COMMON FIXED POINTS VIA WEAKLY BIASED GREGUŠ TYPE MAPPINGS

LJ. B. ĆIRIĆ AND J. S. UME

ABSTRACT. In this paper we investigate generalized Greguš type mappings. We proved some common fixed point theorems for four mappings, using the concept of weakly biased mappings.

1. Introduction

Generalizing the concept of commuting mapping, Sessa [11] introduced concept of weakly commuting mappings, and Jungck [5] the concept of compatible mappings. Further generalization of compatible mappings are given by Jungck et al. [6], Pathak and Khan [10] and Pathak et al. [9]. Recently Jungck and Pathak [7] introduced the concept of biased mappings, very general notion of compatible mappings.

Definition 1.1. [7] Let A and S be self-maps of a metric space (X, d). The pair $\{A, S\}$ is S-*biased* iff whenever $\{x_n\}$ is a sequence in X and Ax_n, $Sx_n \to t \in X$, then

$$\alpha d(SAx_n, Sx_n) \leq \alpha d(ASx_n, Ax_n) \text{ if } \alpha = \liminf \text{ and if } \alpha = \limsup .$$

Definition 1.2. [7] Let A and S be self-maps of X. The pair $\{A, S\}$ is *weakly* S-*biased* iff $Ap = Sp$ implies $d(SAp, Sp) \leq d(ASp, Ap)$.

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Clearly, every biased mappings are weakly biased mappings (see Proposition 1.1 in [7]).

Greguš, Jr. in [4] obtained a fixed point theorem for non-expansive type mappings on normed spaces. This result has been found very useful and has many generalizations (see [1]–[3], [8], [12]). The purpose of this note is to use the concept of weakly biased mappings and to prove some common fixed point theorems for generalized Greguš-type mappings, defined by the non-expansive condition (1) bellow. Our results generalize recent results of Shahzad and Sahar [12] and Pathak and Fisher [8].

2. Main results

Theorem 2.1. *Let A, B, S and T be selfmappings of a normed space X and let C be a closed and convex subset of X satisfying the following condition:*

$$\begin{aligned}||Sx-Ty||^p \le \alpha ||Ax-By||^p + (1-\alpha)\max\{\lambda ||Sx-By||^p, \lambda ||Ty-Ax||^p\} \\ +r\cdot \min\{||Ax-Sx||^p, ||By-Ty||^p\}\end{aligned} \tag{1}$$

for all x, $y \in C$, where $0<\alpha<1$, $0<\lambda<1$, $p>0$, $r\ge 0$ and suppose that

$$A(C) \supseteq (1-k)A(C) + kS(C), \tag{2}$$

$$B(C) \supseteq (1-k')B(C) + k'T(C), \tag{3}$$

for some fixed k, k' such that $0<k<1$, $0<k'<1$. If for some $x_0 \in C$, a sequence $\{x_n\}$ in C defined inductively for $n=0,1,2,\dots$ by

$$Ax_{2n+1} = (1-k)Ax_{2n} + kSx_{2n}, \tag{4}$$

$$Bx_{2n+2} = (1-k')Bx_{2n+1} + k'Tx_{2n+1} \tag{5}$$

converges to a point $z \in C$, if A and B are continuous at z, and if $\{S, A\}$ is weakly A-biased, $\{T, B\}$ is weakly B-biased, then A, B, S and T have a unique common fixed point $\omega = Tz$ in C. Further, if A and B are continuous at ω, then S and T are continuous at ω.

Proof. First, we prove that

$$Az = Bz = Sz = Tz. \tag{6}$$

From (4) it follows that

$$kSx_{2n} = Ax_{2n+1} - (1-k)Ax_{2n},$$

and since $0 < k < 1$, $x_n \to z$ and A is continuous at z,

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Ax_n = Az. \tag{7}$$

Similarly, we get

$$\lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Bx_n = Bz. \tag{8}$$

Assume that $Az \neq Bz$. Then, using (1) with $x = x_{2n}$ and $y = x_{2n+1}$, we obtain

$$\begin{aligned} ||Sx_{2n} - Tx_{2n+1}||^p &\leq \alpha||Ax_{2n} - Bx_{2n+1}||^p \\ &+ (1-\alpha)\lambda \max\{||Sx_{2n} - Bx_{2n+1}||^p, ||Tx_{2n+1} - Ax_{2n}||^p\} \\ &+ r \cdot \min\{||Ax_{2n} - Sx_{2n}||^p, ||Bx_{2n+1} - Tx_{2n+1}||^p\}. \end{aligned}$$

Letting $n \to \infty$, by virtue of (7) and (8), it follows that

$$||Az - Bz||^p \leq (1 - (1-\alpha)(1-\lambda))||Az - Bz||^p,$$

a contradiction, as $(1-\alpha)(1-\lambda) > 0$. Thus, $Az = Bz$.

Now suppose that $Tz \neq Az$. Then from (1) we have

$$\begin{aligned} ||Sx_{2n} - Tz||^p \leq \alpha||Ax_{2n} - Bz||^p + (1-\alpha)\lambda \max\{||Sx_{2n} - Bz||^p, ||Tz - Ax_{2n}||^p\} \\ + r \cdot \min\{||Ax_{2n} - Sx_{2n}||^p, ||Bz - Tz||^p\}. \end{aligned}$$

Letting $n \to \infty$, we get, as $Bz = Az$ and $||Ax_{2n} - Sx_{2n}|| \to 0$,

$$||Az - Tz||^p \leq (1-\alpha)\lambda||Az - Tz||^p,$$

a contradiction. Thus, $Az = Tz$. Similarly, $Sz = Bz$. Therefore, we proved that $Az = Bz = Sz = Tz$.

Set

$$\omega = Az = Bz = Sz = Tz.$$

Since $\{S, A\}$ is weakly A-biased, we have

$$||ASz - Az|| \leq ||SAz - Sz||,$$

that is,

$$||A\omega - \omega|| \leq ||S\omega - \omega||.$$

We show that $S\omega = \omega$, and hence $A\omega = \omega$. From (1) we get

$$\begin{aligned} ||S\omega - \omega||^p &= ||S\omega - Tz||^p \leq \alpha||A\omega - \omega||^p \\ &+ (1-\alpha)\lambda \max\{||S\omega - \omega||^p, ||\omega - A\omega||^p\} + r||Bz - Tz||^p \\ &\leq (1 - (1-\alpha)(1-\lambda))||S\omega - \omega||^p. \end{aligned}$$

This implies $||S\omega - \omega||^p = 0$. Hence $S\omega = \omega$ and so $A\omega = \omega$. Similarly, we can prove that $T\omega = B\omega = \omega$. Therefore, we have

$$\omega = A\omega = B\omega = S\omega = T\omega. \tag{9}$$

Now we prove that, if A and B are continuous at ω, then S and T are continuous at ω. Let $\{y_n\}$ be an arbitrary sequence in C converging to ω. From (1) we have

$$\begin{aligned} ||Sy_n - S\omega||^p &= ||Sy_n - T\omega||^p \leq \alpha||Ay_n - B\omega||^p \\ &+ (1-\alpha)\lambda \max\{||Sy_n - B\omega||^p, ||T\omega - Ay_n||^p\} + r||B\omega - T\omega||^p. \end{aligned}$$

Hence we get, by (9),

$$||Sy_n - S\omega||^p \le (\alpha + (1-\alpha)\lambda)\max\{||Sy_n - S\omega||^p, ||Ay_n - A\omega||^p\}.$$

Hence, as $0 < \alpha + (1-\alpha)\lambda < 1$,

$$||Sy_n - S\omega||^p \le ||Ay_n - A\omega||^p.$$

Letting $n \to \infty$ we obtain, as A is continuos,

$$\lim_{n\to\infty} Sy_n = S\omega.$$

Thus, S is continuous at ω. Similarly, we can prove that T is continuous at ω.

The uniqueness of the common fixed point follows from (1). For, if $\omega' = A\omega' = B\omega' = S\omega' = T\omega'$, then we have

$$||\omega - \omega'||^p = ||S\omega - T\omega'||^p \le (1 - (1-\alpha)(1-\lambda))||\omega - \omega'||^p.$$

This implies $\omega' = \omega$. □

If in Theorem 2.1 $r = 0$, $S = T$ and $A = B$, then we have the following corollary.

Corollary 2.2. *Let T and A be two self-mappings of a normed space X and let C be a closed and convex subset of X satisfying the following condition:*

$$\begin{aligned} ||Tx - Ty||^p &\le \alpha||Bx - By||^p \\ &+ (1-\alpha)\max\{\lambda||Tx - By||^p, \lambda||Ty - Bx||^p\}, \\ &\quad B(C) \supseteq (1-k)B(C) + kT(C) \end{aligned}$$

for all x, $y \in C$, where $0 < \alpha < 1$, $0 < \lambda < 1$, $p > 0$, and for some fixed k such that $0 < k < 1$. Suppose, for some $x_0 \in C$, the sequence $\{x_n\}$ in C defined inductively for $n = 0, 1, 2, \ldots$ by

$$Bx_{n+1} = (1-k)Bx_n + kTx_n$$

converges to a point z in C and the pair $\{T, B\}$ is B-biased. If B is continuous at z, then B and T have a unique common fixed point. Further, if B is continuous at Bz, then T is continuous at a common fixed point.

Remark 2.3. Corollary 2.1 with $\lambda = \frac{1}{2}$, C bounded and the pair $\{T, B\}$ is B-biased, becomes Theorem 2.11 of Shahzad and Sahar in [12]. Thus, Corollary 2.2 is a generalization of Theorem 2.1 in [12].

Remark 2.4. When $B = I$, the identity mapping, and $\lambda = \frac{1}{2}$, then our Corollary 2.2 becomes Corollary 2.3 of Shahzad and Sahar in [12].

Theorem 2.5. *Let A, B, S and T be self-mappings of a normed space X. Let C be a closed and convex subset of X such that*

$$A(C) \supseteq (1-k)A(C) + kS(C), \tag{10}$$

$$B(C) \supseteq (1-k')B(C) + k'T(C), \tag{11}$$

where $0 < k < 1$, $0 < k' < 1$ and such that

$$\begin{aligned} ||Sx - Ty||^p \leq \varphi\Big(& \frac{2\alpha||Ax - By||^{2p}}{||Sx - By||^p + ||Ty - Ax||^p} + \\ & +(1-\alpha)\max\{||Sx - By||^p, ||Ty - Ax||^p\}\Big) + \\ & + r \cdot \min\{||Ax - Sx||^p, ||By - Ty||^p\} \end{aligned} \tag{12}$$

for all x, $y \in C$ for which

$$\max\{||Sx - By||, ||Ty - Ax||\} \neq 0,$$

where $0 < \alpha < 1$, $p > 0$, $r \geq 0$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is upper semicontinuous function such that $\varphi(t) < t$ for all $t > 0$. If for some $x_0 \in C$, a sequence $\{x_n\}$ in C defined inductively for $n = 0, 1, 2, \ldots$ by

$$Ax_{2n+1} = (1-k)Ax_{2n} + kSx_{2n}, \tag{13}$$

$$Bx_{2n+2} = (1-k')Bx_{2n+1} + k'Tx_{2n+1} \tag{14}$$

converges to a point z in C, if A and B are continuous at z, and if $\{S, A\}$ is weakly A-biased, $\{T, B\}$ is weakly B-biased, then A, B, S and T have a unique common fixed point $\omega = Az$ in C. Further, if A and B are continuous at Az, then S and T are continuous at a common fixed point.

Proof. Similarly as in Theorem 2.1 we can prove that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_{2n} = Az, \tag{15}$$

$$\lim Bx_n = \lim_{n\to\infty} Tx_{2n+1} = Bz. \tag{16}$$

If we suppose that $Az \neq Bz$, then for large enough n, $||Sx_{2n} - Bx_{2n+1}|| > 0$. Thus, from (12) we have

$$\begin{aligned} ||Sx_{2n} - Tx_{2n+1}||^p \leq \varphi\Big(& \frac{2\alpha||Ax_{2n} - Bx_{2n+1}||^{2p}}{||Sx_{2n} - Bx_{2n+1}||^p + ||Tx_{2n+1} - Ax_{2n}||^p} + \\ & + \ (1-\alpha)\max\{||Sx_{2n} - Bx_{2n+1}||^p, ||Tx_{2n+1} - Ax_{2n}||^p\}\Big) + \\ & + r \cdot \min\{||Ax_{2n} - Sx_{2n}||^p, ||Bx_{2n+1} - Tx_{2n+1}||^p\}. \end{aligned} \tag{17}$$

Since (15) and (16) imply that argument t_n of $\varphi(t_n)$ in (17) tends to $||Az - Bz||^p$ as $n \to \infty$ and as $\varphi(t)$ is upper semicontinuous, letting $n \to \infty$ in (17) we get

$$||Az - Bz||^p \leq \varphi(||Az - Bz||^p) < ||Az - Bz||^p,$$

a contradiction. Thus, $Az = Bz$.
Now, if we assume that $||Az - Tz|| > 0$, then for large enough n, $||Ax_{2n} - Tz|| > 0$. Thus, from (12) we obtain

$$||Sx_{2n} - Tz||^p \leq \varphi\Big(\frac{2\alpha||Ax_{2n} - Bz||^{2p}}{||Sx_{2n} - Bz||^p + ||Ax_{2n} - Tz||^p} + \\ + (1-\alpha)\max\{||Sx_{2n} - Bz||^p, ||Ax_{2n} - Tz||^p\}\Big) + \\ + r \cdot \min\{||Ax_{2n} - Sx_{2n}||^p, ||Bz - Tz||^p\}.$$

Letting $n \to \infty$ we get, as $||Ax_{2n} - Sx_{2n}|| \to 0$,

$$||Az - Tz||^p \leq \varphi((1-\alpha)||Az - Tz||^p) < (1-\alpha)||Az - Tz||^p,$$

a contradiction. Thus, $Az = Tz$. Similarly $Sz = Bz$. Therefore, we proved that

$$\omega = Az = Bz = Sz = Tz.$$

Since the pair $\{S, A\}$ is weakly A-biased and $\{T, B\}$ is weakly B-biased, similarly as in Theorem 2.1 we can prove that

$$\omega = A\omega = B\omega = S\omega = T\omega. \tag{18}$$

Now we prove that, if A and B are continuous at ω, then S and T are continuous at a common fixed point ω. We show that

$$||Sx - S\omega|| \leq ||Ax - A\omega|| \tag{19}$$

for all $x \in C$.

Suppose that $||Sx - S\omega|| > ||Ax - A\omega||$. Then from (12) and (18) we have, as $\varphi(t) < t$,

$$||Sx - S\omega||^p = ||Sx - T\omega||^p < \alpha||Ax - A\omega||^p + (1-\alpha)||Sx - S\omega||^p < ||Sx - S\omega||^p,$$

a contradiction. Thus (19) holds. Since A is continous at ω, (19) implies that S is contiunuos at ω. Similarly it can be proved that T is contiunuos at ω. The uniqueness of a common fixed point follows from (12). $\square$

Remark 2.6. In Theorem 2.6 of Shahzad and Sahar in [12], the argument of a function $\varphi(t)$ is

$$t = \frac{\alpha||Ax - By||^{2p}}{\max\{||Sx - By||^p, ||Ty - Ax||^p\}} + \min\{||Sx - By||^p, ||Ty - Ax||^p\},$$

and coefficient r is zero. It is easy to verify that Theorem 2.5 remains true with this argument of $\varphi(t)$ and $r > 0$.

Remark 2.7. If $S = T$ and $A = B$ in Theorem 2.5, then we have the corollary, which generalizes Corollary 2.7 in [12]. Further, if $A = B = I$, the identity mapping on X, then we obtain the corollary which generalizes Corollary 2.8 in [12], and if in addition $\varphi(t) = \lambda t$; $0 < \lambda < 1$, then we have the corollary which generalizes Corollary 2.9 in [12]. For details, we refer to [12], and for many illustrative examples, to [7]–[10] and [12].

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LJ. B. Ćirić, University of Belgrade, Aleksinackih rudara 12/35, 11080 Belgrade, Serbia, *e-mail*: `lciric@mas.bg.ac.yu`

J. S. Ume, Department of Aplied Mathematics, Changwon National University, Changwon, 641-773, Korea, *e-mail*: `jsume@sarim.changwon.ac.kr`