Acta Math. Univ. Comenianae
Vol. LXXIV, 1(2005), pp. 143–148

α-FUZZY FIXED POINTS FOR α-FUZZY MONOTONE MULTIFUNCTIONS

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Abstract. In this note, we prove the existence of maximal, minimal, greatest and least α-fuzzy fixed points for α-fuzzy monotone multifunctions.

1. Introduction

Let X be a nonempty set. A fuzzy subset A of X is a function of X into $[0,1]$ (see [**14**]). A fuzzy multifunction is a map $T : X \to [0,1]^X$ such that for every $x \in X$, $T(x)$ is a nonempty fuzzy set. Let $\alpha \in]0,1]$ and let $T : X \to [0,1]^X$ be a fuzzy multifunction. We say that an element x of X is an α-fuzzy fixed point of T if $T(x)(x) = \alpha$. When $\alpha = 1$, the element x is called a fixed point of T.

During the last few decades several authors established fixed points theorems in fuzzy setting, see for example [**1**] – [**12**]. Recently, in [**9**], we introduced the notion of α-fuzzy ordered sets in which we established some fixed points theorems for fuzzy monotone multifunctions.

The aim of this note is to study the existence of α-fuzzy fixed points for α-fuzzy monotone multifunctions. First, we prove the existence of maximal and minimal α-fuzzy fixed points (see Theorems 3.1 and 3.3). Second, we establish the existence of greatest and least α-fuzzy fixed points (see Theorems 4.1 and 4.2).

2. Preliminaries

First, we recall the definition of α-fuzzy order.

Definition 2.1. [**9**] Let X be a nonempty set and $\alpha \in]0,1]$. An α-fuzzy order on X is a fuzzy subset r_α of $X \times X$ satisfying the following three properties:

(i) for all $x \in X$, $r_\alpha(x,x) = \alpha$, (α-fuzzy reflexivity);
(ii) for all $x, y \in X$, $r_\alpha(x,y) + r_\alpha(y,x) > \alpha$ implies $x = y$. (α-fuzzy antisymmetry);
(iii) for all $x, z \in X$, $r_\alpha(x,z) \geq \sup_{y \in X} [\min\{r_\alpha(x,y), r_\alpha(y,z)\}]$ (α-fuzzy transitivity).

Received February 4, 2004.
2000 *Mathematics Subject Classification.* Primary 04A72, 03E72, 06A06, 47H10.
Key words and phrases. Fuzzy set, α-fuzzy order relation, monotone multifunction, fixed point.

The pair (X, r_α), where r_α is a α-fuzzy order on X is called a r_α-fuzzy ordered set. An α-fuzzy order r_α is said to be total if for all $x \neq y$ we have either $r_\alpha(x,y) > \frac{\alpha}{2}$ or $r_\alpha(y,x) > \frac{\alpha}{2}$. A r_α-fuzzy ordered set X on which the order r_α is total is called r_α-fuzzy chain.

Let (X, r_α) be a nonempty r_α-fuzzy ordered set and A be a subset of X.

An element u of X is said to be a r_α-upper bound of A if $r_\alpha(x,u) > \frac{\alpha}{2}$ for all $x \in A$.

If x is a r_α-upper bound of A and $x \in A$, then it is called a greatest element of A.

An element m of A is called a maximal element of A if there is $x \in A$ such that $r_\alpha(m,x) > \frac{\alpha}{2}$, then $x = m$.

An element l of X is said to be a r_α-lower bound of A if $r_\alpha(l,x) > \frac{\alpha}{2}$ for all $x \in A$.

If l is a r_α-lower bound of A and $l \in A$, then it is called the least element of A.

An element n of A is called a minimal element of A if there is $x \in A$ such that $r_\alpha(x,n) > \frac{\alpha}{2}$, then $x = n$. As usual,

$\sup_{r_\alpha}(A) :=$ the least element of r_α-upper bounds of A (if it exists),
$\inf_{r_\alpha}(A) :=$ the greatest element of r_α-lower bounds of A (if it exists),
$\max_{r_\alpha}(A) :=$ the greatest element of A (if it exists),
$\min_{r_\alpha}(A) :=$ the least element of A (if it exists).

Next, we shall give four examples of α-fuzzy orders.

Examples.

1. Let $X = \{0,1,2\}$ and r_α be the α-fuzzy order relation defined on X by:

$$r_\alpha(0,0) = r_\alpha(1,1) = r_\alpha(2,2) = \alpha,$$

$$\begin{cases} r_\alpha(0,2) = 0.55\alpha \\ r_\alpha(2,0) = 0.1\alpha \end{cases} \quad \begin{cases} r_\alpha(2,1) = 0.2\alpha \\ r_\alpha(1,2) = 0.6\alpha \end{cases} \quad \begin{cases} r_\alpha(1,0) = 0.7\alpha \\ r_\alpha(0,1) = 0.15\alpha. \end{cases}$$

As properties of r_α, we have $\inf_{r_\alpha}(X) = 0$ and $\sup_{r_\alpha}(X) = 2$.

2. Consider the α-fuzzy order relation r_α defined on $X = \{0,1,2\}$ by:

$$r_\alpha(0,0) = r_\alpha(1,1) = r_\alpha(2,2) = \alpha,$$

$$\begin{cases} r_\alpha(0,2) = 0.6\alpha \\ r_\alpha(2,0) = 0.2\alpha \end{cases} \quad \begin{cases} r_\alpha(2,1) = 0.2\alpha \\ r_\alpha(1,2) = 0.3\alpha \end{cases} \quad \begin{cases} r_\alpha(1,0) = 0.3\alpha \\ r_\alpha(0,1) = 0.55\alpha. \end{cases}$$

In this case, we have $\inf_{r_\alpha}(X) = 0$ and $\sup_{r_\alpha}(X)$ do not exist in X. Note that 1 and 2 are two maximal elements in (X, r_α).

3. Let r_α be the α-fuzzy order defined on $X = \{0,1,2\}$ by:

$$r_\alpha(0,0) = r_\alpha(1,1) = r_\alpha(2,2) = \alpha,$$

$$\begin{cases} r_\alpha(0,2) = 0.65\alpha \\ r_\alpha(2,0) = 0.15\alpha \end{cases} \quad \begin{cases} r_\alpha(2,1) = 0.1\alpha \\ r_\alpha(1,2) = 0.7\alpha \end{cases} \quad \begin{cases} r_\alpha(1,0) = 0.15\alpha \\ r_\alpha(0,1) = 0.10\alpha. \end{cases}$$

Then, $\sup_{r_\alpha}(X) = 2$ and $\inf_{r_\alpha}(X)$ do not exist in X. In addition, 1 and 0 are two minimal elements in (X, r_α).

4. Let r_α be the α-fuzzy order defined on $X = \{0, 1, 2\}$ by:

$$r_\alpha(0,0) = r_\alpha(1,1) = r_\alpha(2,2) = \alpha,$$

$$\begin{cases} r_\alpha(0,2) = 0.8\alpha \\ r_\alpha(2,0) = 0.15\alpha \end{cases} \qquad \begin{cases} r_\alpha(2,1) = 0.20\alpha \\ r_\alpha(1,2) = 0.30\alpha \end{cases} \qquad \begin{cases} r_\alpha(1,0) = 0.30\alpha \\ r_\alpha(0,1) = 0.20\alpha. \end{cases}$$

In this case, $\inf_{r_\alpha}(X)$ and $\sup_{r_\alpha}(X)$ do not exist in X. Also, 1 is a maximal and minimal element of (X, r_α).

Next, we recall some definitions and results for subsequent use.

Definition 2.2. [**9**] Let (X, r_α) be a nonempty r_α-fuzzy ordered set. The inverse α-fuzzy relation s_α of r_α is defined by $s_\alpha(x,y) = r_\alpha(y,x)$, for all $x, y \in X$.

Let us not that by [**9**, Proposition 3.5], if r_α is an α-fuzzy order, then s_α is also an α-fuzzy order.

In [**10**], we proved the following lemma.

Lemma 2.3. *Let (X, r_α) be a r_α-fuzzy order set and s_α be the inverse fuzzy order relation of r_α. Then,*

(i) *If a nonempty subset A of X has a r_α-supremum, then A has a s_α-infimum and $\inf_{s_\alpha}(A) = \sup_{r_\alpha}(A)$.*
(ii) *If a nonempty subset A of X has a r_α-infimum, then A has a s_α-supremum and $\inf_{r_\alpha}(A) = \sup_{s_\alpha}(A)$.*

The following α-fuzzy Zorn's Lemma is given in [**9**].

Lemma 2.4. *Let (X, r_α) be a nonempty α-fuzzy ordered sets. If every nonemty r_α-fuzzy chain in X has a r_α-upper bound, then X has a maximal element.*

Let $T : X \to [0,1]^X$ be a fuzzy multifunction. Then, for every $x \in X$, we define the following subset of X by setting:

$$T_x^\alpha = \{y \in X : T(x)(y) = \alpha\}.$$

In this note, we shall use the following definition of α-fuzzy monotonicity.

Definition 2.5. Let (X, r_α) be a nonempty r_α-fuzzy ordered set. A fuzzy multifunction $T : X \to [0,1]^X$ is said to be r_α-fuzzy monotone if the two following properties are satisfied:

(i) for all $x \in X$, $T_x^\alpha \neq \emptyset$;
(ii) if $r_\alpha(x,y) > \frac{\alpha}{2}$ and $x \neq y$, for $x, y \in X$, then for all $a \in T_x^\alpha$ and $b \in T_y^\alpha$, we have $r_\alpha(a,b) > \frac{\alpha}{2}$.

We denote by $\mathcal{F}_T^\alpha$ the set of all α-fuzzy fixed points of T.

3. Maximal and minimal α-fuzzy fixed points

In this section, we investigate the existence of maximal and minimal α-fuzzy fixed points of α-fuzzy monotone multifunctions. First, we shall show the following:

Theorem 3.1. *Let (X, r_α) be an α-fuzzy ordered set with the property that every nonempty r_α-fuzzy chain in (X, r_α) has a r_α-supremum. Let $T : X \to [0,1]^X$ be a r_α-fuzzy monotone multifunction. If there exist $a, b \in X$ such that $T(a)(b) = \alpha$ and $r_\alpha(a,b) > \frac{\alpha}{2}$, then the set $\mathcal{F}_T^\alpha$ of all α-fuzzy fixed points of T is nonempty and has a maximal element.*

Proof. Let H_α be the fuzzy ordered subset of X defined by

$$H_\alpha = \left\{ x \in X : \text{ there exists } y \in X, T(x)(y) = \alpha \text{ and } r_\alpha(x,y) > \frac{\alpha}{2} \right\}.$$

Since $a \in H_\alpha$, then the subset H_α is nonempty.

Claim 1. The subset H_α has a maximal element. Indeed, if C is a nonempty r_α-fuzzy chain in H_α and $s = \sup_{r_\alpha}(C)$, then we distinguish the following two cases.

First case: $s \in C$, then $s \in H_\alpha$.

Second case: $s \notin C$. Then, for every $c \in C$, $r_\alpha(c,s) > \frac{\alpha}{2}$ and $c \neq s$. By our definition $T_s^\alpha \neq \emptyset$. Then, there exists $z \in X$ such that $T(s)(z) = \alpha$. Since $c \in H_\alpha$, there exists $d \in X$ such that $T(c)(d) = \alpha$ and $r_\alpha(c,d) > \frac{\alpha}{2}$. As T is r_α-fuzzy monotone, we get $r_\alpha(d,z) > \frac{\alpha}{2}$. By α-fuzzy transitivity, we obtain $r_\alpha(c,z) > \frac{\alpha}{2}$. As c is a general element of C, then z is a r_α-upper bound of C. On the other hand, we know that $s = \sup_{r_\alpha}(C)$. Hence, $r_\alpha(s,z) > \frac{\alpha}{2}$. From this we deduce that $s \in H_\alpha$. Therefore every nonemty r_α-fuzzy chain in H_α has a r_α-upper bound in H_α. By Lemma 2.4, H_α has a maximal element, say m.

Claim 2. The element m is a maximal α-fuzzy fixed point of T. Indeed, by *Claim 1*, $m \in H_\alpha$. Hence, there exists $y \in X$ such that $T(m)(y) = \alpha$ and $r_\alpha(m,y) > \frac{\alpha}{2}$. On the other hand, by our hypothesis, $T_y^\alpha \neq \emptyset$. Therefore, there exists $t \in X$ such that $T(y)(t) = \alpha$. From r_α-fuzzy monotonicity of T we get $r_\alpha(y,t) > \frac{\alpha}{2}$. So, $y \in H_\alpha$. By *Claim 1*, m is a maximal element of H_α. From this and since $T(m)(y) = \alpha$, $r_\alpha(y,m) > \frac{\alpha}{2}$ and $y \in H_\alpha$, we deduce that we have $y = m$. So, $T(m)(m) = \alpha$. Thus, $m \in \mathcal{F}_T^\alpha$. Now, let $x \in \mathcal{F}_T^\alpha$. Then, $x \in H_\alpha$. So, $\mathcal{F}_T^\alpha \subseteq H_\alpha$. As $m \in \mathcal{F}_T^\alpha$, then m is a maximal element of $\mathcal{F}_T^\alpha$. □

In order to establish the existence of a minimal α-fuzzy fixed, we shall need the following lemma:

Lemma 3.2. *Let (X, r_α) be a r_α-fuzzy order set and s_α be the inverse fuzzy relation of r_α. Then, every r_α-fuzzy monotone multifunction is also s_α-fuzzy monotone.*

Proof. Let $T : X \to [0,1]^X$ be a r_α-fuzzy monotone multifunction. Now, let $x, y \in X$ such that $x \neq y$ and $s_\alpha(x,y) > \frac{\alpha}{2}$. Then, we have $r_\alpha(y,x) > \frac{\alpha}{2}$. Since T is r_α-fuzzy monotone, then for all $a, b \in X$ such that $T(x)(a) = \alpha$ and $T(y)(b) = \alpha$, we get $r_\alpha(b,a) > \frac{\alpha}{2}$. Therfore, we obtain $s_\alpha(a,b) > \frac{\alpha}{2}$. □

By using Lemmas 2.3 and 3.2 and Theorem 3.1, we obtain the following result.

Theorem 3.3. *Let (X, r_α) be a r_α-fuzzy ordered set with the property that every nonempty r_α-fuzzy chain has a r_α-infimum. Let $T : X \to [0,1]^X$ be a r_α-fuzzy monotone multifunction. Assume that there exist $a, b \in X$ such that $T(a)(b) = \alpha$*

and $r_\alpha(b,a) > \frac{\alpha}{2}$. *Then, the set* $\mathcal{F}_T^\alpha$ *of all* α*-fuzzy fixed points of* T *is nonempty and has a minimal element.*

Proof. Let s_α be the inverse fuzzy order relation of r_α. From Lemma 2.3, every nonempty s_α-fuzzy chain has a s_α-supremum. On the other hand, by Lemma 3.2, we know that T is s_α-fuzzy monotone. From this and $s_\alpha(a,b) > \frac{\alpha}{2}$, by Theorem 3.1, we deduce that T has a maximal α-fuzzy fixed point, l say, in (X, s_α). Let $x \in \mathcal{F}_T^\alpha$ such that $r_\alpha(x,l) > \frac{\alpha}{2}$. Then, $s_\alpha(l,x) > \frac{\alpha}{2}$. Since l is a maximal α-fuzzy fixed point of T in (X, s_α), then $l = x$. Therefore, l is a minimal α-fuzzy fixed point of T in (X, r_α). □

4. Greatest and least α-fuzzy fixed points

In this section, we shall establish the existence of the greatest and the least α-fuzzy for α-fuzzy monotone multifunctions. First, we shall prove the following:

Theorem 4.1. *Let* (X, r_α) *be a* r_α*-fuzzy ordered set with the property that every nonempty fuzzy ordered subset of* X *has a* r_α*-supremum. Let* $T : X \to [0,1]^X$ *be a* r_α*-fuzzy monotone multifunction. If there exist* $a, b \in X$ *such that* $T(a)(b) = \alpha$ *and* $r_\alpha(a,b) > \frac{\alpha}{2}$, *then* T *has the greatest* α*-fuzzy fixed point. Moreover, we have*

$$\max(\mathcal{F}_T^\alpha) = \sup_{r_\alpha} \left\{ x \in X : \textit{ there exists } y \in X, T(x)(y) = \alpha \textit{ and } r_\alpha(x,y) > \frac{\alpha}{2} \right\}.$$

Proof. Let P_α be the fuzzy ordered subset defined by

$$P_\alpha = \left\{ x \in X : \text{ there exists } y \in X, T(x)(y) = \alpha \text{ and } r_\alpha(x,y) > \frac{\alpha}{2} \right\}.$$

As $a \in P_\alpha$, then the subset P_α is nonempty. Let $g = \sup_{r_\alpha}(P_\alpha)$.
Claim 1. We have: $g \in P_\alpha$. Indeed, assume on the contrary that $g \notin P_\alpha$. Then for all $x \in P_\alpha$, we have $x \neq g$. As by our definition $T_g^\alpha \neq \emptyset$, then there exists $z \in T_g^\alpha$. Let $x \in P_\alpha$. Hence, there exists $y \in T_x^\alpha$ such that $r_\alpha(x,y) > \frac{\alpha}{2}$. From α-fuzzy monotonicity of T, we obtain $r_\alpha(y,z) > \frac{\alpha}{2}$. By α-fuzzy transitivity, we get $r_\alpha(x,z) > \frac{\alpha}{2}$. As x is a general element of P_α, so z is a r_α-upper bound of P_α. On the other hand; by our hypothesis; we have $g = \sup_{r_\alpha}(P_\alpha)$. Then, $r_\alpha(g,z) > \frac{\alpha}{2}$. Thus, $g \in P_\alpha$. That is a contradiction, and our claim is proved.
Claim 2. We have: $\left\{z \in X : T(g)(z) = \alpha \text{ and } r_\alpha(g,z) > \frac{\alpha}{2}\right\} = \{g\}$. By absurd, suppose that there exists $z \in T_g^\alpha$ such that $r_\alpha(g,z) > \frac{\alpha}{2}$ and $z \neq g$. As T is r_α-fuzzy monotone and $T_z^\alpha \neq \emptyset$, then there exists $l \in T_z^\alpha$ such that $r_\alpha(z,l) > \frac{\alpha}{2}$. Therefore, $z \in P$ and $r_\alpha(z,g) > \frac{\alpha}{2}$. Hence, we get $r_\alpha(z,g) + r_\alpha(g,z) > \alpha$. From this and α-fuzzy antisymmetry, we obtain $g = z$. That is a contradiction with the fact that $z \neq g$ and our Claim is proved.
Claim 3. The element g is the greatest α-fuzzy fixed point of T. Indeed, as $g \in P_\alpha$, then there exists $z \in T_g^\alpha$ such that $r_\alpha(g,z) > \frac{\alpha}{2}$. Then by *Claim 2*, we deduce that $z = g$ and g is a α-fuzzy fixed point of T. On the other hand, let x be an α-fuzzy fixed point of T. So $x \in P_\alpha$. Thus, $\mathcal{F}_T^\alpha \subseteq P_\alpha$. Hence, g is a r_α-upper bound of $\mathcal{F}_T^\alpha$. As $g \in \mathcal{F}_T^\alpha$, therefore, g is the greatest element of $\mathcal{F}_T^\alpha$. □

Combining Lemmas 2.3 and 3.2 and Theorem 4.1, we get the following:

Theorem 4.2. *Let (X, r_α) be a r_α-fuzzy ordered set with the property that every nonempty fuzzy ordered subset of X has a r_α-infimum. Let $T : X \to [0,1]^X$ be a r_α-fuzzy monotone multifunction. Assume that there is $a, b \in X$ such that $T(a)(b) = \alpha$ and $r_\alpha(b,a) > \frac{\alpha}{2}$. Then, T has a least α-fuzzy fixed point. Furthermore, we have*

$$\min(\mathcal{F}_T^\alpha) = \inf_{r_\alpha}\Big\{x \in X :\ \textit{there exists } y \in X, T(x)(y) = \alpha \textit{ and } r_\alpha(y,x) > \frac{\alpha}{2}\Big\}.$$

Proof. Let s_α be the inverse α-fuzzy order of r_α. From Lemma 2.3, every nonempty fuzzy ordered subset of X has an infimum in (X, s_α). By Lemma 3.2, T is s_α-fuzzy monotone. Since $r_\alpha(b,a) > \frac{\alpha}{2}$, then $s_\alpha(a,b) > \frac{\alpha}{2}$. From this and by Theorem 4.1 we deduce that the fuzzy multifunction T has a greatest α-fuzzy fixed point in (X, s_α), m, say. Therefore, m is the least α-fuzzy fixed point of T in (X, r_α). Since m is the greatest α-fuzzy fixed of T in (X, s_α), then by Theorem 4.1, we have

$$m = \sup_{s_\alpha}\Big\{x \in X :\ \text{there exists } y \in X, T(x)(y) = \alpha \text{ and } s_\alpha(x,y) > \frac{\alpha}{2}\Big\}.$$

Therefore, by Lemma 2.3, we conclude that

$$m = \inf_{r_\alpha}\Big\{x \in X :\ \text{there exists } y \in X, T(x)(y) = \alpha \text{ and } r_\alpha(y,x) > \frac{\alpha}{2}\Big\}.$$

□

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