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MINIMALITY AND PREHOMOGENEITY

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Abstract. We introduce two new types of minimality; Namely: minimal set and minimal preopen set. Several results concerning these types and the known type (minimal open set) are obtained. We also introduce prehomogeneity concept as a generalization of homogeneity. Several results concerning it are given, some of which related to minimality concepts. Various counter examples relevant to the relations obtained in this paper are given.

1. Introduction

Let (X,τ) be a space and $A \subseteq X$. We denote the complement of A in X by $X-A$, the closure and the interior of A respectively by $\overline{A}$ and $\mathrm{Int}(A)$, the relative topology on A by $\tau\,|A$, A is *preopen* [**13**] if $A \subseteq \mathrm{Int}(\overline{A})$. $\mathrm{PO}\,(X,\tau)$ is the family of all preopen sets in X. The topology on X with the subbase $\mathrm{PO}(X,\tau)$ will be denoted by τ^* and is called *the topology generated by preopen sets* [**14**]. A is *α-set* [**17**] if $A \subseteq \mathrm{Int}\left(\overline{\mathrm{Int}\,(A)}\right)$. The family of all α-sets in a space (X,τ), denoted by τ^α is again a topology on X satisfying $\tau \subseteq \tau^\alpha$. A function $f:(X,\tau)\to(X,\tau)$ is *preirresolute* [**4**] if $f^{-1}(A) \in \mathrm{PO}\,(X,\tau_1)$ for all $A \in \mathrm{PO}\,(Y,\tau_2)$. f is *a prehomeomorphism* [**14**] if f is bijective and $A \in \mathrm{PO}(X,\tau_1)$ iff $f(A) \in \mathrm{PO}(Y,\tau_2)$, i.e., f is a bijection and both f and f^{-1} are perirresolute. f is an *α-homeomorphism* [**14**] if f is bijective and $A \in \tau^\alpha$ iff $f(A) \in \tau^\alpha$. If (X,τ) is a space, then $\mathrm{PH}\,(X,\tau)$ will denote the group of all prehomeomorphisms from (X,τ) onto itself.

The homogeneity concept was introduced by W. Sierpinski [**19**] in 1920 as follows: A space (X,τ) is homogeneous if for any two points $x,y \in X$ there exists an autohomeomorphism f on (X,τ) such that $f(x)=y$. Seven years earlier, L. Brouwer had shown that if A and B are two countable dense subsets of the n-dimensional Euclidean space $\mathbb{R}^n$, then there is an autohomeomorphism on $\mathbb{R}^n$ that takes A to B. He needed this result in his development of dimension theory. Many modifications of homogeneity were introduced and studied [**1**] – [**3**], [**5**] – [**7**], [**9**], [**15**], [**18**]. In [**10**], we fuzzified homogeneity.

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Mathematicians generalized many concepts of general topology using preopen sets. In a part of this paper we find it is suitable to generalize homogeneity using preopen sets. In the first part of this paper we introduce the concepts of minimal sets and minimal preopen sets. Those concepts also play a vital rule in studying prehomogeneity.

The following lemmas will be used in the sequel.

Lemma 1.1. [**4**] *Let* (X,τ) *be a space. If* $A \in \tau^{\alpha}$ *and* $B \in \mathrm{PO}(X,\tau)$*, then* $A \cap B \in \mathrm{PO}(X,\tau)$.

Lemma 1.2. [**4**] *Let* (X,τ) *be a space. If* $A \subseteq B \subseteq X$ *and* $A \in \mathrm{PO}(X,\tau)$*, then* $A \in \mathrm{PO}(B, \tau|B)$.

Lemma 1.3. [**11**] *Let* (X,τ) *be a space. Then* $\mathrm{PO}(X,\tau) = \mathrm{PO}(X,\tau^{\alpha})$.

Recall that a space (X,τ) is *locally indiscrete* if every open subset of X is closed.

Lemma 1.4. [**4**] *For a space* (X,τ) *the following are equivalent.*

(i) (X,τ) *is locally indiscrete.*
(ii) *Every singleton in* X *is preopen.*
(iii) *Every subset of* X *is preopen.*

Lemma 1.5. [**16**] *Let* (X,τ) *be a space and let* A *be a minimal open set in* X*. Then*

$$A = \bigcap \{O : O \text{ is open in } X \text{ with } x \in O\}$$

for any element x *of* A.

Lemma 1.6. [**16**] *Let* (X,τ) *be a space. If* A *is a minimal open set in* X*, then every subset of* A *is preopen.*

Lemma 1.7. [**12**] *Every homeomorphism is a prehomeomorphism but not conversely.*

Lemma 1.8. [**12**] *If* X, Y *are* T_1 *spaces, then the classes of prehomeomorphisms and* α*-homeomorphisms from* X *onto* Y *coincide.*

Recall that a partition base for the space (X,τ) is just a base for τ for which their elements form a partition on X.

Lemma 1.9. [**8**] *Let* (X,τ) *be a space which contains a minimal open set. Then the following are equivalent.*

(a) (X,τ) *is a homogeneous space.*
(b) (X,τ) *has a partition base consisting of minimal open sets all of which is homeomorphic to one another.*

2. Minimal Sets

Let (X, τ) be a space. For each $x \in X$, denote by U_x the intersection of all open sets in X containing x, i.e.,

$$U_x = \bigcap \{O : O \text{ is open in } X \text{ with } x \in O\}.$$

Definition 2.1. Let (X, τ) be a space and let $A \subseteq X$, then A is called a minimal set in X if there exists $x \in X$ such that $A = U_x$.

Definition 2.2. [**16**] Let (X, τ) be a space. A non empty open set A of X is called a minimal open set in X if any open set in X which is contained in A is $\emptyset$ or A, i.e., $\tau\,|A$ is the indiscrete topology on A.

Definition 2.3. Let (X, τ) be a space. A non empty preopen set A of X is called a minimal preopen set in X if any preopen set in X which is contained in A is $\emptyset$ or A.

Theorem 2.4. *Let (X, τ) be a space. If A is a minimal open set in X, then A is a minimal set in X.*

Proof. Lemma 1.5. □

The converse of Theorem 2.4 is not true as the following example shows.

Example 2.5. Let $\mathbb{R}$ be the real line with the usual topology.
Since $0 \in U_0 \subseteq \bigcap\limits_{n=1}^{\infty} \left(\frac{-1}{n}, \frac{1}{n} \right) = \{0\}$, $U_0 = \{0\}$. But $\{0\}$ is not even preopen in $\mathbb{R}$.

The following result is a characterization of locally indiscrete spaces in terms of minimal preopen sets. The proof follows easily and is left to the reader.

Theorem 2.6. *For a space (X, τ) the following are equivalent*
(i) *(X, τ) is locally indiscrete.*
(ii) *The set of all minimal sets in X form a partition base for (X, τ).*

The following result shows that minimal preopen sets are singletons.

Theorem 2.7. *Let (X, τ) be a space and $A \subseteq X$, then A is a minimal preopen set in X iff $A \in \mathrm{PO}(X, \tau)$ and A is a singleton.*

Proof. Suppose that A is a minimal preopen set in X. Choose $x \in A$. We are going to show that $\{x\}$ is preopen. By Lemma 1.1, it follows that $\left(X - \overline{\{x\}}\right) \cap A$ is preopen. Therefore, since A is a minimal preopen set in X and $\left(X - \overline{\{x\}}\right) \cap A \subseteq A$ with $\left(X - \overline{\{x\}}\right) \cap A \neq A$, it follows that $\left(X - \overline{\{x\}}\right) \cap A = \emptyset$. Thus, $A \subseteq \overline{\{x\}}$ and hence $\overline{A} \subseteq \overline{\{x\}}$. Therefore, $\{x\} \subseteq A \subseteq \mathrm{Int}\left(\overline{A}\right) \subseteq \mathrm{Int}\left(\overline{\{x\}}\right)$ and hence $\{x\}$ is preopen. Thus, by preopen minimality of A, $A = \{x\}$ and hence A is a singleton.
Conversely, if $A \in \mathrm{PO}(X, \tau)$ and A is a singleton, then it is clear that A is a minimal preopen set. □

Corollary 2.8. *If A is a minimal open set in the space (X, τ), then for each $a \in A$, $\{a\}$ is a minimal preopen set in X.*

Proof. Lemma 1.6 and Theorem 2.7. □

Corollary 2.8, shows that if a space have the property 'having a minimal open set', then it must have the property 'having a minimal preopen set'. The following example shows that the converse is not true. It also shows that the converse of Corollary 2.8 is not true.

Example 2.9. Let $\mathcal{R}$ be the set of real numbers with the topology τ having the family $\{[-a, a] : a \in \mathbb{R} \text{ and } a > 1\}$ as a base. Then $\{0\}$ is a minimal preopen set in $\mathbb{R}$, but $(\mathbb{R}, \tau)$ has no minimal open set. On the other hand, it is easy to see that $\{a\}$ is a minimal preopen set in $\mathbb{R}$ for each point $a \in U_0 = [-1, 1]$ and U_0 is not a minimal open set in $\mathbb{R}$.

Theorem 2.10. *Let (X, τ) be a space and $x \in X$, then $\{x\}$ is a minimal preopen subset of X iff $\{y\} \in \mathrm{PO}(X, \tau)$ for each $y \in U_x$.*

Proof. Let $y \in U_x$, then $x \in \overline{\{y\}}$ and so $\overline{\{x\}} \subseteq \overline{\{y\}}$. Since $\{x\} \in \mathrm{PO}(X, \tau)$, there exists $U \in \tau$ such that $\{x\} \subseteq U \subseteq \overline{\{x\}}$. Thus, $\{y\} \subseteq U_x \subseteq U \subseteq \overline{\{x\}} \subseteq \overline{\{y\}}$ and hence $\{y\} \in \mathrm{PO}(X, \tau)$.
Conversely, if $\{y\} \in \mathrm{PO}(X, \tau)$ for each $y \in U_x$, then in particular, $\{x\} \in \mathrm{PO}(X, \tau)$ and hence $\{x\}$ is a minimal preopen subset of X. □

Corollary 2.11. *Let (X, τ) be a space. If $\{x\}$ is a minimal preopen subset of X, then the minimal set U_x is preopen.*

Corollary 2.12. *Let (X, τ) be a space. If $\{x\}$ is a minimal preopen subset of X, then the subspace $(U_x, \tau|U_x)$ is locally indiscrete.*

Proof. Lemmas 1.2 and 1.4 and Theorem 2.10. □

The following example shows that in Corollary 2.12, the condition '$\{x\}$ is a minimal preopen subset of X' cannot be dropped although the minimal set $U_x \in \mathrm{PO}(X, \tau)$. It also shows that the converse of Corollary 2.11, is not true.

Example 2.13. Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Then $U_b = \{a, b\} \in \mathrm{PO}(X, \tau)$ but $\{b\} \notin \mathrm{PO}(X, \tau)$. On the other hand, $\tau|U_b = \{\emptyset, U_b, \{a\}\}$, and hence the subspace $(U_b, \tau|U_b)$ is not locally indiscrete.

Theorem 2.14. *Let (X, τ) be a space. If $\{x\}$ is a minimal preopen set and there is a non empty open subset $V \subseteq X$ such that $V \subseteq U_x$, then $U_x = V$.*

Proof. Let V be a non empty open set in X such that $V \subseteq U_x$. Choose $y \in V$. Since $y \in U_x$, $x \in \overline{\{y\}}$. Also, since $\{x\} \in \mathrm{PO}(X, \tau)$, there exists $U \in \tau$ such that $\{x\} \subseteq U \subseteq \overline{\{x\}}$. Since $x \in U \cap \overline{\{y\}}$, $y \in U$ and so $y \in \overline{\{x\}}$. Therefore, $x \in V$ and hence $U_x \subseteq V$. Thus, $U_x = V$. □

Corollary 2.15. *Let (X, τ) be a space. If $\{x\}$ is a minimal preopen set in X, then either U_x is a minimal open set in X or $\mathrm{Int}(U_x) = \emptyset$.*

In corollary 2.15, the statement '$\mathrm{Int}\,(U_x) = \emptyset$', cannot be replaced by the statement 'U_x is a minimal preopen set' as Example 2.9 shows.

Theorem 2.16. *Let (X, τ) be a space and $x \in X$. Then U_x is a minimal open set in X iff $\{x\}$ is a minimal preopen set in X and* $\mathrm{Int}\,(U_x) \neq \emptyset$.

Proof. If U_x is a minimal open set, then it is clear that $\mathrm{Int}\,(U_x) \neq \emptyset$. On the other hand, by Corollary 2.8, it follows that $\{x\}$ is a minimal preopen set in X.

Conversely, if $\{x\}$ is a minimal preopen set with $\mathrm{Int}\,(U_x) \neq \emptyset$, then by Corollary 2.15, U_x is a minimal open set. □

3. Prehomogeneous spaces

Definition 3.1. A space (X, τ) is prehomogeneous if for any two points $x, y \in X$ there exists $f \in \mathrm{PH}\,(X, \tau)$ such that $f\,(x) = y$.

Theorem 3.2. *Every homogeneous space is prehomogeneous.*

Proof. Lemma 1.7. □

Theorem 3.3. *Every locally indiscrete space is a prehomogeneous.*

Proof. Let (X, τ) be a locally indiscrete space and let $x_1, x_2 \in X$. Define $f : (X, \tau) \to (X, \tau)$ by $f\,(x_1) = x_2$, $f\,(x_2) = x_1$ and $f\,(x) = x$ elsewhere. Then by Lemma 1.4 it is easy to see that $f \in \mathrm{PH}\,(X, \tau)$ with $f\,(x_1) = x_2$. Therefore, (X, τ) is a prehomogeneous space. □

Corollary 3.4. *Let (X, τ) be a space. If $\{x\}$ is a minimal preopen set in X. Then the subspace $(U_x, \tau\,|U_x)$ is prehomogenous.*

Proof. Corollary 2.12 and Theorem 3.3. □

Corollary 3.5. *Let (X, τ) be a space. If A is a minimal open subset of X. Then the subspace $(A, \tau\,|A)$ is prehomogenous.*

Proof. Since A is a minimal open set in X, then by Theorem 2.4, it follows that A is a minimal set in X and so there exists $x \in X$ such that $A = U_x$. Since $x \in A$, it follows by Corollary 2.8, that $\{x\}$ is a minimal preopen set and so according to Corollary 3.4 we get the result. □

The following example shows that the converse of Theorem 3.2 is not true.

Example 3.6. Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Then by Theorem 3.3, the space (X, τ) is prehomogeneous but using Lemma 1.9, it follows that (X, τ) is not homogeneous.

One may ask the following question: Does there exist a non-homogeneous and perhomogeneous space that is not locally indiscrete?

The following example answers this question.

Example 3.7. Consider the set of natural numbers $\mathbb{N}$ with the topology $\tau = \{\emptyset\} \cup \{\{n, n+1, n+2, ...\} : n \in \mathbb{N}\}$. Then this space is prehomogeneous and non homogeneous but not locally indiscrete.

Proof. Note that $\mathrm{PO}(N,\tau) = \{\emptyset\} \cup \{A \subseteq N : A \text{ is infinite}\}$. Therefore, for any two points $n, m \in N$, it is easy to see that the function $f : (N,\tau) \to (N,\tau)$ where $f(n) = m$, $f(m) = n$, and $f(x) = x$ elsewhere is a prehomeomorphism with $f(n) = m$ and hence (N,τ) is a prehomogeneous space. Now it is not difficult to see that there is no homeomorphism between (N,τ) and itself which takes 1 to 2 and hence (N,τ) is not homogeneous. Finally, it is clear that the space (N,τ) is not locally indiscrete. □

Now we state one of our main results in this section.

Theorem 3.8. *Let (X,τ) be a space which contains a minimal preopen set. Then the following are equivalent.*

(i) *(X,τ) is prehomogeneous.*
(ii) *(X,τ) is locally indiscrete.*

Proof. $(i) \Rightarrow (ii)$ Suppose that (X,τ) is prehomogeneous and let $\{x\}$ be a minimal preopen set in X. Let $y \in X$, then since (X,τ) is a prehomogeneous space there exists $f \in \mathrm{PH}(X,\tau)$ such that $f(y) = x$. Since f is preirresolute, $\{y\} = f^{-1}(\{x\}) \in \mathrm{PO}(X,\tau)$. Therefore, every singleton in X is preopen and hence by Lemma 1.4, it follows that (X,τ) is a locally indiscrete space.
$(ii) \Rightarrow (i)$ Theorem 3.3. □

Example 3.7, shows that the condition 'containing a minimal preopen set' in Theorem 3.8 cannot be dropped.

Corollary 3.9. *Let (X,τ) be a space which contains a minimal open set. Then the following are equivalent*

(i) *(X,τ) is prehomogeneous.*
(ii) *(X,τ) is locally indiscrete.*

Proof. Corollary 2.8 and Theorem 3.8. □

Corollary 3.10. *Let (X,τ) be a space for which X is finite or τ is finite. Then (X,τ) is prehomogeneous iff (X,τ) is locally indiscrete.*

Theorem 3.11. *Let (X,τ) be a space. Then (X,τ) is prehomogeneous iff (X,τ^{α}) is prehomogeneous.*

Proof. Suppose that (X,τ) is a prehomogeneous space and let $x_1, x_2 \in X$, then there exists $f \in \mathrm{PH}(X,\tau)$ such that $f(x_1) = x_2$. Now, by Lemma 1.3, it follows that $f \in \mathrm{PH}(X,\tau^{\alpha})$ and hence (X,τ^{α}) is prehomogeneous. □

Similarly, if (X,τ^{α}) is prehomogeneous then we can show that (X,τ) is prehomogeneous.

Corollary 3.12. *Let (X,τ) be a space. If (X,τ^{α}) is homogeneous, then (X,τ) is a prehomogeneous.*

Proof. Since (X,τ^{α}) is homogeneous, it follows by Theorem 3.2 that (X,τ^{α}) is prehomogeneous. Therefore, by Theorem 3.11, it follows that (X,τ) is a prehomogeneous. □

In fact, Example 3.6 shows that the converse of Corollary 3.12 is not true. However, in T_1 spaces we have the following result.

Theorem 3.13. *If (X, τ) is a T_1 space. Then the following are equivalent.*
(i) (X, τ^α) *is homogeneous.*
(ii) (X, τ) *is prehomogeneous.*

Proof. $(i) \Rightarrow (ii)$ Corollary 3.12.
$(ii) \Rightarrow (i)$ Lemma 1.8. □

Theorem 3.14. *Let (X, τ) be a space. If (X, τ) is prehomogeneous then (X, τ^*) is homogeneous.*

Proof. Suppose that (X, τ) is prehomogeneous. Let $x_1, x_2 \in X$, then since (X, τ) is prehomogeneous, there is $f \in \mathrm{PH}(X, \tau)$ such that $f(x_1) = x_2$. Let B be a basic open set in (X, τ^*), then $B = \bigcap_{i=1}^{n} A_i$ where $A_i \in \mathrm{PO}(X, \tau)$ for all i. Since $f \in \mathrm{PH}(X, \tau)$, $f^{-1}(A_i) \in \mathrm{PO}(X, \tau)$ for all i and hence $f^{-1}(B) = \bigcap_{i=1}^{n} f^{-1}(A_i) \in \tau^*$. Therefore, f is continuous. Similarly we can show f^{-1} is continuous. Therefore, $f : (X, \tau^*) \to (X, \tau^*)$ is a homeomorphism with $f(x_1) = x_2$ and hence (X, τ^*) is a homogeneous space. □

The following example shows that the converse of Theorem 3.14 is not true.

Example 3.15. Consider the set $X = \{a, b, c, d\}$ with the topology $\tau = \{\emptyset, X, \{a, b\}, \{a, b, c\}\}$. Then τ^* is the discrete topology on X and hence by Lemma 1.9, it follows that (X, τ^*) is a homogeneous space. On the other hand, using Corollary 3.10, it is easy to see that (X, τ) is not prehomogeneous.

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