X_{q_1} \preceq X_{q_2}.$$

Investigations of $\mathcal{M}$-representations for the class $\mathcal{M}$ of all homeomorphic embeddings are of rather old origin. In 1926, C. Kuratovski and W. Sierpiński proved in [**4**] that the ordinal $\mathfrak{c}^+$ has such a representation within the class of subspaces of the real line and C. Kuratowski proved in [**3**] that the antichain on $2^{\mathfrak{c}}$ points also has such representation within this class. After more than sixty years, this field of problems was revisited in [**5**], [**6**], [**7**], [**8**]. In [**5**], such a representation was constructed for every poset(= partially ordered set) of cardinality at most $\mathfrak{c}$ and,

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in [**8**], for the set $\exp \mathfrak{c}$ of all subsets of $\mathfrak{c}$ ordered by the inclusion. The result of [**8**] implies those of [**5**] and [**3**] because, for every infinite set A, every poset $(P, \leq)$ with $|P| \leq |A|$ and the antichain on $2^{|A|}$ points can be embedded into $(\exp\ A, \subseteq)$.

In [**6**], [**7**], representations of quosets(= quasiordered sets) are investigated (with respect to the homeomorphic embeddings). Given an infinite cardinal number α, the authors of [**7**] construct a T_0-space X with $|X| \leq \delta(\alpha)$, where $\delta(\alpha)$ denotes the smallest cardinal number δ for which there exist α distinct cardinals (not necessarily infinite) smaller than δ, such that every quoset $(Q, \leq)$ with $|Q| \leq \alpha$ has a representation by the subspaces of X (with respect to the homeomorphic embeddings). In the final comment, they say that it would be good to have such spaces with better separation axioms and a lower cardinality than $\delta(\alpha)$ (which is satisfactorily small for $\alpha = \omega$ but rather large for α uncountable). We present here such a space X with $|X| = \alpha$ and X strongly paracompact whenever $\omega \leq \alpha \leq \mathfrak{c}$ and metrizable whenever $\alpha \geq \mathfrak{c}$.

In fact, we present stronger results: we investigate also smaller systems of subspaces of X (e.g. all *closed* subspaces of X) and $\mathcal{M}$-representations also for other classes of maps, namely

- $\mathcal{M}_1$ = the class of all one-to-one continuous maps,
- $\mathcal{M}_2$ = the class of all homeomorphic embeddings,
- $\mathcal{M}_3$ = the class of all homeomorphisms onto closed subspaces,
- $\mathcal{M}_4$ = the class of all homeomorphisms onto clopen[1] subspaces.

For $i \leq j$, an $\mathcal{M}_i\mathcal{M}_j$ *− representation of a quoset* $(Q, \leq)$ *within a class* $\mathbb{C}$ *of spaces* is any system $\{X_q | q \in Q\}$ of non-homeomorphic spaces in $\mathbb{C}$ such that

- if $q_1 \leq q_2$, then there exists $f : X_{q_1} \to X_{q_2}$ in $\mathcal{M}_j$ and
- if $q_1 \not\leq q_2$, then no $f : X_{q_1} \to X_{q_2}$ is in $\mathcal{M}_i$.

The following theorem is an easy application of the ideas of [**7**] and the well-known results (see below):

Theorem 1. *For any infinite cardinal α,* $(\exp \alpha, \subseteq)$ *has an $\mathcal{M}_1\mathcal{M}_4$-representation within the class of all clopen subspaces of a suitable space X with $|X| = \alpha$ which is*

- *strongly paracompact whenever $\omega \leq \alpha \leq \mathfrak{c}$,*
- *metrizable whenever $\alpha \geq \mathfrak{c}$.*

As mentioned above, all posets of cardinalities at most α and the antichain on 2^α points can be embedded into $(\exp \alpha, \subseteq)$, hence they have $\mathcal{M}_1\mathcal{M}_4$-representation within clopen subspaces of the above X.

To formulate the theorems about the representability of quosets, let T_α denote the quoset obtained from $(\exp \alpha, \subseteq)$ by splitting any element into 2^α distinct but mutually comparable elements. More precisely, T_α is the set $\exp \alpha \times \exp \alpha$ with the quasiorder $\leq$ given by the rule

$$(A_1, A_2) \leq (B_1, B_2) \quad \text{if and only if} \quad A_1 \subseteq B_1.$$

[1] closed-and-open

Theorem 2. *For every $\alpha \geq \mathfrak{c}$ there exists a metrizable space X such that $|X| = \alpha$ and T_α has an $\mathcal{M}_1\mathcal{M}_4$-representation within the clopen subspaces of X. For $\alpha = \mathfrak{c}$, X can be moreover strongly paracompact.*

Theorem 3. *For every α with $\omega \leq \alpha \leq \mathfrak{c}$ there exists a strongly paracompact space X such that $|X| = \alpha$ and T_α has an $\mathcal{M}_2\mathcal{M}_3$-representation within the set of closed subspaces of X.*

Proofs of these theorems are presented in the section below. Although, for $\alpha \geq \mathfrak{c}$, Theorem 2 implies the statement of Theorem 1, we give a separate proof of Theorem 1 because of its simplicity.

2. The Proofs

Proof of Theorem 1. a) Let $\omega \leq \alpha \leq \mathfrak{c}$: By [**2**], there exists a set of cardinality $\mathfrak{c}$ of non-principal ultrafilters on ω which are mutually incomparable in the Rudin-Keisler order of ultrafilters, i.e. there exists a system $\{\mathcal{F}_i | i \in \mathfrak{c}\}$ of non principal ultrafilters on ω such that, denoting by P_i the subspace $P_i = \omega \cup \{\mathcal{F}_i\}$ of the compactification $\beta\omega$, every continuous map $P_i \to P_j$ is constant on a set $F \in \mathcal{F}_i$ whenever $i \neq j$. Then the space $X = \coprod_{i\in\alpha} P_i$, where $\amalg$ denotes the coproduct(= the sum = the disjoint union as clopen subspaces), has the required properties: if $A \subseteq \alpha$, we put $X_A = \coprod_{i\in A} P_i$. Then, clearly, $\{X_A | A \subseteq \alpha\}$ forms an $\mathcal{M}_1\mathcal{M}_4$-representation of $(\exp\alpha, \subseteq)$.
b) Let $\alpha \geq \mathfrak{c}$: We put again $X = \coprod_{i\in\alpha} P_i$ and $X_A = \coprod_{i\in A} P_i$; but now, $\mathcal{P} = \{P_i | i \in \alpha\}$ is a system of metrizable spaces such that $|P_i| = \alpha$ and every continuous map $P_i \to P_j$ is constant whenever $i \neq j$ (and then $\{X_A | A \subseteq \alpha\}$ is an $\mathcal{M}_1\mathcal{M}_4$-representation of $(\exp\alpha, \subseteq)$ again). Such a system $\mathcal{P}$ does exist. More strongly,

$$(*)\begin{cases}\text{for every cardinal nuber } \alpha \geq \mathfrak{c} \text{ there exists a set } \mathcal{P} \text{ of the cardinality } 2^\alpha\\ \text{consisting of metrizable spaces of the cardinality } \alpha \text{ such that if } X, Y \in \mathcal{P}\\ \text{and } f: X \to Y \text{ is a continuous map, then either } f \text{ is constant or } X = Y\\ \text{and } f \text{ is the identity.}\end{cases}$$

This is explicitly stated in [**12**, p. 510] where this construction is completly described (for all the corresponding proofs see [**9**, pp 139, 215–219 and 222–226], but $(*)$ is not explicitly stated there). The construction also implies that for $\alpha = \mathfrak{c}$, the spaces in $\mathcal{P}$ are separable; hence X, being a coproduct of $\mathfrak{c}$ metrizable separable spaces, is strongly paracompact. □

Proof of Theorem 2. We use the system $\mathcal{P}$ satisfying $(*)$ of the previous proof again and we use also a compact metric zero-dimensional space K of the cardinality at most $\mathfrak{c}$ homeomorphic to the coproduct of its three copies $K \amalg K \amalg K$ but not homeomorphic to $K \amalg K$. Such a space was constructed in [**1**]. Hence

1. if $P_1, P_2 \in \mathcal{P}, P_1 \neq P_2$, then there exists no continuous one-to-one map of any of the spaces $P_1, P_1 \times K, P_1 \times (K \amalg K)$ into any of the spaces $P_2, P_2 \times K, P_2 \times (K \amalg K)$ (because K is zero-dimensional while the spaces in $\mathcal{P}$ must be connected)

2. although $P_2 \times K$ is homeomorphic to a clopen subspace of $P_2 \times (K \amalg K)$ and vice versa, $P_2 \times K$ and $P_2 \times (K \amalg K)$ are not homeomorphic. In fact, since every continuous map $f : P_2 \to P_2$ has to be either the identity or a constant, by $(*)$, the existence of a homeomorphism of $P_2 \times K$ onto $P_2 \times (K \amalg K)$ would imply the existence of a homeomorphism of K onto $K \amalg K$.

Let $\widetilde{\mathcal{P}} = \{P_{i,j} | i \in \alpha, j = 1, 2\}$ be a subsystem of $\mathcal{P}$. Then our required space is

$$X = \coprod_{i \in \alpha} P_{i,1} \amalg \coprod_{i \in \alpha} (P_{i,2} \times K)$$

i.e. $|X| = \alpha$ and T_α has an $\mathcal{M}_1\mathcal{M}_4$-representation within clopen subspaces of X: for $(A_1, A_2) \in T_\alpha$ we put

$$X_{(A_1,A_2)} = \coprod_{i \in A_1} P_{i,1} \amalg \coprod_{i \in A_2} (P_{i,2} \times K) \amalg \coprod_{i \in \alpha \setminus A_2} (P_{i,2} \times h(K \amalg K)),$$

where h is a homeomorphism of $K \amalg K \amalg K$ onto K. Then, clearly, $\{X_{(A_1,A_2)} | (A_1, A_2) \in T_\alpha\}$ is an $\mathcal{M}_1\mathcal{M}_4$-representation of T_α. $\square$

Proof of Theorem 3. As in part a) of the proof of Theorem 1, we use the incomparable ultrafilters again; but now, we denote the subspace $\omega \cup \{\mathcal{F}\}$ of $\beta\omega$ by $P_{\mathcal{F}}$. We use also the construction of [**11**] of a countable strongly paracompact space S homomorphic to $S \times S \times S$ but not to $S \times S$. We recall it briefly: first, for every triple $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$ of non-principal ultrafilters on ω, the space $P_{\mathcal{F}_0,\mathcal{F}_1,\mathcal{F}_2}$ is constructed in [**11**] as follows:

$$\widetilde{P}_0 = P_{\mathcal{F}_0}, \quad \widetilde{P}_1 = P_{\mathcal{F}_0} \amalg \coprod (\omega \times P_{\mathcal{F}_1}),$$

$$\widetilde{P}_n = P_{\mathcal{F}_0} \amalg (\omega \times P_{\mathcal{F}_1}) \amalg \coprod_{k=2}^{n} (\omega^k \times P_{\mathcal{F}_2}) \quad \text{for } n \geq 2.$$

Now, let $P_0 = P_{\mathcal{F}_0}$ and let P_n be the quotient space of $\widetilde{P}_n$ obtained by identifying each point $m \in \omega \subseteq P_{\mathcal{F}_0}$ with the point $(m, \mathcal{F}_1) \in \omega \times P_{\mathcal{F}_1}$ and, for $n > 1$, each point $(m_1, m_2) \in \omega \times \omega \subseteq \omega \times P_{\mathcal{F}_1}$ with the point $(m_1, m_2, \mathcal{F}_2) \in \omega^2 \times P_{\mathcal{F}_2}$ and, for $n > 2$, each point $(m_1, \dots, m_k) \in \omega^{k-1} \times \omega \subseteq \omega^{k-1} \times P_{\mathcal{F}_2}$ with the point $(m_1, \dots, m_k, \mathcal{F}_2) \in \omega^k \times P_{\mathcal{F}_2}$, $k = 3, 4, \dots, n$. We may suppose that $P_0 \subseteq P_1 \subseteq \subseteq P_2 \subseteq \cdots$ and $P_{\mathcal{F}_0,\mathcal{F}_1,\mathcal{F}_2}$ is $\bigcup_{n=0}^{\infty} P_n$ with the inductively generated topology (in the modern description, see [**14**], $P_{\mathcal{F}_0,\mathcal{F}_1,\mathcal{F}_2}$ is precisely the space $Seq(u_t)$ with $u_t = \mathcal{F}_0$ whenever the length $|t|$ is 0, $u_t = \mathcal{F}_1$ whenever $|t| = 1$ and $u_t = \mathcal{F}_2$ in all the other cases).

Let $\{\mathcal{F}_{j,n} | j \in \{0, 1, 2\}; n \in \omega\}$ be a collection of pairwise incomparable non-principal ultrafilters on ω. As in [**11**], let us denote the space $P_{\mathcal{F}_{0,n},\mathcal{F}_{1,n},\mathcal{F}_{2,n}}$ by Q_n and its point $\mathcal{F}_{0,n}$ by O_n. For every map $a : \omega \to \omega$ put

$$\widetilde{Q}_a = \prod_{n \in \omega} Q_n^{a(n)}$$

(where $Q_n^{a(n)} = \{O_n\}$ whenever $a(n) = 0$) and denote by O_a its point with all the coordinates equal to the corresponding O_n's. Let Q_a be the subspace of $\widetilde{Q}_a$ consisting of all the points which differ from O_a only in finitely many coordinates. The space S (homeomorphic to $S \times S \times S$ but not to $S \times S$) is a coproduct of ω copies of Q_a for every a in a countable set $A \subseteq \omega^\omega$ satisfying $A = A + A + A$ and $A \cap (A + A) = \emptyset$ (where $A + A = \{a + b | a, b \in A\}$, $\big(a + b\big)(n) = a(n) + b(n)$); such a set A does exist, see [**10**] .

Now, let a cardinal number α with $\omega \le \alpha \le \mathfrak{c}$ be given. Let $\{\mathcal{F}_i, \mathcal{F}_{i,j,n} | i \in \alpha; j \in \{0, 1, 2\}; n \in \omega\}$ be a system of mutually incomparable non-principal ultrafilters on ω. We put

$$X = \coprod_{i \in \alpha} P_{\mathcal{F}_i} \amalg \coprod_{i \in \alpha} S_i$$

where S_i is the space obtained by the above described construction from the system $\{\mathcal{F}_{i,j,n} | j \in \{0, 1, 2\}; n \in \omega\}$. Then X has the required properties, i.e. T_α has an $\mathcal{M}_2\mathcal{M}_3$-representation within closed subspaces of X. In fact, for $(A_1, A_2) \in T_\alpha$, we put

$$X_{(A_1, A_2)} = \coprod_{i \in A_1} P_{\mathcal{F}_i} \amalg \coprod_{i \in A_2} S_i \amalg \coprod_{i \in \alpha \setminus A_2} h_i(S_i \times S_i \times \{s_i\})$$

where h_i is a homeomorphism of $S_i \times S_i \times S_i$ onto S_i and s_i is an arbitrarily chosen point in S_i. Then $\{X_{(A_1, A_2)} | (A_1, A_2) \in T_\alpha\}$ is an $\mathcal{M}_2\mathcal{M}_3$-representation of T_α by closed subspaces of X. This follows easily from the incomparability of the above ultrafilters $\mathcal{F}_i$, $\mathcal{F}_{i,j,n}$ using the following Lemma 5 of [**11**]:

> Let $\{R_n | n \in \omega\}$ be arbitrary spaces and $\pi_k : \prod_{n \in \omega} R_n \to R_k$ be the projections. For any non-principal ultrafilter $\mathcal{F}$ on ω and any homeomorphism h of $P_{\mathcal{F}}$ into the space $\prod_{n \in \omega} R_n$ there exists $n \in \omega$ such that $\pi_n \circ h$ is nonconstant on any $F \in \mathcal{F}$.

Clearly, for every clopen subset $\mathcal{U}$ of S_i, every point $x \in \mathcal{U}$ lies in a copy of $P_{\mathcal{F}_{i,j,n}}$ contained in $\mathcal{U}$, for some $j \in \{0, 1, 2\}$ and $n \in \omega$, such that the copy is closed in S_i and x plays the rôle of the point $\mathcal{F}_{i,j,n}$ in it. Hence, for every $i \in \alpha$, there exists no homeomorphism of S_i (or of P_i or $h_i(S_i \times S_i \times \{s_i\})$) into the coproduct of all the other summands in the definition of X (or of $X_{(A_1, A_2)}$). Thus if $A_1 \not\subseteq B_1$, then $X_{(A_1, A_2)}$ does not admit any homeomorphism into $X_{(B_1, B_2)}$; and $X_{(A, B_1)}$ is not homeomorphic to $X_{(A, B_2)}$ whenever $B_1 \ne B_2$ because, for $i \in (B_1 \setminus B_2) \cup (B_2 \setminus B_1)$, S_i is not homeomorphic to $h_i(S_i \times S_i \times \{s_i\})$. $\square$

Concluding remarks. Questions and results concerning $\mathcal{M}$-representations (or $\mathcal{M}\mathcal{M}'$-representations with $\mathcal{M} \supseteq \mathcal{M}'$) within various classes of spaces form a very extensive field. Some results of this kind can be found in [**13**], along with an initial attack on “simultaneous representations” (i.e. representations of more than one quasiordered set by a single system of spaces).

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