# ON THE STRONG STABILITY OF A NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL SYSTEM

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ABSTRACT. In this paper we provide sufficient conditions for strong stability of the trivial solution of the systems (1) and (2).

## 1. INTRODUCTION

In [3], T. Hara, T. Yoneyama and T. Itoh proved sufficient conditions for uniform stability, asymptotic stability, uniform asymptotic stability and exponential asymptotic stability of trivial solution of a nonlinear Volterra integro-differential system of the form

$$x' = A(t)x + \int_0^t F(t, s, x(s))ds \tag{1}$$

The purpose of our paper is to provide sufficient conditions for strong stability of trivial solution of (1), as a perturbed system of

$$x' = A(t)x. \tag{2}$$

We investigate conditions on the fundamental matrix $Y(t)$ for linear system (2) and on the function $F(t, s, x)$ under which the trivial solution of (1) or (2) is strongly stable on $\mathbb{R}_+$.

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## 2. Definitions, notations and hypotheses

Let $\mathbb{R}^n$ denote the Euclidean $n$-space. For $x \in \mathbb{R}^n$, let $\|x\|$ be the norm of $x$. For an $n \times n$ matrix $A$, we define the norm $|A|$ of $A$ by

$$|A| = \sup_{\|x\| \le 1} \|Ax\|.$$

In equation (1) we consider that $A$ is a continuous $n \times n$ matrix on $\mathbb{R}_+$ and $F : D \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $D = \{(t,s) \in \mathbb{R}^2; 0 \le s \le t < \infty\}$, is a continuous $n$-vector such that $F(t,s,0) = 0$ for $(t,s) \in D$.

**Definition 2.1.** The solution $x(t)$ of (1) is said to be *strongly stable* (Ascoli, [1]) on $\mathbb{R}_+$ if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that any solution $\widetilde{x}(t)$ of (1) which satisfies the inequality $\|\widetilde{x}(t_0) - x(t_0)\| < \delta$ for some $t_0 \ge 0$, exists and satisfies the inequality $\|\widetilde{x}(t) - x(t)\| < \varepsilon$ for all $t \ge 0$.

**Remark 2.1.** For definitions of other types of stability, see [2, page 51].

**Remark 2.2.** It is easy to see that strong stability is not equivalent with none of these types of stability.

## 3. The Main Results

The following result [2] is well-known.

**Theorem 3.1.** *Let $Y(t)$ be a fundamental matrix for (2). Then, the trivial solution of (2) is strongly stable on $\mathbb{R}_+$ if and only if there exists a positive constant $K$ such that*

$$|Y(t)Y^{-1}(s)| \le K \qquad \text{for all } \; 0 \le s,t < \infty$$

*or, equivalently,*

$$|Y(t)| \le K \qquad \text{and} \qquad |Y^{-1}(t)| \le K \qquad \text{for all } \; t \ge 0.$$

Let $Y(t)$ be a fundamental matrix for (2). Consider the following hypotheses:

$\mathbf{H}_1:$ There exist a continuous function $\varphi : \mathbb{R}_+ \longrightarrow (0,\infty)$ and the constants $p_1 \geq 1$, $K_1 > 0$ for

$$\int_0^t \big(\varphi(s)|Y(t)Y^{-1}(s)|\big)^{p_1}\, ds \leq K_1, \qquad \text{for all } t \geq 0.$$

$\mathbf{H}_2:$ There exist a continuous function $\varphi : \mathbb{R}_+ \longrightarrow (0,\infty)$ and the constants $p_2 \geq 1$, $K_2 > 0$ for

$$\int_0^t \big(\varphi(s)|Y^{-1}(t)Y(s)|\big)^{p_2}\, ds \leq K_2, \qquad \text{for all } t \geq 0.$$

$\mathbf{H}_3:$ There exist a continuous function $\varphi : \mathbb{R}_+ \longrightarrow (0,\infty)$ and the constants $p_3 \geq 1$, $K_3 > 0$ for

$$\int_0^t \big(\varphi(s)|Y^{-1}(s)Y(t)|\big)^{p_3}\, ds \leq K_3, \qquad \text{for all } t \geq 0.$$

$\mathbf{H}_4:$ There exist a continuous function $\varphi : \mathbb{R}_+ \longrightarrow (0,\infty)$ and the constants $p_4 \geq 1$, $K_4 > 0$ for

$$\int_0^t \big(\varphi(s)|Y(s)Y^{-1}(t)|\big)^{p_4}\, ds \leq K_4, \qquad \text{for all } t \geq 0.$$

**Theorem 3.2.** *Suppose that the fundamental matrix $Y(t)$ for (2) satisfies one of the following conditions:*

$\mathbf{C}_1:$ $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *are true.*
$\mathbf{C}_2:$ $\mathbf{H_1}$ *and* $\mathbf{H_4}$ *are true.*
$\mathbf{C}_3:$ $\mathbf{H_2}$ *and* $\mathbf{H_3}$ *are true.*
$\mathbf{C}_4:$ $\mathbf{H_3}$ *and* $\mathbf{H_4}$ *are true.*

*Then, the trivial solution of (2) is strongly stable on $\mathbb{R}_+$.*

*Proof.* We will prove that $Y(t)$ and $Y^{-1}(t)$ are bounded on $\mathbb{R}_+$.
First of all, we consider the case $\mathbf{C_2}$. For the beginning we prove that $Y(t)$ is bounded on $\mathbb{R}_+$.

Let $q(t) = \varphi^{p_1}(t)|Y(t)|^{-p_1}$ for $t \geq 0$. From the identity

$$\left(\int_0^t q(s)ds\right) Y(t) = \int_0^t (\varphi(s)Y(t)Y^{-1}(s))(q(s)(\varphi(s))^{-1}Y(s))ds, \qquad t \geq 0,$$

it follows that

$$\left(\int_0^t q(s)ds\right) |Y(t)| \leq \int_0^t \left(\varphi(s)|Y(t)Y^{-1}(s)|\right) \left(q(s)(\varphi(s))^{-1}|Y(s)|\right) ds, \qquad t \geq 0. \tag{3}$$

In case $p_1 = 1$, we have that $q(s)(\varphi(s))^{-1}|Y(s)| = 1$. From (3) and the hypothesis $\mathbf{H_1}$ it follows that

$$\left(\int_0^t q(s)ds\right) |Y(t)| \leq \int_0^t \varphi(s)|Y(t)Y^{-1}(s)|ds \leq K_1, \qquad t \geq 0.$$

In case $p_1 > 1$, we have that $q(s)(\varphi(s))^{-1}|Y(s)| = (q(s))^{\frac{1}{q_1}}$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$. From (3), it follows that

$$\left(\int_0^t q(s)ds\right) \varphi(t)(q(t))^{-\frac{1}{p_1}} \leq \int_0^t (\varphi(s)|Y(t)Y^{-1}(s)|)(q(s))^{\frac{1}{q_1}} ds,$$

for all t $\geq 0$.

Using the Hölder inequality, we obtain

$$\left(\int_0^t q(s)ds\right)\varphi(t)(q(t))^{-\frac{1}{p_1}}$$
$$\leq \left(\int_0^t (\varphi(s)|Y(t)Y^{-1}(s)|)^{p_1} ds\right)^{\frac{1}{p_1}} \left(\int_0^t q(s)ds\right)^{\frac{1}{q_1}}, \qquad t \geq 0.$$

Using the hypothesis $\mathbf{H_1}$, we obtain that

$$\left(\int_0^t q(s)ds\right)^{\frac{1}{p_1}} \varphi(\mathrm{t})(\mathrm{q}(\mathrm{t}))^{-\frac{1}{p_1}} \leq K_1^{\frac{1}{p_1}}, \qquad t \geq 0$$

or

$$\left(\int_0^t q(s)ds\right)|Y(t)|^{p_1} \le K_1, \qquad\qquad t \ge 0.$$

Thus, for $p_1 \ge 1$, the function $|Y(t)|$ satisfies the inequality

$$|Y(t)| \le K_1^{\frac{1}{p_1}}\left(\int_0^t q(s)ds\right)^{-\frac{1}{p}1}, \qquad\qquad t \ge 0.$$

Denote $Q(t) = \displaystyle\int_0^t q(s)ds$ for $t \ge 0$. Thus, we have

$$|\ Y(t)\ | \le K_1^{\frac{1}{p_1}}\ (Q(t))^{-\frac{1}{p_1}}, \qquad\qquad \text{for } t \ge 0.$$

Because

$$Q'(t) = q(t) \ge K_1^{-1}(\varphi(t))^{p_1}Q(t) \qquad\qquad \text{for } t \ge 0,$$

we have that

$$Q(t) \ge Q(1)\,\mathrm{e}^{K_1^{-1}\int_1^t \varphi^{p_1}(s)ds}, \qquad\qquad \text{for } t \ge 1.$$

It follows that

$$|Y(t)| \le K_1^{\frac{1}{p_1}}(Q(1))^{-\frac{1}{p_1}}\mathrm{e}^{-(p_1K_1)^{-1}\int_1^t \varphi^{p_1}(s)ds}, \qquad\qquad \text{for } t \ge 1.$$

Because $|Y(t)|$ is a continuous function on $[0,1]$, it follows that there exists a positive constant $M_1$ such that $|Y(t)| \le M_1$ for $t \ge 0$.

In what follows we prove that $Y^{-1}(t)$ is bounded on $\mathbb{R}_+$.

Let $q(t) = \varphi^{p_4}(t)|Y^{-1}(t)|^{-p_4}$ for $t \ge 0$. From the identity

$$\left(\int_0^t q(s)ds\right)Y^{-1}(t) = \int_0^t (q(s)(\varphi(s))^{-1}Y^{-1}(s))(\varphi(s)Y(s)Y^{-1}(t))ds, \qquad t \ge 0$$

it follows that

$$\left(\int_0^t q(s)ds\right)|Y^{-1}(t)| \le \int_0^t \left(q(s)(\varphi(s))^{-1}|Y^{-1}(s)|\right)\left(\varphi(s)|Y(s)Y^{-1}(t)|\right)ds, \qquad t \ge 0. \tag{4}$$

In case $p_4 = 1$, we have that $q(s)(\varphi(s))^{-1}|Y^{-1}(s)| = 1$.

From (4) and the hypothesis $\mathbf{H_4}$, it follows that

$$\left(\int_0^t q(s)ds\right)|Y^{-1}(t)| \le \int_0^t \varphi(s)||Y(s)Y^{-1}(t)|ds \le K_4, \qquad t \ge 0.$$

In case $p_4 > 1$, we have that

$$q(s)(\varphi(s))^{-1}|Y^{-1}(s)| = (q(s))^{\frac{1}{q_4}}, \qquad s \ge 0.$$

where $\dfrac{1}{p_4} + \dfrac{1}{q_4} = 1$.

From (4) it follows that

$$\left(\int_0^t q(s)ds\right)|Y^{-1}(t)| \le \int_0^t q^{\frac{1}{q_4}}(s)\left(\varphi(s)|Y(s)Y^{-1}(t)|\right)ds$$

for all $t \ge 0$.

Using the Hölder inequality, we obtain that

$$\left(\int_0^t q(s)ds\right)|Y^{-1}(t)| \le \left(\int_0^t \left(\varphi(s)|Y(s)Y^{-1}(t)|\right)^{p_4} ds\right)^{\frac{1}{p_4}} \left(\int_0^t q(s)ds\right)^{\frac{1}{q_4}}, \qquad t \ge 0.$$

Using the hypothesis $H_4$, we have

$$\left(\int_0^t q(s)ds\right)|Y^{-1}(t)| \le \left(\int_0^t q(s)ds\right)^{\frac{1}{q_4}} K_4^{\frac{1}{p_4}}, \qquad t \ge 0$$

or

$$\left(\int_0^t q(s)ds\right)^{\frac{1}{p_4}} |Y^{-1}(t)| \le K_4^{\frac{1}{p_4}}, \qquad t \ge 0.$$

Thus, for $p_4 \ge 1$, the function $|Y^{-1}(t)|$ satisfies the inequality

$$|\,Y^{-1}(t)| \le K_4^{\frac{1}{p_4}} \left(\int_0^t q(s)ds\right)^{-\frac{1}{p_4}}, \qquad t \ge 0.$$

Denote $Q(t) = \int_0^t q(s)ds$ for $t \geq 0$. Thus, we have

$$|Y^{-1}(t)| \leq K_4^{\frac{1}{p_4}} (Q(t))^{-\frac{1}{p_4}}, \qquad t \geq 0.$$

Because

$$Q'(t) = q(t) \geq \varphi^{p_4}(t) K_4^{-1} Q(t), \qquad t \geq 0,$$

we have

$$Q(t) \geq Q(1) \mathrm{e}^{K_4^{-1} \int_1^t \varphi^{p_4}(s)ds}, \qquad t \geq 1.$$

It follows that

$$|Y^{-1}(t)| \leq K_4^{\frac{1}{p_4}} (Q(1))^{-\frac{1}{p_4}} \mathrm{e}^{-(p_4 K_4)^{-1} \int_1^t \varphi^{p_4}(s)ds}, \qquad t \geq 1.$$

Because $|Y^{-1}(t)|$ is a continuous function on $[0, 1]$, it follows that there exists a positive constant $M_2$ such that $|Y^{-1}(t)| \leq M_2$ for $t \geq 0$.

Hence, the conclusion follows immediately from Theorem 3.1.

Finally, in the cases $\mathbf{C_1}$, $\mathbf{C_3}$ or $\mathbf{C_4}$, the proof is similarly.

The proof is now complete. □

**Remark 3.1.** The function $\varphi$ can serve to weaken the required hypotheses on the fundamental matrix $Y$.

**Theorem 3.3.** *If*

*1. the fundamental matrix $Y(t)$ of the equation (2) satisfies*

$$|Y(t)Y^{-1}(s)| \leq K$$

*for all $0 \leq s, t < +\infty$, where $K$ is constant,* *2. the function $F$ satisfies the condition*

$$\|F(t,s,x)-F(t,s,y)\| \leq f(t,s)\|x-y\|$$

*for $0 \leq s \leq t < +\infty$ and for all $x, y \in \mathbb{R}^n$, where $f$ is a continuous nonnegative function on $D$ such that*

$$M = \int_0^\infty \int_0^t f(t,s)\, ds\, dt < K^{-1},$$

*then, for all $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and $\rho > 0$, there exists a unique solution of (1) on $\mathbb{R}_+$ such that $x(t_0) = x_0$ and $\|x(t)\| \leq \rho$ for all $t \in [0, t_0]$, if $\| x_0\|$ is sufficiently small.*

*Proof.* It is well-known that the problem

$$x' = A(t)x + \int_0^t F(t,s,x(s))ds, \qquad x(t_0) = x_0$$

can be reduced by means of variation of constants to the nonlinear integral system

$$x(t) = Y(t)Y^{-1}(t_0)x_0 + \int_{t_0}^t Y(t)Y^{-1}(s) \int_0^s F(s,u,x(u))\, du\, ds, \qquad t \geq 0. \tag{5}$$

We introduce the Fréchet space $C_c$ of all continuous maps from $\mathbb{R}_+$ into $\mathbb{R}^n$ with the seminorms $\|x|_\tau = \sup\limits_{0 \leq t \leq \tau} \|x(t)\|$, $\tau \geq 0$. Thus, convergence in $C_c$ is equivalent to the usual convergence over all compact intervals of $\mathbb{R}_+$.

For $t_0 \geq 0$ and $\rho > 0$, let $x_0 \in \mathbb{R}^n$ be such that $\|x_0\| < \rho(1-KM)K^{-1}$. Let $S_\rho$ be the set

$$S_\rho = \{x \in C_c \ ; \ \|x\|_{t_0} \leq \rho, \ \ \|x\|_\tau \leq \rho \mathrm{e}^{KM} \text{ for } \tau > t_0\}.$$

We consider the following operator $T$ from $S_\rho$ into $C_c$ :

$$(Tx)(t) = Y(t)Y^{-1}(t_0)x_0 + \int_{t_0}^{t} Y(t)Y^{-1}(s) \int_0^s F(s,u,x(u))\,du\,ds, \qquad t \geq 0.$$

For $x \in S_\rho$ and $t \in [0, t_0]$, we have

$$\begin{aligned}\|(Tx)(t)\| \leq\ & K\|x_0\| + K \int_t^{t_0} \int_0^s f(s,u)\|x(u)\|\,du\,ds \\ \leq\ & K\|x_0\| + K \sup_{0\leq t\leq t_0} \|x(t)\| \int_0^{t_0} \int_0^s f(s,u)\,du\,ds \\ \leq\ & K\rho(1-KM)K^{-1} + K\rho M = \rho.\end{aligned}$$

For $x \in S_\rho$ and $t > t_0$, using the same kind of arguments as above, we obtain

$$\|(Tx)(t)\| \leq \rho \mathrm{e}^{KM}.$$

Thus, $TS_\rho \subset S_\rho$.

Let $x, y \in S_\rho$. For $t \in [0, t_0]$, we have

$$\begin{aligned}
&\|(Tx)(t) - (Ty)(t)\| \\
&\quad = \left\| \int_{t_0}^{t} Y(t)Y^{-1}(s) \int_0^s (F(s,u,x(u)) - F(s,u,y(u)))\, du\, ds \right\| \\
&\quad \leq \int_t^{t_0} \left| Y(t)Y^{-1}(s) \right| \int_0^s \|F(s,u,x(u)) - F(s,u,y(u))\|\, du\, ds \\
&\quad \leq K \int_t^{t_0} \int_0^s f(s,u) \|x(u) - y(u)\|\, du\, ds \\
&\quad \leq K \sup_{0 \leq u \leq t_0} \|x(u) - y(u)\| \int_t^{t_0} \int_0^s f(s,u)\, du\, ds \\
&\quad \leq KM \|x - y\|_{t_0}.
\end{aligned}$$

Then,

$$\|Tx - Ty\|_{t_0} \leq KM \|x - y\|_{t_0}.$$

Similarly, for $\tau > \mathrm{t}_0$, we have

$$\|Tx - Ty\|_\tau \leq KM \|x - y\|_\tau.$$

Hence, $T$ is a contraction. By the Banach's Theorem for Fréchet spaces [4], $S_\rho$ contains a unique fixed point $\widetilde{x} = T\widetilde{x}$, i. e., the equation (1) has a unique solution $\widetilde{x}(t)$ on $\mathbb{R}_+$ such that $\widetilde{x}(t_0) = x_0$ and $\|\widetilde{x}(t)\| \leq \rho$ for all t $\in [0, t_0]$ and $\|\widetilde{x}(t)\| \leq \rho \mathrm{e}^{KM}$ for all $t \geq 0$, if $\|x_0\|$ is sufficiently small.

Now, we suppose that $x(t)$ is a solution in $C_c$ of (5) such that $\|x(t)\| \leq \rho$ for $t \in [0, t_0]$ and $\|x_0\| \leq \rho(1-KM)K^{-1}$. For $t \geq t_0$ we have

$$\begin{aligned}
\|x(t)\| &= \|Y(t)Y^{-1}(t_0)x_0 + \int_{t_0}^{t} Y(t)Y^{-1}(s)\int_0^s F(s,u,x(u))\,du\,ds\| \\
&\leq K\|x_0\| + K\int_{t_0}^{t}\int_0^s f(s,u)\|x(u)\|\,du\,ds \\
&= K\|x_0\| + K\int_{t_0}^{t}\int_0^{t_0} f(s,u)\|x(u)\|\,du\,ds + K\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)\|x(u)\|\,du\,ds \\
&\leq K\|x_0\| + K\rho\int_{t_0}^{t}\int_0^{t_0} f(s,u)\,du\,ds + K\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)\|x(u)\|\,du\,ds \\
&\leq K\rho(1-KM)K^{-1} + K\rho M + K\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)\|x(u)\|\,du\,ds \\
&= \rho + K\int_{t_0}^{t}\int_{t_0}^{s} \mathrm{f(s,u)}\|x(u)\|\,du\,ds.
\end{aligned}$$

It is easy to see that the function $Q(t) = \int_{t_0}^{t}\int_{t_0}^{s} f(s,u)\|x(u)\|\,du\,ds$ is continuously differentiable and increasing on $[t_0, \infty)$.

For $t \geq t_0$, we have

$$\begin{aligned} Q'(t) = & \int_{t_0}^{t} f(t,u)\|x(u)\|du \\ \leq & \int_{t_0}^{t} f(t,u)(\rho + KQ(u))du = \rho \int_{t_0}^{t} f(t,u)du + K \int_{t_0}^{t} f(t,u)Q(u)du. \end{aligned}$$

Then,

$$\begin{aligned} & \left[ Q(t)e^{-K \int_{t_0}^{t} \int_{t_0}^{s} f(s,u)duds} \right]' \\ = & \ \mathrm{e}^{-K \int_{t_0}^{t} \int_{t_0}^{s} f(s,u)\,du\,ds} \left[ Q'(t) - KQ(t) \int_{t_0}^{t} f(t,u)du \right] \\ \leq & \ \mathrm{e}^{-K \int_{t_0}^{t} \int_{t_0}^{s} f(s,u)\,du\,ds} \left[ \rho \int_{t_0}^{t} f(t,u)du + K \int_{t_0}^{t} f(t,u)(Q(u) - Q(t))du \right] \\ \leq & \ \mathrm{e}^{-K \int_{t_0}^{t} \int_{t_0}^{s} f(s,u)\,du\,ds} \left[ \rho \int_{t_0}^{t} f(t,u)du \right] = \left[ -\rho K^{-1} \mathrm{e}^{-K \int_{t_0}^{t} \int_{t_0}^{s} f(s,u)\,du\,ds} \right]'. \end{aligned}$$

By integrating from $t_0$ to $t \geq t_0$, we have

$$Q(t)\mathrm{e}^{-K\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)\,du\,ds} - Q(t_0) \leq -\rho K^{-1}\mathrm{e}^{-K\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)duds} + \rho K^{-1}.$$

We deduce that

$$\|x(t)\| \leq \rho + KQ(t) \qquad \text{for } t \geq t_0,$$

and then

$$\|x(t)\| \leq \rho \mathrm{e}^{KM} \qquad \text{for } t \geq t_0.$$

This shows that $x \in S_\rho$ and then $x = \widetilde{x}$. Thus, for all $t_0 \geq 0$, $x_0 \in \mathbb{R}^{\mathrm{n}}$ and $\rho > 0$, there exists a unique solution of (1) on $\mathbb{R}_+$ such that $x(t_0) = x_0$ and $\|x(t)\| \leq \rho$ for all $t \in [0, t_0]$, if $\|x_0\|$ is sufficiently small. The proof is complete. □

**Theorem 3.4.** *If the hypotheses of Theorem 3.3 are satisfied, then the trivial solution of (1) is strongly stable on $\mathbb{R}_+$.*

*Proof.* Let $\varepsilon > 0$ be arbitrary and let $\delta(\varepsilon) = \varepsilon(1-KM)K^{-1}\mathrm{e}^{-KM}$, $\mathrm{t}_0 \geq 0$ and let $x_0 \in \mathbb{R}^n$ satisfy $\|x_0\| < \delta(\varepsilon)$.
Applying Theorem 3.3, we deduce that there exists a unique solution $x(t)$ on $\mathbb{R}_+$ of (1) with $x(t_0) = x_0$ such that $x \in S_{\varepsilon \mathrm{e}^{-KM}}$, i. e., $\|x(t)\| \leq \varepsilon$ for $t \geq 0$.
This proves that the trivial solution of (1) is strongly stable on $\mathbb{R}_+$. The proof is complete. □

**Example 3.1.** Let $a, b : \mathbb{R}_+ \to \mathbb{R}$ be continuous and let the system (2) with

$$A(t) = \begin{pmatrix} a(t) & -b(t) \\ b(t) & a(t) \end{pmatrix}.$$

It is easy to see that

$$Y(t) = r(t)\begin{pmatrix} -\cos\theta(t) & -\sin\theta(t) \\ -\sin\theta(t) & \cos\theta(t) \end{pmatrix},$$

where

$$r(t) = \mathrm{e}^{\int_0^t a(u)du} \qquad \text{and} \qquad \theta(t) = \int_0^t b(u)du,$$

is a fundamental matrix of (2).

We have

$$|Y(t)Y^{-1}(s)| \leq \sqrt{2}\mathrm{e}^{\int_s^t a(u)du} \qquad \text{for all } t, s \geq 0.$$

In [3], it is proved that if there exists $\lambda > 0$ such that

$$a(t) \leq -\lambda \qquad \text{for all } t \geq 0.$$

then the system (2) is uniformly asymptotically stable on $\mathbb{R}_+$.

We remark that if there exist $C \geq 0$ and $\lambda > 0$ such that

$$\int_s^t a(u)du \leq C - \lambda(t-s) \qquad \text{for all } t \geq s \geq 0,$$

then we have the same conclusion.

In addition, if there exists $L > 0$ such that

$$|\int_s^t a(u)du| \leq L \qquad \text{for all } t, s \geq 0,$$

then the system (2) is strongly stable on $\mathrm{R}_+$.

Now, we consider

$$F(t,s,x) = \mathrm{e}^{-\alpha t+s}\begin{pmatrix} \sin x_1 + t\arctan x_2 \\ s\sin x_1 - \arctan x_2 \end{pmatrix},$$

where $\alpha \in \mathbb{R}$.

It is easy to see that the function $F$ satisfies the conditions of Theorem 3.3 for $\alpha$ sufficiently large positive number.

In these conditions for $A(t)$ and $F$, for all $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and $\rho > 0$, there exists a unique solution $x(t)$ of (1) on $\mathbb{R}_+$ such that $x(t_0) = x_0$ and $\|x(t)\| \leq \rho$ for all $t \in [0, t_0]$, if $\|x_0\|$ is sufficiently small.

In addition, the trivial solution of (1) is strongly stable on $\mathbb{R}_+$.

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