DEGENERATE DIFFUSIVE SEIR MODEL WITH LOGISTIC POPULATION CONTROL

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ABSTRACT. In this paper we analyze the global existence and eventually uniform bound and the existence of periodic solution for a reaction diffusion system with degenerate diffusion arising in modelling the spatial spread of an epidemic disease. We also obtain the existence of the global attractor.

1. Introduction

In this paper we shall be concerned with a degenerate parabolic system of the form

$$
(1)\qquad \begin{cases}
\partial_t U_1 - \Delta U_1^{m_1} = & -\gamma(U_1,U_2,U_3,U_4) + \sum_{i=1}^4 b_{1i}U_i + \delta U_4 - \nu U_1 \\
& -(k_1P+m_1)U_1 + F_1(x,t) = \ f_1(x,t,U_1,U_2,U_3,U_4), \\
\partial_t U_2 - \Delta U_2^{m_2} = & \gamma(U_1,U_2,U_3,U_4) + b_{22}U_2 - (k_2P+m_2+\lambda+\mu)U_2 \\
& +F_2(x,t) = \ f_2(x,t,U_1,U_2,U_3,U_4), \\
\partial_t U_3 - \Delta U_3^{m_3} = & b_{33}U_3 + \lambda\pi U_2 - (k_3P+\alpha+m_3+m+\mu)U_3 \\
& +F_3(x,t) = \ f_3(x,t,U_1,U_2,U_3,U_4), \\
\partial_t U_4 - \Delta U_4^{m_4} = & b_{44}U_4 + (1-\pi)\lambda U_2 + \alpha U_3 + \nu U_1 - \delta U_4 \\
& -(k_4P+m_4)U_4 + F_4(x,t) = \ f_4(x,t,U_1,U_2,U_3,U_4).
\end{cases}
$$

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in $\Omega \times (0, +\infty)$, subject to the initial conditions

$$U_i(x, 0) = U_{i,0}(x) \geq 0, \qquad x \in \Omega; \qquad\qquad i = 1, \ldots, 4. \tag{2}$$

and to the Neumann boundary conditions

$$\frac{\partial U_i^{m_i}}{\partial \eta}(x, t) = 0, \qquad x \in \partial\Omega, \qquad t > 0, \qquad\qquad i = 1, \ldots, 4. \tag{3}$$

Herein, Ω is an open, bounded and connected domain in $\mathbb{R}^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$; Δ is the Laplace operator in $\mathbb{R}^N$. Powers m_i verify $m_i > 1$, $i = 1, \ldots, 4$. Finally P is the total mass of the population $P = \sum_{i=1}^{4} U_i$, and F_i, $i = 1, \ldots, 4$ are nonnegative and continuous function on $\Omega \times (0, +\infty)$.

In the spatially homogeneous case this problem can be reduced to one of the models of propagation of an epidemic disease devised in Kermack and McKendricks [20], namely

$$\begin{cases} S' &= -\gamma S I, \\ I' &= +\gamma S I - \lambda I, \\ R' &= +\lambda I. \end{cases}$$

This basic model served as a starting point for many further developments, both from epidemiological or mathematical point of view see Busenberg and Cooke [5] or Capasso [6] and their references. Thus, system (1) leads to so-called $(S - E - I - R)$ models : $U_1 = S$ is the distribution of susceptible individuals in a given population, $\gamma(S, E, I, R)$ is the incidence term or number of susceptible individuals infected by contact with an infective individual $U_3 = I$ per time unit and becoming exposed $U_2 = E$, while $U_4 = R$ is the density of removed or resistant (immune) individuals. Then $b_{i,j}$ (resp. m_i) is the natural birth-rate (resp. death-rate), λ (resp. α) is the inverse of the duration of the exposed stage (resp. infective stage) or rate at which exposed individuals enter the infective class (resp. infective individuals who do not die from the disease recover), m is the additional mortality due to infection in the infective class, immunity is lost at rate δ, F_i represents an eventually source term and the quadratic term accounts for the damping of growth due to resource limitation of the habitat or environment. The last two parameters are control parameters: first ν is a vaccination rate; next, for a population of animals, as it is considered here as in Anderson et al. [4], Fromont et al. [15], Courchamp et al. [8] or Langlais and Suppo [22], μ is an elimination rate of exposed and infective individuals. Lastly, as it is suggested by the FeLV, a retrovirus of domestic cats (*Felis catus*) see [15], one also introduces a parameter π measuring the proportion of exposed individuals which actually develop the disease after the exposed stage, the remaining proportion $1-\pi$ becoming resistant.

The nonlinear incidence term γ takes various forms as it can be found from the literature; at least two of them are widely used in applications

$$\gamma(S,E,I,R) = \begin{cases} \gamma S I, & [4,\ 6,\ 20], \quad \text{mass action in } [5,\ 6] \text{ or} \\ & \qquad\qquad \text{pseudo-mass action } [19,\ 10]. \\ \gamma \dfrac{SI}{S+E+I+R}, & [8,\ 15,\ 22], \quad \text{proportionate mixing in } [5] \\ & \qquad\qquad \text{or true mass action } [19,\ 10]. \end{cases}$$

We refer to De Jong et al., [19] and Diekmann et al. [10] for a discussion supporting the second one in populations of varying size and Fromont et al. [16] for a specific discussion in the case of a cat population. See Capasso and Serio [7] and Capasso [6] for more general incidence terms.

System (1)–(3) is uniformly parabolic in the region $D = \cap_{i=1}^{4}[U_i \neq 0]$ and degenerate into first order equations on $Q_T \setminus D$. Note that degenerate diffusion is a good approach in modeling slow diffusion of individuals in the spatial spread of an epidemic disease, see Okubo [24].

A mathematical analysis of the model of Kermack and McKendricks for spatially structured populations with linear diffusion, i.e. $m_i = 1,\ i = 1 \ldots 4$, is performed in Webb [28]. Nonlinear but nondegenerate diffusion terms are introduced in Fitzgibbon et al. [14]. Global existence and large time behavior results are derived therein. Homogeneous Neumann boundary conditions correspond to isolated populations.

A comprehensive analysis of generic $(S-E-I-R)$ models with linear diffusion is initiated in Fitzgibbon and Langlais [12] and Fitzgibbon et al. [13]. These models include a logistic effect on the demography, yielding $L^1(\Omega)$ a priori estimates on solutions independent of the initial data for large time; this allows to use a bootstrapping argument to show global existence and exhibit a global attractor in $(C(\overline{\Omega}))^4$.

For degenerate reaction-diffusion equations, the case of mass action incidence was studied by Aliziane and Moulay [3] and they established the long time behavior of the solution of the SIS model, Aliziane and Langlais [2] studied the SEIR model without logistic effect on the demography and they established global existence result of the solution and the long time behavior of the solution. Finally Hadjadj et al. [18] studied the case where the source term depends on gradient of solution, they resolved the problem of existence of globally bounded weak solutions or blow-up, depending on the relations between the parameters that appear in the problem.

This paper is organized as follows: in Section 2 notion of a weak solution is introduced and we state our mean results, in Section 3 we will construct our solution as a limit of solutions of quasilinear and nondegenerate problems depending on a parameter ε, derive uniform a priori estimates on these solutions, and prove existence, uniqueness and regularity results in Section 4. In Section 5 we prove the existence of periodic solution of (1)–(3) under periodic assumption on F. Finally in the last section we obtain the existence of a global attractor.

2. Main results

2.1. Basic assumptions and notations

Herein, Ω is an open, bounded and connected domain of the N-dimensional Euclidian space $\mathbb{R}^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$, a $(N-1)$-dimensional manifold so that locally Ω lies on one side of $\partial\Omega$; $x = (x_1, \ldots, x_N)$ is the generic element of $\mathbb{R}^N$. Next we shall denote the gradient with respect to x by ∇ and the Laplace operator in $\mathbb{R}^N$ by Δ, $\operatorname{sign}_\varepsilon$ is a smooth approximation of the function signum (sign), finally if r is a real number then we set $r^+ = \sup(r, 0)$, $r^- = \sup(-r, 0)$.

Then we set $\Omega \times (0,T) = Q_T$ and for $0 \le \tau < T$, $\Omega \times (\tau, T) = Q_{\tau,T}$. The norm in $L^p(\Omega)$ is $\| \ \|_{p,\Omega}$ and the norm in $L^p(Q_{\tau,T})$ is $\| \ \|_{p,Q_{\tau,T}}$ for $1 \le p \le +\infty$.

Next we shall assume throughout this paper

(H0) $U_{i,0} \in C(\bar{\Omega})$, $\quad U_{i,0}(x) \ge 0,\ x \in \Omega, \qquad i = 1, \dots, 4.$
(H1) Powers m_i verify $m_i > 1,\ i = 1, \dots, 4.$
(H2) $\mu, \alpha, \nu, m, \lambda, \pi, b_{ii}, b_{1i}, k_i,$
$i = 1, \dots, 4$ are nonnegative constants, $k_i > 0$, $i = 1, \dots, 4$ and $0 \le \pi \le 1$.
(H3) $\gamma : \mathbb{R}_+^4 \longrightarrow \mathbb{R}_+$ is a locally lipschitz continuous function with polynomial growth and $\gamma(0, U_2, U_3, U_4) = 0$ on $\mathbb{R}_+^3$.
(H4) There exists nonnegative constants C_1, C_2 and $0 \le r \le 1$ such that

$$\gamma(U_1, U_2, U_3, U_4) \le (C_1 + C_2 \sum_{i=1}^{4} U_i^r) \text{ on } \mathbb{R}_+^4.$$

(H5) F_i, $i = 1, \dots, 4$ are nonnegative continuous and bounded function on $\Omega \times (0, +\infty)$.

Remark. The assumption $\gamma(0, U_2, U_3, U_4) = 0$ is a natural assumption for our motivating problem: no new exposed individuals when there is no susceptible ones. $(H4)$ removes mass action incidence terms.

2.2. Main results

System (1) is degenerate: when $U_i = 0$ the equation for U_i degenerates into first order equation. Hence classical solutions cannot be expected for Problem (1)–(3). A suitable notion of generalized solutions is required. We adopt the notion of weak solution introduced in Oleinik et al. [25].

Definition 2.1. A quadruple (U_1, U_2, U_3, U_4) of nonnegative and continuous functions $U_i : \Omega \times [0, +\infty) \to [0, +\infty)$, $i = 1, \ldots, 4$, is a weak solution of Problem (1)–(3) in Q_T, $T > 0$ if for each $i = 1, \ldots, 4$ and for each $\varphi_i \in C^1(\bar{Q}_T)$, such that $\dfrac{\partial \varphi_i}{\partial \eta} = 0$ on $\partial\Omega \times (0, T)$.

(i) ∇U^{m_i} exists in the sense of distribution and $\nabla U_i^{m_i} \in L^2(Q_T)$,
(ii) U_i verifies the identity

$$\begin{aligned}(4)\qquad &\int_\Omega U_i(x,T)\varphi_i(x,T)dx + \int_{Q_T} \nabla U_i^{m_i} \nabla\varphi_i(x,t)dxdt \\ &= \int_{Q_T} (\partial_t \varphi_i U_i - f_i\varphi_i)(x,t)dxdt + \int_\Omega U_{i,0}(x)\varphi_i(x,0)dx,\end{aligned}$$

We are now ready to state our first result.

Theorem 2.2. *For each quadruple of continuous nonnegative initial functions* $(U_{1,0}, U_{2,0}, U_{3,0}, U_{4,0})$ *there exists a unique weak solution* (U_1, U_2, U_3, U_4) *of Problem* (1) – (3) *on* Q_∞

i) $U_{i,0} \in C((0,+\infty); \bar{\Omega}) \cap L^\infty(Q_\infty)$, *and* $U_i^{m_i} \in H^1(Q_{\tau,T})$ *for all* , $0 < \tau < T$, $i = 1, \ldots, 4$.
ii) *There exists a nonnegative constant* K *such that*

$$(5)\qquad \int_\Omega \sum_{i=1}^4 |U_{1,i} - U_{2,i}|(x,t)dx \le (1 + Kte^{Kt}) \int_\Omega |U_{1,i,0} - U_{2,i,0}|(x)dx,$$

for all $t > 0$, *where* $U_{j,i}$ *is solution of* (1)–(3) *with initial data* $U_{j,i,0}$.

The proof is found in Section 4.
Now we look at the existence of periodic nonnegative solution of (1).

Theorem 2.3. *Assume*

$$(HP) \textit{ There exists a positive constant } T^* \textit{ so that } F_i(x,t+T^*) = F_i(x,t).$$

Then there exists a solution (U_1, U_2, U_3, U_4) *to* (1)–(3) *so that for* $t \geq 0$, $x \in \Omega$, *we have*

$$U_i(x,t+T^*) = U_i(x,t),\ i = 1,\dots,4.$$

The proof is found in Section 5.

3. Auxiliary problem and a priori estimates

In this section we consider an auxiliary problem depending on a small parameter ε, with $0 < \varepsilon \leq 1$. Namely let us introduce in $\Omega \times (0,+\infty)$ the quasilinear nondegenerate initial and boundary value problem

$$(6)\qquad \begin{cases} \partial_t U_1 - \Delta d_1(U_1) = & -\gamma((U_1-\varepsilon)^+, U_2, U_3, U_4)) + \sum_{i=1}^{4} b_{1i}(U_i-\varepsilon) + \delta(U_4-\varepsilon) \\ & -\nu(U_1-\varepsilon) - (k_1(P-4\varepsilon)+m_1)(U_1-\varepsilon) + F_1(x,t), \\ \partial_t U_2 - \Delta d_2(U_2) = & \gamma((U_1-\varepsilon)^+, U_2, U_3, U_4) + b_{21}(U_2-\varepsilon) \\ & -(k_2(P-4\varepsilon)+m_2+\lambda+\mu)(U_2-\varepsilon) + F_2(x,t), \\ \partial_t U_3 - \Delta d_3(U_3) = & b_{31}(U_3-\varepsilon) + \lambda\pi(U_2-\varepsilon) \\ & -(k_3(P-4\varepsilon)+\alpha+m_3+\mu)(U_3-\varepsilon) + F_3(x,t), \\ \partial_t U_4 - \Delta d_3(U_4) = & b_{41}(U_4-\varepsilon) + (1-\pi)\lambda(U_2-\varepsilon) + \alpha(U_3-\varepsilon) + \nu(U_1-\varepsilon) \\ & -\delta(U_4-\varepsilon) - (k_4(P-4\varepsilon)+m_4)(U_4-\varepsilon) + F_4(x,t). \end{cases}$$

$$(7)\qquad \begin{cases} U_{i,\epsilon}(x,0) = U_{i,0,\epsilon}(x) \geq 0, \quad x \in \Omega; \\ \dfrac{\partial d_i(U_{i,\varepsilon})}{\partial \eta}(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \end{cases} \qquad i = 1,\dots,4.$$

Herein $(r)^+$ is the nonnegative part of the real number r; for each $i = 1, \dots, 4$
$d_i : \mathbb{R} \longrightarrow (\frac{\varepsilon}{2}, +\infty)$ is a smooth and increasing functions with

$$d_i(u) = u^{m_i}, \quad \varepsilon \le u; \tag{8}$$

$(U_{i,0,\varepsilon})_{i=1,\dots,4}$ is a quadruple of smooth functions over $\bar{\Omega}$ such that

$$\begin{cases} U_{i,0,\varepsilon}(x) \ge \varepsilon, \quad x \in \Omega, \quad 0 < \varepsilon \le 1; \\ \displaystyle\int_\Omega (U_{i,0,\varepsilon}(x) - \varepsilon)dx = \int_\Omega U_{i,0}(x)dx \qquad i = 1, \dots, 4; \\ U_{i,0,\varepsilon} \longrightarrow U_{i,0} \text{ in } C(\bar{\Omega}), \quad \text{as } \varepsilon \longrightarrow 0; \end{cases} \tag{9}$$

we refer to [1] for a construction of such a set of initial data. From standard results, [21] or [26], local existence and uniqueness of a quadruple $(U_{1,\varepsilon}, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})$, a classical solution of (6)–(7) in some maximal interval $[0, T_{\max,\varepsilon})$ is granted.

Looking at the equation for $U_{i,\varepsilon}$ it is checked that $([\varepsilon, \ +\infty[)^4$ is an invariant region (see [26]), thus $0 < \epsilon \le U_{i,\varepsilon}(x,t), \ x \in \Omega, \ 0 < t < T_{\max,\varepsilon}$. As a consequence $U_{i,\varepsilon}$ is the solution of the initial and boundary value problem

$$\begin{cases} \partial_t U_1 - \Delta U_1^{m_1} = & -\gamma((U_1 - \varepsilon), U_2, U_3, U_4)) + \displaystyle\sum_{i=1}^{4} b_{1i} U_i + \delta(U_4 - \varepsilon) \\ & -\nu(U_1 - \varepsilon) - (k_1 P + m_1)(U_1 - \varepsilon) + F_1(x,t), \\ \partial_t U_2 - \Delta U_2^{m_2} = & \gamma((U_1 - \varepsilon), U_2, U_3, U_4) + b_{21}(U_2 - \varepsilon) \\ & -(k_2 P + m_2 + \lambda + \mu)(U_2 - \varepsilon) + F_2(x,t), \\ \partial_t U_3 - \Delta U_3^{m_3} = & b_{31}(U_3 - \varepsilon) + \lambda\pi(U_2 - \varepsilon) - (k_3 P + \alpha + m_3 + \mu)(U_3 - \varepsilon) \\ & +F_3(x,t), \\ \partial_t U_4 - \Delta U_4^{m_4} = & b_{41}(U_4 - \varepsilon) + (1 - \pi)\lambda(U_2 - \varepsilon) + \alpha(U_3 - \varepsilon) + \nu(U_1 - \varepsilon) \\ & -\delta(U_4 - \varepsilon) - (k_4 P + m_4)(U_4 - \varepsilon) + F_4(x,t). \end{cases} \tag{10}$$

in $\Omega \times (0, +\infty)$, together with (7).

Lemma 3.1. *The solution of* (10) *subject to* (7) *is global (that is $T_{\max,\varepsilon} = \infty$) and there exist a constant C independent of ε, $0 < \varepsilon < 1$, such that*

$$\|u_{i\varepsilon}(t,.)\|_{L^\infty} \le C(\|u_0\|_{L^\infty}), \quad \text{for all } t > 0, \ i = 1, \dots, 4. \tag{11}$$

Moreover, there exists a positive function F not depending on ε and on u_0 such that

$$\|u_{i\varepsilon}(t,.)\|_{L^\infty} \le F(\xi) \qquad \text{for all } t \ge \xi > 0, \tag{12}$$

Proof. Let us multiply each equation in $U_{i,\varepsilon}$ by $U_{i,\varepsilon}^{p-1}$,integrate over Ω and use $(H4)$ we get

$$\begin{cases} \frac{1}{p}\frac{d}{dt}\int_\Omega U_{1,\varepsilon}^p dx \le \int_\Omega \sum_{i=1}^{4} b_{1i} U_{i,\varepsilon} U_{1,\varepsilon}^{p-1} + \delta U_{4,\varepsilon} U_{1,\varepsilon}^{p-1} - \nu U_{1,\varepsilon}^p dx \\ \qquad - \int_\Omega (k_1 P_\varepsilon + m_1) U_{1,\varepsilon}^p + F_1(x,t) U_{1,\varepsilon}^{p-1} dx, \\ \frac{1}{p}\frac{d}{dt}\int_\Omega U_{2,\varepsilon}^p dx \le \int_\Omega (C_1 + C_2 \sum_{i=1}^{4} U_{i,\varepsilon}^r) U_{2,\varepsilon}^{p-1} + b_{21} U_{2,\varepsilon}^p dx \\ \qquad - \int_\Omega (k_2 P_\varepsilon + m_2 + \lambda + \mu) U_{2,\varepsilon}^p + F_2(x,t) U_{2,\varepsilon}^{p-1} dx, \\ \frac{1}{p}\frac{d}{dt}\int_\Omega U_{3,\varepsilon}^p dx \le \int_\Omega b_{31} U_{3,\varepsilon}^p + \lambda\pi U_{2,\varepsilon} U_{3,\varepsilon}^{p-1} - (k_3 P_\varepsilon + \alpha + m_3 + \mu) U_{3,\varepsilon}^p dx \\ \qquad + \int_\Omega F_3(x,t) U_{3,\varepsilon}^{p-1} dx, \\ \frac{1}{p}\frac{d}{dt}\int_\Omega U_{4,\varepsilon}^p dx \le \int_\Omega b_{41} U_{4,\varepsilon}^p + (1-\pi)\lambda U_{2,\varepsilon} U_{4,\varepsilon}^{p-1} + \alpha U_{3,\varepsilon} U_{4,\varepsilon}^{p-1} - \delta U_{4,\varepsilon}^p dx \\ \qquad + \int_\Omega \nu U_{1,\varepsilon} U_{4,\varepsilon}^{p-1} - (k_4 P_\varepsilon + m_4) U_{4,\varepsilon}^p + F_4(x,t) U_{4,\varepsilon}^{p-1} dx. \end{cases} \tag{13}$$

Now by Hölder and Jensen, Young inequalities one can deduce

$$\int_\Omega U_{i,\varepsilon}^r U_{2,\varepsilon}^{p-1}(x,t)dx \le \left\{(1-r)|\Omega|^{\frac{1}{p}} + r\|U_{i,\varepsilon}(\cdot,t)\|_{p,\Omega}\right\}\|U_{2,\varepsilon}(\cdot,t)\|_{p,\Omega}^{p-1}.$$

$$\int_\Omega U_{i,\varepsilon}^{p+1} \ge \left(\frac{1}{|\Omega|}\right)^{\frac{1}{p}}\left(\int_\Omega U_{i,\varepsilon}^p\right)^{\frac{p+1}{p}}$$

and

$$(14)\qquad \begin{cases} \frac{d}{dt}\|U_{1,\varepsilon}(\cdot,t)\|_{p,\Omega} \le \sum_{i=1}^4 b_{1i}\|U_{i,\varepsilon}(\cdot,t)\|_{p,\Omega} - (\nu+m_1)\|U_{1,\varepsilon}(\cdot,t)\|_{p,\Omega} \\ \qquad\qquad +\delta\|U_{4,\varepsilon}(\cdot,t)\|_{p,\Omega} + \|F_1(\cdot,t)\|_{p,\Omega} - k_1|\Omega|^{\frac{-1}{p}}\|U_{1,\varepsilon}(\cdot,t)\|_{p,\Omega}^2 \\ \frac{d}{dt}\|U_{2,\varepsilon}(\cdot,t)\|_{p,\Omega} \le rC_2\sum_{i=1}^4 \|U_{i,\varepsilon}(\cdot,t)\|_{p,\Omega} + (b_{22}-m_2-\lambda-\mu)\|U_{2,\varepsilon}(\cdot,t)\|_{p,\Omega} \\ \qquad\qquad +(C_1+(1-r)C_2)|\Omega|^{\frac{1}{p}} + \|F_2(\cdot,t)\|_{p,\Omega} \\ \qquad\qquad -k_1|\Omega|^{\frac{-1}{p}}\|U_{2,\varepsilon}(\cdot,t)\|_{p,\Omega}^2 \\ \frac{d}{dt}\|U_{3,\varepsilon}(\cdot,t)\|_{p,\Omega} \le (b_{33}-\alpha-m_3)\|U_{3,\varepsilon}(\cdot,t)\|_{p,\Omega} + \lambda\pi\|U_{2,\varepsilon}(\cdot,t)\|_{p,\Omega} \\ \qquad\qquad +\|F_3(\cdot,t)\|_{p,\Omega} - k_3|\Omega|^{\frac{-1}{p}}\|U_{3,\varepsilon}(\cdot,t)\|_{p,\Omega}^2 \\ \frac{d}{dt}\|U_{4,\varepsilon}(\cdot,t)\|_{p,\Omega} \le (b_{44}-\delta-m_4)\|U_{4,\varepsilon}(\cdot,t)\|_{p,\Omega} + \nu\|U_{1,\varepsilon}(\cdot,t)\|_{p,\Omega} \\ \qquad\qquad +\alpha\|U_{3,\varepsilon}(\cdot,t)\|_{p,\Omega} + (1-\pi)\lambda\|U_{2,\varepsilon}(\cdot,t)\|_{p,\Omega} \\ \qquad\qquad +\|F_4(\cdot,t)\|_{p,\Omega} - k_4|\Omega|^{\frac{-1}{p}}\|U_{4,\varepsilon}(\cdot,t)\|_{p,\Omega}^2. \end{cases}$$

Adding these inequalities and use Jensen's and Young inequalities another time to get

$$\frac{d}{dt}\sum_{i=1}^{4}\|U_{4,\varepsilon}(\cdot,t)\|_{p,\Omega} \leq B_{0,p} - B_{1,p}\left(\sum_{i=1}^{4}\|U_{i,\varepsilon}(\cdot,t)\|_{p,\Omega}\right)^2, \tag{15}$$

with

$$\begin{aligned} B_{0,p} &= (C_1 + (1-r)C_2)|\Omega|^{\frac{1}{p}} + \sum_{i=1}^{4}\sup{}_t\|F_4(\cdot,t)\|_{p,\Omega} \\ &\quad + \ 2\frac{\left(\sum_{i=1}^{4}(b_{1,i}+b_{ii}-m_i)+rC_2\right)^2}{\min_i(k_i)}|\Omega|^{\frac{1}{p}}, \\ B_{1,p} &= \frac{\min_i(k_i)|\Omega|^{\frac{-1}{p}}}{8}. \end{aligned}$$

Finally let $y(t)=\sum_{i=1}^{4}\|U_{i,\varepsilon}(\cdot,t)\|_{p,\Omega}$, and $B_0 = \lim_{p\to+\infty} B_{0,p}$ and $B_1 = \lim_{p\to+\infty} B_{1,p}$ then $y(t)$ then $y(t)$ verifies

$$y'(t) \leq \quad B_{0,p} - B_{1,p}y^2,$$

and by standard argument see [11, Lemma 1] we get

$$y(t) \leq \left(\frac{B_{0,p}}{B_{1,p}}\right)^{\frac{1}{2}} + \frac{B_{1,p}}{t}. \tag{16}$$

and

$$y(t) \leq \max\left(y(0), \left(\frac{B_{0,p}}{B_{1,p}}\right)^{\frac{1}{2}}\right).$$

Going back to the definition of $y(t)$ one can find

$$\sum_{i=1}^{4} \|U_{i,\varepsilon}(\cdot,t)\|_{p,\Omega} \leq \max\left(\sum_{i=1}^{4} \|U_{i,0,\varepsilon}\|_{p,\Omega}, \left(\frac{B_{0,p}}{B_{1,p}}\right)^{\frac{1}{2}}\right).$$

To conclude, one observes that $U_{i,\varepsilon}$ being continuous on $\bar{\Omega} \times [0, T_{\max,\varepsilon})$ it follows

$$\lim_{p\to+\infty} \|U_{i,\varepsilon}(\cdot,t)\|_{p,\Omega} = \|U_{i,\varepsilon}(\cdot,t)\|_{\infty,\Omega}.$$

Hence

$$\sum_{i=1}^{4} \|U_{i,\varepsilon}(\cdot,t)\|_{\infty,\Omega} \leq \max\left(\sum_{i=1}^{4} \|U_{i,0,\varepsilon}\|_{\infty,\Omega}, \left(\frac{B_0}{B_1}\right)^{\frac{1}{2}}\right), \tag{17}$$

and $T_{\max,\varepsilon} = +\infty$. □

Remark. Estimation (16) implies that for each $\eta > 0$ there exists a constant $C(\eta)$ independent on initial data such that

$$\sum_{i=1}^{4} \|U_{i,\varepsilon}(\cdot,t)\|_{\infty,\Omega} \leq C(\eta), \text{ for all } t \geq \eta > 0 \tag{18}$$

Lemma 3.2. *For all $T > 0$ there exists a nondecreasing function C_1 independent of ε, $0 < \varepsilon < 1$ such that*

$$\int_{Q_T} U_{i,\varepsilon}^2(x,T)dx + \int_{Q_T} |\nabla U_{i,\varepsilon}^{m_i}|^2(x,t)dx\,dt \leq C_1(T), \quad T > 0, \;\; i = 1, \dots, 4; \tag{19}$$

Proof. The first term is bounded as an immediate consequence of Lemma 3.1 because $U_{i,\varepsilon}$ is uniformly bounded from below independently of ε. The boundeness of the second term is obtained by multiplying the equation for $U_{i,\varepsilon}$ by $U_{i,\varepsilon}^{m_i}$ and integrating over $\Omega \times (0,T)$ and use the same artifices as in the proof of Lemma 3.1. □

Lemma 3.3. *For all $T > 0$ there exists a nondecreasing function C_1 independent of ε, $0 < \varepsilon < 1$ such that*

$$\int_{Q_T} \left(\partial_t U_{i,\varepsilon}^{\frac{m_i+1}{2}}\right)^2 (x,t)dx\,dt + \int_\Omega |\nabla U_{i,\varepsilon}^{m_i}|^2(x,T)dx \le C_1(T), \qquad T > 0, \tag{20}$$

$i = 1, \dots, 4.$

Proof. Let us multiply by $\partial_t U_{i,\varepsilon}^{m_i}$ the equation for $U_{i,\varepsilon}$ and integrate over $\Omega \times (\tau, T)$, $0 < \tau < T$; then one finds

$$\begin{aligned}\left(\frac{2}{m_i+1}\right)^2 \int_{Q_{\tau,T}} \left(\partial_t U_{i,\varepsilon}^{\frac{m_i+1}{2}}\right)^2 (x,s)dx\,ds + \frac{1}{2}\|\nabla U_{i,\varepsilon}^{m_i}(.,T)\|_{2,\Omega}^2 \\ \le \int_{Q_{\tau,T}} f_i\left(U_{1,\varepsilon}, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon}\right) \partial_t U_{i,\varepsilon}^{m_i}(x,s)dx\,ds + \frac{1}{2}\|\nabla U_{1,\varepsilon}^{m_i}(.,\tau)\|_{2,\Omega}^2.\end{aligned}$$

By Lemma 3.1 f_i, $i = 1, \dots, 4$ are bounded and we can use Young's inequality to get

$$\begin{aligned}\int_{Q_{\tau,T}} f_i(U_{1,\varepsilon}, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})\partial_t U_{i,\varepsilon}^{m_i}(x,s)dx\,ds + \frac{1}{2}\|\nabla U_{i,\varepsilon}^{m_i}(.,T)\|_{2,\Omega}^2 \\ \le \frac{2}{(m_i+1)^2}\int_{Q_{\tau,T}} (\partial_t U_{i,\varepsilon}^{\frac{m_i+1}{2}})^2(x,s)dx\,ds + \frac{Tm_i^2}{2}|\Omega|\|f_i\|_\infty \|U_{i,\varepsilon}^{m_i}\|_\infty.\end{aligned}$$

Reporting this inequality into the previous and integrating in τ over $(0,T)$, and Lemma 3.3 follows by Lemma 3.2. $\square$

4. Existence and continuous dependence on data

In this section we supply a quick proof of Theorem 2.2.

4.1. Existence

Let us fix $T > 0$. From the estimates established in the previous section one has: for each $i = 1, \dots, 4$ $(U_{i,\varepsilon})_{0<\varepsilon\leq 1}$ and $(\nabla U_{i,\varepsilon}^{m_i})_{0<\varepsilon\leq 1}$ are respectively bounded in $L^2(Q_T)$ and $(L^2(Q_T))^N$. Then there exists two sequences which one still denotes $(U_{i,\varepsilon})_{0<\varepsilon\leq 1}$ and $(\nabla U_{i,\varepsilon}^{m_i})_{0<\varepsilon\leq 1}$ such that for $i = 1, \dots, 4$ as $\varepsilon \to 0$: $(U_{i,\varepsilon})_{0<\varepsilon\leq 1}$ is weakly convergent to some U_i in $L^2(Q_T)$ and $(\nabla U_{i,\varepsilon}^{m_i})_{0<\varepsilon\leq 1}$ is weakly convergent to some V_i in $(L^2(Q_T))^N$.

On the other hand $(U_{i,\epsilon})_{0<\epsilon\leq 1}$ is bounded in $L^\infty(Q_T)$; using a weak formulation of the equation for $U_{i,\epsilon}$ one can invoke the results in Di Benedetto [9] to get: $(U_{i,\varepsilon})_{0<\varepsilon\leq 1}$ is a relatively compact subset of $C(\overline{\Omega} \times (0,T])$. It follows that actually $(U_{i,\varepsilon})_{0<\epsilon\leq 1}$ is convergent to U_i in $C(\overline{\Omega} \times (0,T])$ and $(U_{i,\varepsilon}^{m_i})_{0<\varepsilon\leq 1}$ is convergent to $U_i^{m_i}$ in $C(\overline{\Omega} \times (0,T])$.

As a first consequence one has: $V_i = \nabla U_i^{m_i}$; next one also has:

$$\gamma(U_{1,\varepsilon} - \varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon}) \;\to\; \gamma(U_1, U_2, U_3, U_4) \;\text{ in }\; C(\overline{\Omega} \times (0,T]) \;\text{ as }\; \epsilon \;\to 0.$$

From standard arguments one may conclude that the quadruple (U_1, U_2, U_3, U_4) is a desirable weak solution. Note that all estimates in Lemmas 3.1–3.3 still valid for (U_1, U_2, U_3, U_4) by passing to limit as ε goes to zero.

The regularity results for $\nabla U_i^{m_i}$ and $\partial_t U_i^{m_i}$ follow from the a priori estimates in Lemma 3.2 and Lemma 3.3.

4.2. Uniqueness and continuous dependence on data

Assume there exists two quadruples $(U_{j,1}, U_{j,2}, U_{j,3}, U_{j,4})_{j=1,2}$, both weak solutions of Problem (1)–(3) with initial data $(U_{j,1,0}, U_{j,2,0}, U_{j,3,0}, U_{j,4,0})_{j=1,2}$. They verify the integral identity, for $i = 1, \ldots, 4$

$$
\begin{aligned}
&\int_\Omega (U_{1,i} - U_{2,i})(x,T)\varphi_i(x,T)dx + \int_{Q_T} \nabla(U_{1,i}^{m_i} - U_{2,i}^{m_i})\nabla\varphi_i(x,t)dxdt \\
&= \int_\Omega (U_{1,i,0} - U_{2,i,0})(x)\varphi_i(x,0)dx + \int_{Q_T} \partial_t\varphi_i(U_{1,i} - U_{2,i})(x,t)dxdt \\
&- \int_{Q_T} [(f_i(U_{1,1}, U_{1,2}, U_{1,3}, U_{1,4}) - f_i(U_{2,1}, U_{2,2}, U_{2,3}, U_{2,4}))\varphi_i]dxdt
\end{aligned}
\tag{21}
$$

for every $\varphi_i \in C^1(\bar{Q}_T)$, such that $\dfrac{\partial\varphi_i}{\partial\eta} = 0$ on $\partial\Omega \times (0,T)$ and $\varphi_i > 0$.

We follow an idea of [23] and introduce a function ψ_i as follows

$$
\psi_i(x,t) = \begin{cases} \dfrac{U_{1,i}^{m_i} - U_{2,i}^{m_i}}{U_{1,i} - U_{2,i}} & \text{if } U_{1,i} \neq U_{2,i}, \\ 0 & \text{otherwise.} \end{cases} \quad i = 1, \ldots, 4.
$$

Let us consider a sequence of smooth functions $(\psi_{i,\varepsilon})_{\varepsilon\geq 0}$ such that $\psi_{i,\varepsilon} \geq \varepsilon$, $\psi_{i,\varepsilon}$ is uniformly bounded in $L^\infty(Q_T)$ and

$$
\lim_{\varepsilon\to 0} \|(\psi_{i,\varepsilon} - \psi_i)/\sqrt{\psi_{i,\varepsilon}}\|_{L^2(Q_T)} = 0.
$$

For any $0 < \varepsilon \leq 1$ let us introduce the adjoint nondegenerate boundary value problem

$$
\begin{cases} \partial_t\varphi_i + \psi_{i,\varepsilon}\Delta\varphi_i = 0 & \text{in } \Omega \times (0,T) \\ \dfrac{\partial\varphi_i}{\partial\eta}(x,t) = 0 & \text{in } \partial\Omega \times (0,T) \quad i = 1, \ldots, 4. \\ \varphi_i(x,T) = \chi_i & \text{in } \Omega \end{cases}
\tag{22}
$$

For any smooth χ_i with $0 \le \chi_i(x,t) \le 1,\ i = 1, \dots, 4$, any $0 < \varepsilon \le 1$ this problem has unique classical solution $\varphi_{i,\varepsilon}$ such that see [23]

$$0 \le \varphi_{i,\varepsilon}(x,t) \le 1$$

$$\int_{Q_T} \psi_{i,\varepsilon}(\Delta\varphi_{i,\varepsilon})^2 dxdt \le K_1,$$

If in (21) we replace φ_i by $\varphi_{i,\varepsilon}$which is the solution of problem (22) with $\chi_i = \operatorname{sign}((U_i - V_i)^+)$ we obtain. $\chi_1(x) = \chi_{1,\varepsilon}(x) = \operatorname{sign}_\varepsilon^+(S_1 - S_2)(x,T)$

$$\begin{aligned}
&\int_\Omega (U_{1,i} - U_{2,i})^+(x,T)\varphi_{i,\varepsilon}(x,T)dx + \int_{Q_T} (\psi_i - \psi_{i,\varepsilon})(U_{1,i} - U_{2,i})\Delta\varphi_{i,\varepsilon} dxdt \\
&= \int_{Q_T} (f_i(U_{1,1}, U_{1,2}, U_{1,3}, U_{1,4}) - f_i(U_{2,1}, U_{2,2}, U_{2,3}, U_{2,4}))\varphi_{i,\epsilon} dxdt \\
&+ \int_\Omega (U_{1,i,0} - U_{2,i,0})(x)\varphi_{i,\varepsilon}(x,0)dx
\end{aligned}$$

Using the local lipschitz continuity of f_i and the properties of $\psi_{i,\epsilon}$ and $\varphi_{i,\varepsilon}$ we deduce by letting $\epsilon \to 0$

$$\int_\Omega (U_{1,i} - U_{2,i})^+(x,T)dx \le K \int_{Q_T} \sum_{i=1}^{4} |U_{1,i} - U_{2,i}| + \int_\Omega |U_{1,i,0} - U_{2,i,0}|(x)dx$$

In a similar fashion we establish an analogous inequality for $(U_i - V_i)^-$ and deduce by Gronwall's Lemma.

$$\int_\Omega \sum_{i=1}^{4} |U_{1,i} - U_{2,i}|(x,T)dx \le (1 + KTe^{KT}) \int_\Omega |U_{1,i,0} - U_{2,i,0}|(x)dx. \tag{23}$$

Uniqueness is immediately deduced.

5. Existence of Periodic solution

We need the following periodicity assumption upon our model:

(HP) There exists a T^* so that $F_i(x, t + T^*) = F_i(x, t)$.

By periodic solution with period T^*, we mean a weak solution of (1) satisfying (3) so that for all $t \geq 0$, $x \in \Omega$, $U_i(x, t + T^*) = U_i(x, t)$, $i = 1, \ldots, 4$.

In order to proof Theorem 2.3 we need the following variant of the Schauder's Fixed Point Theorem which is given in [17].

Theorem 5.1. *(Schauder's Fixed Point). Let X be a Banach space, $K \subset X$ be a convex set in X and $J : K \longrightarrow K$ be a continuous mapping such that the image $J(K)$ is precompact. Then J has a fixed point in K.*

In the present context, let $X = (L^2(\Omega))^4$ and

$$K = \left\{ (U_1, U_2, U_3, U_4); U_i \in L^2(\Omega),\ U_i \geq 0 \text{ s. t. } \sum_{i=1}^{4} U_i(x) \leq B \text{ a. e. } x \in \Omega. \right\}$$

with $B = \left(\dfrac{B_0}{B_1} \right)^{\frac{1}{2}}$, found in the proof of Lemma 3.1, K is a convex set in X.

For $U_0 = (U_{1,0}, U_{2,0}, U_{3,0}, U_{4,0}) \in X$, let $J(U_0) = U(\cdot, T^*)$, with U solution of problem (1)–(3). Then by lemma 3.2 and (17), we have $J(K) \subset K$ and by (23) there exists a constant C dependent only on B, k', T^* and , $|\Omega|$ with k' is the lipschitz constant of the vector field $(f_i)_i$ such that

$$\|J(U) - J(U')\|_X \leq C \|U - U'\|_X^{\frac{1}{4}} \text{ for all } U, U' \in K$$

and J is continuous from K into K.

Now Let $(U_n)_n$ be a bounded sequence in K, then by Lemma 3.2 and Lemma 3.3 for each $i = 1, \ldots, 4$, $J(U_n)_i{}^{m_i}$ is bounded in $H^1(\Omega)$, then there exists a sequence which still denoted U_n such $J(U_n)_i{}^{m_i}$ converges in $L^2(\Omega)$ and almost every where in Ω, finally thanks to Lebesgue dominate convergence theorem to deduce with (17) that $J(U_n)$ converges in X, and $J(K)$ is precompact. By schauder's fixed point theorem there exists $U^* \in K$ such that $J(U^*) = U^*$.

Now let $U(t,x)$ be the solution of of problem (1) – (3) with $U_0 = U^*$ and set $V(t,x) = U(t+T^*, x)$ then U, V are solutions of problem (1)–(3) with same initial datas then by uniqueness $U(t,x) = U(t+T^*, x)$ and U is the desired periodic solution of (1).

6. Global attractor

Let us consider the following problem

$$\begin{cases} \partial_t U_i - \Delta\left(|U_i|^{m_i} \operatorname{sign} U_i\right) = f_i(x,t,U_1,U_2,U_3,U_4), & (x,t) \in \Omega \times (0,+\infty) \\ \dfrac{\partial\left(|U_i|^{m_i} \operatorname{sign} U_i\right)}{\partial \eta}(x,t) = 0, & x \in \partial\Omega,\ t>0,\ i=1,\ldots,4. \\ U_i(x,0) = U_{i,0}(x), & x \in \Omega;\ i=1,\ldots,4. \end{cases} \tag{24}$$

Problem (24) admits a unique weak solution verifying (17), (18), (19), (20) and (23). The construction of the solution is obtained in the same manner as below with slight modification. See [18] for more details. This yields that the PDE system (24) defines a nonlinear semigroup $\{S(t)\}$ as follows $S(t)(U_{1,0}, U_{2,0}, U_{3,0}, U_{4,0}) = (U_1(t), U_2(t), U_3(t), U_4(t))$ and $S(0) = I$ the identity map. we have the a continuous dynamical system on the set of bounded vector valued function. [27, Theorem 1.1] can be applied to prove that there exist a global attractor $\mathcal{A}$ of the above dynamical system to which all the trajectories of this dynamical system will eventually converges, namely we have the following

Theorem 6.1. *Let $X = (L^\infty(\Omega))^4$ with the metric inherited from $L^2(\Omega)$ then the semigroup $\{S(t)\}_{t\geq 0}$ defined above posses a global attractor $\mathcal{A} \subset (H^1(\Omega) \cap L^\infty(\Omega))^4$.*

Proof. From (18) we can proof easily that $\|U(\cdot,t)\|_{L^2(\Omega)}$ and $\|\nabla U^{m_i}(\cdot,t)\|_{L^2(\Omega)}$ are bounded independently of the initial data for $t \geq \eta > 0$, and we see that $S(t)$ defined on $X = (L^\infty(\Omega))^4$ is a compact mapping on X with the L^2 norm and and admits an absorbing set in X which absorbs any bounded set B in X after some finite time. Therefore, [27, Theorem 1.1] can be applied to exhibit global attractor which is bounded in $(H^1(\Omega)\cap L^\infty(\Omega))^4$. $\square$

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