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A NEW EXTENSION OF HILBERT'S INEQUALITY FOR MULTIFUNCTIONS WITH BEST CONSTANT FACTORS

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Abstract. The aim of this paper is to establish a new extension of Hilbert's inequality and Hardy-Hilbert's inequality for multifunctions with best constant factors. Also, we present some applications for Hilbert's inequality which give new integral inequalities.

1. Introduction

Hilbert's inequality has a great interest in analysis and its applications (see [**10**], [**11**]). The original Hilbert's inequality can be stated as follows

If $f(x),\ g(x) \geq 0$, such that $0 < \int_0^\infty f^2(x)\mathrm{d}x < \infty$ and $0 < \int_0^\infty g^2(x)\mathrm{d}x < \infty$, then (see [**6**])

$$\int\limits_0^\infty \int\limits_0^\infty \frac{f(x)\ g(y)}{x+y}\mathrm{d}x\,\mathrm{d}y < \pi \left\{ \int\limits_0^\infty f^2(x)\mathrm{d}x \int\limits_0^\infty g^2(x)\mathrm{d}x \right\}^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible. This inequality was extended by Hardy-Riesz as (see [**5**]):

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x),\ g(x) \geq 0$, such that $0 < \int_0^\infty f^p(x)\mathrm{d}x < \infty$ and $0 < \int_0^\infty g^q(x)\mathrm{d}x < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)\ g(y)}{x+y}\mathrm{d}x\,\mathrm{d}y < \frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \int_0^\infty f^p(x)\mathrm{d}x \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)\mathrm{d}x \right\}^{\frac{1}{q}}, \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible.

Hardy-Hilbert's integral inequality is important in analysis and its applications (see[**10**], [**11**]). In recent years, the various improvements and extensions on the inequality (1) and (2) appeared in some papers (such as [**1**]–[**4**], [**7**], [**9**], [**12**]–[**14**]) and bibliography therein. They focalize on changing the denominator of the function of the left-hand side of (2). Such as the denominator $(x + y)$ is replaced by $(Ax + By)^\lambda$ in paper [**13**], the denominator $(x + y)$ is replaced by

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(x^t+y^t) (t is a parameter which is independent of x and y) in paper [**7**]. Generally, the denominator $(x+y)$ is replaced by $(xu(x)+yv(y))^\lambda$ in paper [**9**].

The main objective of this paper is to build some new Hilbert-type integral inequalities with best constant factors which are extensions of above results for multi-functions f, g and h. Moreover the denominator is $(m(x)+n(y)+r(z))$, where m, n and r are arbitrary functions.

2. Main results

We need the formula of the β function as (see [**8**]):

$$\beta(u,v)=\int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}\mathrm{d}t=\beta(v,u) \qquad u,v>0. \tag{3}$$

Before stating our results we need the following lemmas.

Lemma 2.1. *Let* $f(x,y,z)\in L^p_{[0,\infty]\times[0,\infty]\times[0,\infty]}$ *and* $g(x,y,z)\in L^q_{[0,\infty]\times[0,\infty]\times[0,\infty]}$, *where* $\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_0^\infty |f(x,y,z)\ g(x,y,z)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\le\left(\int_0^\infty\int_0^\infty\int_0^\infty |f(x,y,z)|^p\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{p}}\left(\int_0^\infty\int_0^\infty\int_0^\infty |g(x,y,z)|^q\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{q}}.\end{aligned}\tag{4}$$

Lemma 2.2. *Let* $f(x,y,z)\in L^p_{[0,\infty]\times[0,\infty]\times[0,\infty]}$, $g(x,y,z)\in L^q_{[0,\infty]\times[0,\infty]\times[0,\infty]}$ *and* $h(x,y,z)\in L^k_{[0,\infty]\times[0,\infty]\times[0,\infty]}$ *where* $\dfrac{1}{p}+\dfrac{1}{q}+\dfrac{1}{k}=1$. *Then*

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_0^\infty |f(x,y,z)\ g(x,y,z)h(x,y,z)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\le\left(\int_0^\infty\int_0^\infty\int_0^\infty |f(x,y,z)|^p\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{p}}\times\left(\int_0^\infty\int_0^\infty\int_0^\infty |g(x,y,z)|^q\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{q}}\\ &\quad\times\left(\int_0^\infty\int_0^\infty\int_0^\infty |h(x,y,z)|^r\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{k}}.\end{aligned}\tag{5}$$

Lemma 2.3. *If* $p_1>1$, $\dfrac{1}{p_1}+\dfrac{1}{q_1}=1$, $p>1$, $\dfrac{1}{p}+\dfrac{1}{q}+\dfrac{1}{k}=1$ *then for* $0<\varepsilon<\dfrac{1}{qk}$, *we have*

$$\int_0^\infty \frac{v^{\frac{1}{p_1qk}-\frac{\varepsilon}{p_1}-1}}{(1+v)^{\frac{1}{qk}}}\mathrm{d}v=\beta\left(\frac{1}{p_1qk},\frac{1}{q_1qk}\right)+0(1) \qquad \varepsilon\to 0^+. \tag{6}$$

Proof. Since

$$
\begin{aligned}
&\left|\int_0^\infty \frac{v^{\frac{1}{p_1qk}-\frac{\varepsilon}{p_1}-1}}{(1+v)^{\frac{1}{qk}}}\mathrm{d}v-\beta\left(\frac{1}{p_1qk},\frac{1}{q_1qk}\right)\right| \\
&\quad=\left|\int_0^\infty \frac{v^{\frac{1}{p_1qk}-\frac{\varepsilon}{p_1}-1}-v^{\frac{1}{p_1qk}-1}}{(1+v)^{\frac{1}{qk}}}\mathrm{d}v\right| \\
&\quad\le \int_0^1 \frac{\left|v^{\frac{1}{p_1qk}-\frac{\varepsilon}{p_1}-1}-v^{\frac{1}{p_1qk}-1}\right|}{(1+v)^{\frac{1}{qk}}}\mathrm{d}v+\int_1^\infty \frac{\left|v^{\frac{1}{p_1qk}-\frac{\varepsilon}{p_1}-1}-v^{\frac{1}{p_1qk}-1}\right|}{(1+v)^{\frac{1}{qk}}}\mathrm{d}v \\
&\quad\le \int_0^1 \left(v^{\frac{1}{p_1qk}-\frac{\varepsilon}{p_1}-1}-v^{\frac{1}{p_1qk}-1}\right)\mathrm{d}v+\int_1^\infty \frac{v^{\frac{1}{p_1qk}-1}-v^{\frac{1}{p_1qk}-\frac{\varepsilon}{p_1}-1}}{v^{\frac{1}{qk}}}\mathrm{d}v \\
&\quad=\frac{1}{\frac{1}{p_1qk}-\frac{\varepsilon}{p_1}}-\frac{1}{\frac{1}{p_1qk}}+\frac{-1}{\frac{1}{q_1qk}}+\frac{1}{\frac{1}{q_1qk}-\frac{\varepsilon}{q_1}}\to 0 \qquad \text{for} \qquad \varepsilon\to 0.
\end{aligned}
$$

□

Lemma 2.4. *If* $p_1>1$, $\frac{1}{p_1}+\frac{1}{q_1}=1$, $p>1$, $\frac{1}{p}+\frac{1}{q}+\frac{1}{k}=1$ *and* $0<\varepsilon<\frac{1}{qk}$, *setting*

$$
\begin{aligned}
J_1 := \int_1^\infty\int_1^\infty &\left(\frac{n(y)}{m(x)+n(y)}\right)^{\frac{1}{qk}}(m(x))^{\frac{1}{p_1qk}-\frac{\varepsilon}{p_1}-1} \\
&\times (n(y))^{-\frac{1}{p_1qk}-\frac{\varepsilon}{q_1}-1}\frac{\mathrm{d}m(x)}{\mathrm{d}x}\,\frac{\mathrm{d}n(y)}{\mathrm{d}y}\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
$$

then we have

$$
\begin{aligned}
&\frac{1}{\varepsilon}\left(\beta\left(\frac{1}{p_1qk},\ \frac{1}{q_1qk}\right)+o(1)\right)-O(1) \\
&\qquad\le J_1\le \frac{1}{\varepsilon}\left(\beta\left(\frac{1}{p_1qk},\frac{1}{q_1qk}\right)+o(1)\right), \qquad \varepsilon\to 0^+.
\end{aligned}
\tag{7}
$$

Proof. For fixed y, setting $m(x)=n(y)v$, then by (6), we obtain

$$
\begin{aligned}
J_1 \;=\; &\int_1^\infty (n(y))^{-\frac{1}{p_1qk}-\frac{\varepsilon}{q_1}-1}\frac{\mathrm{d}n(y)}{\mathrm{d}y} \\
&\times\left[\int_1^\infty \left(\frac{n(y)}{m(x)+n(y)}\right)^{\frac{1}{qk}}(m(x))^{\frac{1}{p_1qk}-\frac{\varepsilon}{p_1}-1}\frac{\mathrm{d}m(x)}{\mathrm{d}x}\mathrm{d}x\right]\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&= \int_1^\infty (n(y))^{-\varepsilon-1}\frac{\mathrm{d}n(y)}{\mathrm{d}y}\left[\int_{\frac{1}{n(y)}}^\infty \frac{v^{\frac{1}{p_1qk}-\frac{\varepsilon}{p_1}-1}}{(1+v)^{\frac{1}{qk}}}\mathrm{d}v\right]\mathrm{d}y\\
&= \int_1^\infty (n(y))^{-\varepsilon-1}\frac{\mathrm{d}n(y)}{\mathrm{d}y}\left[\int_0^\infty \frac{v^{\frac{1}{p_1qk}-\frac{\varepsilon}{p_1}-1}}{(1+v)^{\frac{1}{qk}}}\mathrm{d}v\right]\mathrm{d}y\\
&\quad -\int_1^\infty (n(y))^{-\varepsilon-1}\frac{\mathrm{d}n(y)}{\mathrm{d}y}\left[\int_0^{\frac{1}{n(y)}} \frac{v^{\frac{1}{p_1qk}-\frac{\varepsilon}{p_1}-1}}{(1+v)^{\frac{1}{qk}}}\mathrm{d}v\right]\mathrm{d}y\\
&\geq \frac{1}{\varepsilon}\left(\beta\left(\frac{1}{p_1qk},\frac{1}{q_1qk}\right)+0(1)\right)-\int_1^\infty (n(y))^{-\varepsilon-1}\frac{\mathrm{d}n(y)}{\mathrm{d}y}\left[\int_0^1 v^{\frac{1}{p_1qk}-\frac{\varepsilon}{p_1}-1}\mathrm{d}v\right]\mathrm{d}y\\
&= \frac{1}{\varepsilon}\left(\beta\left(\frac{1}{p_1qk},\frac{1}{q_1qk}\right)+0(1)\right)-\frac{1}{\varepsilon}\frac{1}{\left(\frac{1}{p_1qk}-\frac{\varepsilon}{p_1}\right)}\\
&= \frac{1}{\varepsilon}\left(\beta\left(\frac{1}{p_1qk},\frac{1}{q_1qk}\right)+o(1)\right)-O(1).
\end{aligned}
$$

By the same way, we have

$$
\begin{aligned}
J_1 &\leq \int_1^\infty\int_0^\infty \left(\frac{n(y)}{m(x)+n(y)}\right)^{\frac{1}{qk}}(m(x))^{\frac{1}{p_1qk}-\frac{\varepsilon}{p_1}-1}\\
&\qquad\qquad \times (n(y))^{-\frac{1}{p_1qk}-\frac{\varepsilon}{q_1}-1}\frac{\mathrm{d}m(x)}{\mathrm{d}x}\frac{\mathrm{d}n(y)}{\mathrm{d}y}\mathrm{d}x\,\mathrm{d}y,\\
&= \frac{1}{\varepsilon}\left(\beta\left(\frac{1}{p_1qk},\frac{1}{q_1qk}\right)+o(1)\right).
\end{aligned}
$$

The lemma is proved. □

Theorem 2.5. *Assume that m, n, r are increasing functions defined on $[0,\infty[$ such that $m(0)=n(0)=r(0)=0$, $\lim_{x\to\infty} m(x)=\lim_{x\to\infty} n(x)=\lim_{x\to\infty} r(x)=\infty$, and f, g, h satisfy*

$$\int_0^\infty (m(x))^{\frac{q_1}{p_1}(1+\frac{1}{pq})}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})}|f(x)|^{\frac{kq_1}{2}}\,\mathrm{d}x<\infty \tag{8}$$

$$\int_0^\infty (m(x))^{\frac{p_1}{q_1}-\frac{1}{qk}}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-p_1}{q_1}}|f(x)|^{\frac{pp_1}{2}}\,\mathrm{d}x<\infty, \tag{9}$$

$$\int_0^\infty (n(y))^{\frac{q_1}{p_1}(1+\frac{1}{qk})}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})}|g(y)|^{\frac{pq_1}{2}}\,\mathrm{d}y<\infty, \tag{10}$$

$$\int_0^\infty (n(y))^{\frac{p_1}{q_1}-\frac{1}{pk}} \left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p_1}{q_1}} |g(y)|^{\frac{qp_1}{2}}\,\mathrm{d}y < \infty, \tag{11}$$

$$\int_0^\infty (r(z))^{\frac{q_1}{p_1}(1+\frac{1}{pk})} \left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})} |h(z)|^{\frac{qq_1}{2}}\,\mathrm{d}z < \infty, \tag{12}$$

$$\int_0^\infty (r(z))^{\frac{p_1}{q_1}-\frac{1}{pk}} \left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac{-p_1}{q_1}} |h(z)|^{\frac{kp_1}{2}}\,\mathrm{d}z < \infty, \tag{13}$$

where $\frac{1}{p}+\frac{1}{q}+\frac{1}{k}=1$, $\frac{1}{p_1}+\frac{1}{q_1}=1$, *then*

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x)g(y)h(z)|}{m(x)+n(y)+r(z)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\quad\le \left(\frac{\pi}{\sin\frac{\pi}{qk}}\beta\left(\frac{1}{q_1qk},\frac{1}{p_1qk}\right)\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^\infty (m(x))^{\frac{p_1}{q_1}-\frac{1}{qk}}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-p_1}{q_1}}|f(x)|^{\frac{pp_1}{2}}\,\mathrm{d}x\right)^{\frac{1}{pp_1}} \\
&\quad\times\left(\int_0^\infty (n(y))^{\frac{q_1}{p_1}(1+\frac{1}{qk})}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})}|g(y)|^{\frac{pq_1}{2}}\,\mathrm{d}y\right)^{\frac{1}{pq_1}} \\
&\quad\times\left(\frac{\pi}{\sin\frac{\pi}{pk}}\beta\left(\frac{1}{q_1pk},\frac{1}{p_1pk}\right)\right)^{\frac{1}{q}} \\
&\quad\times\left(\int_0^\infty (n(y))^{\frac{p_1}{q_1}-\frac{1}{pk}}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p_1}{q_1}}|g(y)|^{\frac{qp_1}{2}}\,\mathrm{d}y\right)^{\frac{1}{qp_1}} \\
&\quad\times\left(\int_0^\infty (r(z))^{\frac{q_1}{p_1}(1+\frac{1}{pk})}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})}|h(z)|^{\frac{qq_1}{2}}\,\mathrm{d}z\right)^{\frac{1}{qq_1}} \\
&\quad\times\left(\frac{\pi}{\sin\frac{\pi}{pq}}\beta\left(\frac{1}{q_1pq},\frac{1}{p_1pq}\right)\right)^{\frac{1}{k}} \\
&\quad\times\left(\int_0^\infty (r(z))^{\frac{p_1}{q_1}-\frac{1}{pk}}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac{-p_1}{q_1}}|h(z)|^{\frac{kp_1}{2}}\,\mathrm{d}z\right)^{\frac{1}{kp_1}} \\
&\quad\times\left(\int_0^\infty (m(x))^{\frac{q_1}{p_1}(1+\frac{1}{pq})}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})}|f(x)|^{\frac{kq_1}{2}}\,\mathrm{d}x\right)^{\frac{1}{kq_1}},
\end{aligned} \tag{14}$$

where the constant factors

$$\frac{\pi}{\sin\frac{\pi}{qk}}\beta\left(\frac{1}{q_1 qk},\frac{1}{p_1 qk}\right), \qquad \frac{\pi}{\sin\frac{\pi}{pk}}\beta\left(\frac{1}{q_1 pk},\frac{1}{p_1 pk}\right), \qquad \frac{\pi}{\sin\frac{\pi}{pq}}\beta\left(\frac{1}{q_1 pq},\frac{1}{p_1 pq}\right)$$

are best possible.

Proof. Since

$$\begin{aligned}
I &= \int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x)\ g(y)h(z)|}{m(x)+n(y)+r(z)}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\
&= \int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x)|^{\frac{1}{2}}\ |g(y)|^{\frac{1}{2}}}{(m(x)+n(y)+r(z))^{\frac{1}{p}}}\left(\frac{m(x)+n(y)}{r(z)}\right)^{\frac{1}{pqk}}\left(\frac{n(y)}{m(x)+n(y)}\right)^{\frac{1}{pqk}}\\
&\qquad\times\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac{1}{p}}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-1}{k}}\\
&\qquad\times\frac{|g(y)|^{\frac{1}{2}}\ |h(z)|^{\frac{1}{2}}}{(m(x)+n(y)+r(z))^{\frac{1}{q}}}\left(\frac{n(y)+r(z)}{m(x)}\right)^{\frac{1}{pqk}}\left(\frac{r(z)}{n(y)+r(z)}\right)^{\frac{1}{pqk}} \qquad (15)\\
&\qquad\times\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{1}{q}}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac{-1}{p}}\\
&\qquad\times\frac{|f(x)|^{\frac{1}{2}}\ |h(z)|^{\frac{1}{2}}}{(m(x)+n(y)+r(z))^{\frac{1}{k}}}\left(\frac{r(z)+m(x)}{n(y)}\right)^{\frac{1}{pqk}}\left(\frac{m(x)}{r(z)+m(x)}\right)^{\frac{1}{pqk}}\\
&\qquad\times\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{1}{k}}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-1}{q}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.
\end{aligned}$$

Applying Hölder's inequality on (15) we get

$$I \le I_1^{\frac{1}{p}} I_2^{\frac{1}{q}} I_3^{\frac{1}{k}} \qquad (16)$$

where

$$\begin{aligned}
I_1 &= \int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x)|^{\frac{p}{2}}\ |g(y)|^{\frac{p}{2}}}{m(x)+n(y)+r(z)}\left(\frac{m(x)+n(y)}{r(z)}\right)^{\frac{1}{qk}}\left(\frac{n(y)}{m(x)+n(y)}\right)^{\frac{1}{qk}} \qquad (17)\\
&\qquad\times\frac{\mathrm{d}r(z)}{\mathrm{d}z}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p}{k}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,
\end{aligned}$$

$$I_2 = \int_0^\infty \int_0^\infty \int_0^\infty \frac{|g(y)|^{\frac{q}{2}} \ |h(z)|^{\frac{q}{2}}}{m(x)+n(y)+r(z)} \left(\frac{n(y)+r(z)}{m(x)}\right)^{\frac{1}{pk}} \left(\frac{r(z)}{n(y)+r(z)}\right)^{\frac{1}{pk}} \times \frac{\mathrm{d}m(x)}{\mathrm{d}x} \left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac{-q}{p}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{18}$$

$$I_3 = \int_0^\infty \int_0^\infty \int_0^\infty \frac{|f(x)|^{\frac{k}{2}} \ |h(z)|^{\frac{k}{2}}}{m(x)+n(y)+r(z)} \left(\frac{r(z)+m(x)}{n(y)}\right)^{\frac{1}{pq}} \left(\frac{m(x)}{r(z)+m(x)}\right)^{\frac{1}{pq}} \times \frac{\mathrm{d}n(y)}{\mathrm{d}y} \left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-k}{q}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{19}$$

Consider the weight coefficient

$$w_1(x,y) = \int_0^\infty \frac{1}{m(x)+n(y)+r(z)} \left(\frac{m(x)+n(y)}{r(z)}\right)^{\frac{1}{qk}} \frac{\mathrm{d}r(z)}{\mathrm{d}z}\mathrm{d}z. \tag{20}$$

Let $v = \dfrac{r(z)}{m(x)+n(y)}$ in (20) then we obtain

$$w_1(x,y) = \frac{\pi}{\sin \dfrac{\pi}{qk}}. \tag{21}$$

Similarly,

$$w_2(y,z) = \int_0^\infty \frac{1}{m(x)+n(y)+r(z)} \left(\frac{n(y)+r(z)}{m(x)}\right)^{\frac{1}{pk}} \mathrm{d}x = \frac{\pi}{\sin \dfrac{\pi}{pk}} \tag{22}$$

and

$$w_3(z,x) = \int_0^\infty \frac{1}{m(x)+n(y)+r(z)} \left(\frac{r(z)+m(x)}{n(y)}\right)^{\frac{1}{pq}} \mathrm{d}x = \frac{\pi}{\sin \dfrac{\pi}{pq}}. \tag{23}$$

Combining (21), (22), (23) and (16) we get

$$\begin{aligned} I \le &\left(\frac{\pi}{\sin \frac{\pi}{qk}} \int_0^\infty \int_0^\infty \left(\frac{n(y)}{m(x)+n(y)}\right)^{\frac{1}{qk}} |f(x)|^{\frac{p}{2}} \ |g(y)|^{\frac{p}{2}} \left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p}{k}} \mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{p}} \\ &\times \left(\frac{\pi}{\sin \frac{\pi}{pk}} \int_0^\infty \int_0^\infty \left(\frac{r(z)}{n(y)+r(z)}\right)^{\frac{1}{pk}} |g(y)|^{\frac{q}{2}} \ |h(z)|^{\frac{q}{2}} \left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac{-q}{p}} \mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{q}} \\ &\times \left(\frac{\pi}{\sin \frac{\pi}{pq}} \int_0^\infty \int_0^\infty \left(\frac{m(x)}{r(z)+m(x)}\right)^{\frac{1}{pq}} |f(x)|^{\frac{k}{2}} \ |h(z)|^{\frac{k}{2}} \left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-k}{q}} \mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{k}}. \end{aligned} \tag{24}$$

Applying Hölder's inequality with $p_1 > 1, \frac{1}{p_1} + \frac{1}{q_1} = 1$ on the first integral on the right side in (24),we have

$$\int_0^\infty\int_0^\infty \left(\frac{n(y)}{m(x)+n(y)}\right)^{\frac{1}{qk}} |f(x)|^{\frac{p}{2}} |g(y)|^{\frac{p}{2}} \left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p}{k}} \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^\infty\int_0^\infty \frac{|f(x)|^{\frac{p}{2}} \left(\frac{n(y)}{m(x)}\right)^{\frac{1}{p_1q_1qk}-\frac{1}{p_1}} m(x)^{\frac{1}{q_1}-\frac{1}{p_1}}}{(m(x)+n(y))^{\frac{1}{p_1qk}}} \left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{1}{p_1}} \left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-1}{q_1}}$$

$$\times \frac{|g(y)|^{\frac{p}{2}} (n(y))^{\frac{1}{qk}+\frac{1}{p_1}-\frac{1}{q_1}}}{(m(x)+n(y))^{\frac{1}{q_1qk}}} \left(\frac{m(x)}{n(y)}\right)^{\frac{1}{p_1q_1qk}-\frac{1}{q_1}}$$

$$\times \left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p}{k}-\frac{1}{p_1}} \left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{1}{q_1}} \mathrm{d}y\mathrm{d}x$$

(25)
$$\leq \left(\int_0^\infty\int_0^\infty \frac{|f(x)|^{\frac{pp_1}{2}}}{(m(x)+n(y))^{\frac{1}{qk}}} \left(\frac{n(y)}{m(x)}\right)^{\frac{1}{q_1qk}-1} (m(x))^{\frac{p_1}{q_1}-1}\right.$$

$$\left.\times \left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-p_1}{q_1}} \frac{\mathrm{d}n(y)}{\mathrm{d}y}\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{p_1}}$$

$$\times \left(\int_0^\infty\int_0^\infty \frac{|g(y)|^{\frac{pq_1}{2}} (n(y))^{\frac{q_1}{qk}+\frac{q_1}{p_1}-1}}{(m(x)+n(y))^{\frac{1}{qk}}} \left(\frac{m(x)}{n(y)}\right)^{\frac{1}{p_1qk}-1}\right.$$

$$\left.\times \left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})} \frac{\mathrm{d}m(x)}{\mathrm{d}x}\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{q_1}}.$$

Let

(26)
$$w_4(x) = \int_0^\infty \frac{1}{\left(1+\frac{n(y)}{m(x)}\right)^{\frac{1}{qk}}} \left(\frac{n(y)}{m(x)}\right)^{\frac{1}{q_1qk}-1} \frac{\mathrm{d}n(y)}{\mathrm{d}y}\mathrm{d}y$$

Putting $v_1 = \dfrac{n(y)}{m(x)}$ in (26) we have

(27)
$$w_4(x) = m(x)\ \beta\left(\frac{1}{q_1qk}, \frac{1}{qk} - \frac{1}{q_1qk}\right) = m(x)\ \beta\left(\frac{1}{q_1qk}, \frac{1}{p_1qk}\right).$$

Similarly

(28)
$$w_5(y) = \int_0^\infty \frac{1}{\left(1+\frac{m(x)}{n(y)}\right)^{\frac{1}{qk}}} \left(\frac{m(x)}{n(y)}\right)^{\frac{1}{p_1qk}-1} \frac{\mathrm{d}m(x)}{\mathrm{d}x}\mathrm{d}x$$

$$= n(y)\ \beta\left(\frac{1}{q_1qk}, \frac{1}{p_1qk}\right).$$

From (27), (28) and (25) we get

$$\int_0^\infty\int_0^\infty \left(\frac{n(y)}{m(x)+y}\right)^{\frac{1}{qk}} |f(x)|^{\frac{p}{2}}\ |g(y)|^{\frac{p}{2}} \left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p}{k}} \mathrm{d}x\,\mathrm{d}y$$

$$\leq \beta\left(\frac{1}{q_1qk},\frac{1}{p_1qk}\right)\left(\int_0^\infty (m(x))^{\frac{p_1}{q_1}-\frac{1}{qk}}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-p_1}{q_1}} |f(x)|^{\frac{pp_1}{2}}\,\mathrm{d}x\right)^{\frac{1}{p_1}} \tag{29}$$

$$\times\left(\int_0^\infty (n(y))^{\frac{q_1}{p_1}(1+\frac{1}{qk})}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})} |g(y)|^{\frac{pq_1}{2}}\,\mathrm{d}y\right)^{\frac{1}{q_1}}.$$

For the second and third integrals on the right side of (24), following the same steps used for obtaining (29), we obtain

$$\int_0^\infty\int_0^\infty \left(\frac{r(z)}{n(y)+r(z)}\right)^{\frac{1}{pk}} |g(y)|^{\frac{q}{2}}\ |h(z)|^{\frac{q}{2}} \left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac{-q}{p}} \mathrm{d}y\mathrm{d}z$$

$$\leq \beta\left(\frac{1}{q_1qk},\frac{1}{p_1qk}\right)\left(\int_0^\infty (n(y))^{\frac{p_1}{q_1}-\frac{1}{pk}}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p_1}{q_1}} |g(y)|^{\frac{qp_1}{2}}\,\mathrm{d}y\right)^{\frac{1}{p_1}} \tag{30}$$

$$\times\left(\int_0^\infty (r(z))^{\frac{q_1}{p_1}(1+\frac{1}{pk})}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})} |h(z)|^{\frac{qq_1}{2}}\,\mathrm{d}z\right)^{\frac{1}{q_1}},$$

and

$$\int_0^\infty\int_0^\infty \left(\frac{m(x)}{r(z)+m(x)}\right)^{\frac{1}{pq}} |h(z)|^{\frac{k}{2}}\ |f(x)|^{\frac{k}{2}} \left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-k}{q}} \mathrm{d}z\mathrm{d}x$$

$$\leq \beta\left(\frac{1}{q_1pq},\frac{1}{p_1pq}\right)\left(\int_0^\infty (r(z))^{\frac{p_1}{q_1}-\frac{1}{pk}}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac{-p_1}{q_1}} |h(z)|^{\frac{kp_1}{2}}\,\mathrm{d}z\right)^{\frac{1}{p_1}} \tag{31}$$

$$\times\left(\int_0^\infty (m(x))^{\frac{q_1}{p_1}(1+\frac{1}{pq})}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})} |f(x)|^{\frac{kq_1}{2}}\,\mathrm{d}x\right)^{\frac{1}{q_1}}.$$

Let

$$\gamma_1=\beta\left(\frac{1}{q_1qk},\frac{1}{p_1qk}\right),\qquad \gamma_2=\beta\left(\frac{1}{q_1qk},\frac{1}{p_1qk}\right),\qquad \gamma_3=\beta\left(\frac{1}{q_1pq},\frac{1}{p_1pq}\right).$$

To prove the constant factors γ_1,γ_2 and γ_3 are best possible. Assume that the constant factor γ_1 is not the best possible, then there exists a positive constant λ_1 with $\lambda_1<\gamma_1$, such that (29) is still valid if we replace γ_1 by λ_1. Without loss of generality, we assume that $m(1) = n(1) = 1$. For $0 < \varepsilon < 1$, setting f_ε and g_ε as $f_\varepsilon(x) = g_\varepsilon(x) = 0$, for $x \in (0,1)$, and for $x \in [1,\infty)$

$$|f_\varepsilon(x)| = (m(x))^{\frac{2}{pp_1}\left(-\frac{p_1}{q_1}+\frac{1}{qk}-\varepsilon-1\right)} \left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{2}{pp_1}\left(\frac{p_1}{q_1}+1\right)}$$

$$|g_\varepsilon(x)| = (n(x))^{\frac{2}{pq_1}\left(-\frac{q_1}{p_1}(1+\frac{1}{qk})-\varepsilon-1\right)} \left(\frac{\mathrm{d}n(x)}{\mathrm{d}x}\right)^{\frac{2}{pq_1}\left(q_1\left(\frac{p}{k}+\frac{1}{p_1}\right)+1\right)},$$

then we obtain

$$\begin{aligned}
&\lambda_1 \left(\int_0^\infty (m(x))^{\frac{p_1}{q_1}-\frac{1}{qk}} \left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-p_1}{q_1}} |f(x)|^{\frac{pp_1}{2}}\,\mathrm{d}x\right)^{\frac{1}{p_1}} \\
&\qquad \times \left(\int_0^\infty (n(y))^{\frac{q_1}{p_1}(1+\frac{1}{qk})} \left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{-q_1(\frac{p}{k}+\frac{1}{p_1})} |g(y)|^{\frac{pq_1}{2}}\,\mathrm{d}y\right)^{\frac{1}{q_1}} \\
&= \lambda_1 \left(\int_1^\infty (m(x))^{-\varepsilon-1}\frac{\mathrm{d}m(x)}{\mathrm{d}x}\mathrm{d}x\right)^{\frac{1}{p_1}} \left(\int_1^\infty (n(y))^{-\varepsilon-1}\frac{\mathrm{d}n(y)}{\mathrm{d}y}\mathrm{d}y\right)^{\frac{1}{q_1}} \\
&= \frac{\lambda_1}{\varepsilon}.
\end{aligned}$$

But we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \left(\frac{n(y)}{m(x)+y}\right)^{\frac{1}{qk}} |f(x)|^{\frac{p}{2}}\ |g(y)|^{\frac{p}{2}} \left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p}{k}} \mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_1^\infty\int_1^\infty \left(\frac{n(y)}{m(x)+y}\right)^{\frac{1}{qk}} (m(x))^{\frac{1}{p_1qk}-\frac{\varepsilon}{p_1}-1}(n(y))^{-\frac{1}{p_1qk}-\frac{\varepsilon}{q_1}-1} \\
&\qquad\qquad \left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)\mathrm{d}x\,\mathrm{d}y = J_1 \\
&\quad \geq \frac{1}{\varepsilon}(\gamma_1 + o(1)) - O(1).
\end{aligned}$$

Hence we find

$$\frac{1}{\varepsilon}(\gamma_1 + o(1)) - O(1) < \frac{\lambda_1}{\varepsilon} \tag{32}$$

or

$$\gamma_1 + o(1) - \varepsilon O(1) < \lambda_1. \tag{33}$$

For $\varepsilon \to 0^+$, it follows that $\gamma_1 \leq \lambda_1$. This contradicts the fact that $\lambda_1 < \gamma_1$. Hence the constant factor γ_1 in (29) is the best possible. Similarly γ_2 and γ_3 are the best possible. Substituting from (29), (30), (31) in (24), the result of the theorem follows. □

Remark 1. Putting $p = q = k = \frac{1}{3}$ in (14), we get a new inequality in the form

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x)\,g(y)h(z)|}{m(x)+n(y)+r(z)}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$\leq \left(\frac{\pi}{\sin\frac{\pi}{9}}\beta\left(\frac{1}{9q_1},\frac{1}{9p_1}\right)\right)\left(\int_0^\infty (m(x))^{\frac{p_1}{q_1}-\frac{1}{9}}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{\frac{-p_1}{q_1}}|f(x)|^{\frac{3p_1}{2}}\,\mathrm{d}x\right)^{\frac{1}{3p_1}}$$
$$\times\left(\int_0^\infty (n(y))^{\frac{10q_1}{9p_1}}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{-q_1(1+\frac{1}{p_1})}|g(y)|^{\frac{3q_1}{2}}\,\mathrm{d}y\right)^{\frac{1}{3q_1}}$$
$$\times\left(\int_0^\infty (n(y))^{\frac{p_1}{q_1}-\frac{1}{9}}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p_1}{q_1}}|g(y)|^{\frac{3p_1}{2}}\,\mathrm{d}y\right)^{\frac{1}{3p_1}} \tag{34}$$
$$\times\left(\int_0^\infty (r(z))^{\frac{10q_1}{9p_1}}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{-q_1(1+\frac{1}{p_1})}|h(z)|^{\frac{3q_1}{2}}\,\mathrm{d}z\right)^{\frac{1}{3q_1}}$$
$$\times\left(\int_0^\infty (r(z))^{\frac{p_1}{q_1}-\frac{1}{9}}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{\frac{-p_1}{q_1}}|h(z)|^{\frac{kp_1}{2}}\,\mathrm{d}z\right)^{\frac{1}{3p_1}}$$
$$\times\left(\int_0^\infty (m(x))^{\frac{10q_1}{9p_1}}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{-q_1(1+\frac{1}{p_1})}|f(x)|^{\frac{3q_1}{2}}\,\mathrm{d}x\right)^{\frac{1}{3q_1}}.$$

Remark 2. Putting $p_1 = q_1 = 2$ in (14) we obtain

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x)\,g(y)h(z)|}{m(x)+n(y)+r(z)}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$\leq \left(\frac{\pi}{\sin\frac{\pi}{qk}}\beta\left(\frac{1}{2qk},\frac{1}{2qk}\right)\right)^{\frac{1}{p}}\left(\int_0^\infty (m(x))^{1-\frac{1}{qk}}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{-1}|f(x)|^q\,\mathrm{d}x\right)^{\frac{1}{2p}}$$
$$\times\left(\int_0^\infty (n(y))^{(1+\frac{1}{qk})}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{-2(\frac{p}{k}+\frac{1}{2})}|g(y)|^p\,\mathrm{d}y\right)^{\frac{1}{2p}} \tag{35}$$
$$\times\left(\frac{\pi}{\sin\frac{\pi}{pk}}\beta\left(\frac{1}{2pk},\frac{1}{2pk}\right)\right)^{\frac{1}{q}}\left(\int_0^\infty (n(y))^{1-\frac{1}{pk}}\left(\frac{\mathrm{d}n(y)}{\mathrm{d}y}\right)^{\frac{-p_1}{q_1}}|g(y)|^p\,\mathrm{d}y\right)^{\frac{1}{2q}}$$
$$\times\left(\int_0^\infty (r(z))^{(1+\frac{1}{pk})}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{-2(\frac{p}{k}+\frac{1}{2})}|h(z)|^q\,\mathrm{d}z\right)^{\frac{1}{2q}}$$

$$\times\left(\frac{\pi}{\sin\frac{\pi}{pq}}\beta\left(\frac{1}{2pq},\frac{1}{2pq}\right)\right)^{\frac{1}{k}}\left(\int_0^\infty (r(z))^{1-\frac{1}{pk}}\left(\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right)^{-1}|h(z)|^k\,\mathrm{d}z\right)^{\frac{1}{2k}}$$
$$\times\left(\int_0^\infty (m(x))^{(1+\frac{1}{pq})}\left(\frac{\mathrm{d}m(x)}{\mathrm{d}x}\right)^{-2\left(\frac{p}{k}+\frac{1}{2}\right)}|f(x)|^k\,\mathrm{d}x\right)^{\frac{1}{2k}},$$

which is a new inequality.

Remark 3. Let $m(x)=x$, $n(y)=y$ and $r(z)=z$ in (34) and (35) we get respectively

$$\int_0^\infty\int_0^\infty\int_0^\infty\frac{|f(x)\,g(y)h(z)|}{x+y+z}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$\leq\frac{\pi}{\sin\frac{\pi}{9}}\beta\left(\frac{1}{9q_1},\frac{1}{9p_1}\right)\left(\left(\int_0^\infty x^{\frac{p_1}{q_1}-\frac{1}{9}}|f(x)|^{\frac{3p_1}{2}}\,\mathrm{d}x\right)\left(\int_0^\infty y^{\frac{p_1}{q_1}-\frac{1}{9}}|g(y)|^{\frac{3p_1}{2}}\,\mathrm{d}y\right)\right.$$
$$\left.\times\left(\int_0^\infty z^{\frac{p_1}{q_1}-\frac{1}{9}}|h(z)|^{\frac{3p_1}{2}}\,\mathrm{d}z\right)\right)^{\frac{1}{3p_1}}$$
$$\times\left(\left(\int_0^\infty x^{\frac{10q_1}{9p_1}}|f(x)|^{\frac{3q_1}{2}}\,\mathrm{d}x\right)\left(\int_0^\infty y^{\frac{10q_1}{9p_1}}|g(y)|^{\frac{3q_1}{2}}\,\mathrm{d}y\right)\right.$$
$$\left.\times\left(\int_0^\infty z^{\frac{10q_1}{9p_1}}|h(z)|^{\frac{3q_1}{2}}\,\mathrm{d}z\right)\right)^{\frac{1}{3q_1}}.$$

and

$$\int_0^\infty\int_0^\infty\int_0^\infty\frac{|f(x)\,g(y)h(z)|}{x+y+z}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$\leq\left(\frac{\pi}{\sin\frac{\pi}{qk}}\beta\left(\frac{1}{2qk},\frac{1}{2qk}\right)\right)^{\frac{1}{p}}\left(\left(\int_0^\infty x^{1-\frac{1}{qk}}|f(x)|^p\,\mathrm{d}x\right)\left(\int_0^\infty y^{\left(1+\frac{1}{qk}\right)}|g(y)|^p\,\mathrm{d}y\right)\right)^{\frac{1}{2p}}$$
$$\times\left(\frac{\pi}{\sin\frac{\pi}{pk}}\beta\left(\frac{1}{2pk},\frac{1}{2pk}\right)\right)^{\frac{1}{q}}\left(\left(\int_0^\infty y^{1-\frac{1}{pk}}|g(y)|^q\,\mathrm{d}y\right)\left(\int_0^\infty z^{1+\frac{1}{pk}}|h(z)|^q\,\mathrm{d}z\right)\right)^{\frac{1}{2q}}$$
$$\times\left(\frac{\pi}{\sin\frac{\pi}{pq}}\beta\left(\frac{1}{2pq},\frac{1}{2pq}\right)\right)^{\frac{1}{k}}\left(\left(\int_0^\infty z^{1-\frac{1}{pk}}|h(z)|^k\,\mathrm{d}z\right)\left(\int_0^\infty x^{1+\frac{1}{pq}}|f(x)|^k\,\mathrm{d}x\right)\right)^{\frac{1}{2k}}.$$

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