MODIFIED MULTISTEP ITERATION FOR APPROXIMATING A GENERAL CLASS OF FUNCTIONS IN LOCALLY CONVEX SPACES

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ABSTRACT. In this paper, we study the convergence of modified multistep iteration and use the scheme to approximate the fixed point of a general class of functions introduced by Bosede and Rhoades [5] in a complete metrisable locally convex space. As corollaries, the convergence results for SP and Mann iterations are also established. Moreover, most convergence results in Banach spaces are generalized to complete metrisable locally convex spaces. Our convergence results generalize and extend the results of Berinde [2], Olaleru [11], Phuengrattana and Suantai [13], Rafiq [14] among others.

1. INTRODUCTION AND PRELIMINARY DEFINITIONS

A locally Convex space (X, u) with topology u is a topological vector space which has a local base of convex neighborhoods of zero [20, Chap. 7]. It is metrisable if it is Hausdorff and has countable zero basis. Consequently, it is metrisable if u can be described by a countable family of continuous seminorms [20]. X is Hausdorff if and only if for each non-zero $x \in X$, there is $p \in Q$ with $p(x) > 0$ [11]. A seminorm p corresponds to each absolutely convex absorbent subset U of X is called the gauge of U defined by $p(x) = \inf\{\lambda : \lambda > 0, x \in \lambda U\}$ and with the property that $\{x : p(x) < 1\} \subseteq U \subseteq \{x : p(x) \leq 1\}$, U is a neighborhood of zero if and only if p is continuous.
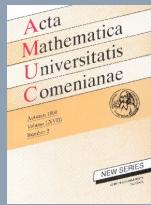
Theorem 1.1 ([**11**]). *The topology of a metrizable locally convex space can always be defined by a metric which is invariant under translation.*

Proof. For details of proof see Olaleru [**11**]. □

Let X be a metrisable topological space and C be a nonempty subset of X and $T: C \to C$ a self map of C. For $x_0 \in C$, the sequence $\{x_n\}_{n=1}^{\infty}$

$$(1.1) \qquad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n,$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called the Mann iterative process [9].

Olaleru [**11**] proved the convergence of Mann iterative process using the Zamfirescu operators [**22**] and generalized several results in literature to complete metrisable locally convex spaces.

For $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$(1.2) \qquad \begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n \\ y_n &= (1 - \beta_n)x_n + \beta_n Tx_n, \end{aligned}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called Ishikawa iterative scheme [**7**].

Observe that if $\beta_n = 0$ for each n, then the Ishikawa iterative scheme (1.2) reduces to the Mann iterative scheme (1.1).

For $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$(1.3) \qquad \begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n Tz_n, \\ z_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n \end{aligned}$$
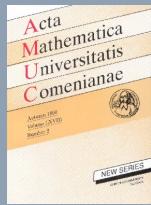
where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called the Noor iterative (or three-step) scheme [10].

Also observe that if $\gamma_n = 0$ for each n, then the Noor iteration process (1.3) reduces to the Ishikawa iterative scheme (1.2).

For $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$
\begin{aligned}
x_{n+1} &= (1 - \alpha_n)y_n + \alpha_n T y_n, \\
y_n &= (1 - \beta_n)z_n + \beta_n T z_n, \\
z_n &= (1 - \gamma_n)x_n + \gamma_n T x_n
\end{aligned}
$$

(1.4)

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ satisfying $\beta_n \leq \alpha_n$, $\gamma_n \leq \alpha_n$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called the SP-iterative (or modified three-step) process [13].

Also observe that if $\gamma_n = 0$ for each n and $z_n = x_n$, then the SP-iteration process (1.4) reduces to the modified Ishikawa iterative scheme

$$
\begin{aligned}
x_{n+1} &= (1 - \alpha_n)y_n + \alpha_n T y_n, \\
y_n &= (1 - \beta_n)x_n + \beta_n T x_n.
\end{aligned}
$$

(1.4b)

If $\gamma_n = 0$ and $\beta_n = 0$ for each n, then the SP-iteration process (1.4) reduces to the Mann iterative process (1.1).

In 2011, Phuengrattana and Suantai [13] used the SP-iterative process (1.4) to approximate the fixed point of continuous functions on an arbitrary interval. They also compared the convergence speed of Mann, Ishikawa, Noor and SP-iterative processes and proved that the SP-iterative process converges faster than the others.
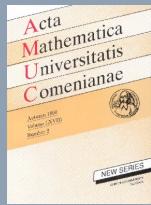
For $x_0 \in C$. The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n^1$$

(1.5)
$$y_n^i = (1 - \beta_n^i)x_n + \beta_n^i T y_n^{i+1},$$

$$y_n^{p-1} = (1 - \beta_n^{p-1})x_n + \beta_n^{p-1} T x_n, \qquad i = 1, 2, \ldots, p-2, \ p \geq 2$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^i\}_{n=0}^{\infty}$, $i = 1, 2, \ldots, p-1$ are real sequences in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called a multistep iteration scheme [**22**].

Observe that the multistep iteration is a generalization of the Noor, Ishikawa and the Mann iterations. In fact, if $p = 1$ in (1.5), we have the Mann iteration (1.2); if $p = 2$ in (1.5), we have the Ishikawa iteration (1.3) and if $p = 3$, we have the Noor iteration (1.4).

Several generalizations of the Banach fixed point theorem have been proved to date (for example, see [**2**], [**15**] and [**22**]). One of the most commonly studied generalization hitherto is the one proved by Zamfirescu [**21**] in 1972, which is stated as follows.

Theorem 1.2 ([**22**]). *Let X be a complete metric space and $T: X \to X$ a Zamfirescu operator satisfying*

(1.6)
$$d(Tx, Ty)$$
$$\leq h \max\{d(x,y), \frac{1}{2}[d(x,Tx) + d(y,Ty)], \frac{1}{2}[d(x,Ty) + d(y,Tx)]\}$$

where $0 \leq h < 1$. Then, T has a unique fixed point and the Picard iteration converges to p for any $x_0 \in X$.

Observe that in a Banach space setting, condition (1.5) implies

(1.7)
$$\|Tx - Ty\| \leq \delta\|x - y\| + 2\delta\|x - Tx\|$$

where $0 \leq \delta < 1$ and $\delta = \max\{h, \frac{h}{2-h}\}$, for details of proof see [**3**].
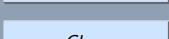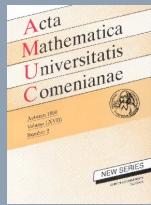
Several papers have been written on the Zamfirescu operators (1.6), (for example, see [2], [15], [18], [22]). The most commonly used methods of approximating the fixed points of the Zamfirescu operators are Picard, Mann [8], Ishikawa [6] and Noor [10] iterative processes. Rhoades [16, 17] used the Zamfirescu contraction condition (1.7) to obtan some convergence results for Mann and Ishikawa iterative processes in a uniformly Banach space. Berinde [1] extended the results of the author [16, 17] to arbitrary Banach space for the same fixed point iteration procedures. Rafiq [15] proved the convergence of Noor iterative process (1.3) using the Zamfirescu operators defined by (1.7). Osilike [12] proved several stability results and generalized several results in literature using the following contractive definition. For each $x, y \in X$, there exist $\delta \in [0, 1)$ and $L \geq 0$ such that

$$(1.8) \qquad\qquad d(Tx, Ty) \leq \delta d(x, y) + L d(x, Tx).$$

In 2003, Imoru and Olatinwo [6] proved some stability results and generalized some known results in the literature using the follwing general contractive definition.
For each $x, y \in X$, there exist $\delta \in [0, 1)$ and a monotone increasing function $\varphi \colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$(1.9) \qquad\qquad d(Tx, Ty) \leq \delta d(x, y) + \varphi(d(x, Tx)).$$

In 2010, Bosede and Rhoades [4] observed that the process of "generalizing" (1.8) could continue ad infinitum. As a result of this observation, the authors [4], introduced the following class of functions and proved the stability of Picard and Mann iterative schemes. That is, if $x = p$ (is a fixed point), then (1.8) becomes

$$(1.10) \qquad\qquad d(p, Ty) \leq \delta d(p, y)$$

for some $0 \leq \delta < 1$ and all $x, y \in X$.
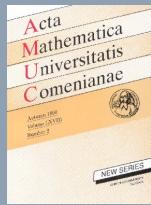
The contractive condition (1.10) is more general than those considered by Imoru and Olatinwo [6], Osilike [12] and several others in the sense that if by replacing L in (1.8) with more complicated expressions, the process of "generalizing" (1.8) could continue ad infinitum. Also, the condition "$\varphi(0) = 0$" usually imposed by Imoru and Olatinwo [6] in the contractive definition (1.9) is no longer necessary in the contractive condition (1.10). However, Bosede [3] also proved strong convergence of the Noor iterative process for this general class of functions.

Definition 1.3. Let X be a metrisable topological space and C be a nonempty subset of X and $T: C \to C$ a self map of C. For $x_0 \in C$, the iteration procedure defined by (1.4) such that the generated sequence $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. Let $\{u_n\}_{n=0}^{\infty}$ be arbitrary sequence in X and set $\varepsilon_n = f_c(u_{n+1} - g(T, u_n))$, for $n \geq 0$. We say the iteration procedure (1.4) is T-stable if and only if $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} u_n = p$.

Definition 1.4. Let $x_0 \in C$. The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

(1.11)
$$x_{n+1} = (1 - \alpha_n)y_n^1 + \alpha_n T y_n^1$$
$$y_n^i = (1 - \beta_n^i)y_n^{i+1} + \beta_n^i T y_n^{i+1},$$
$$y_n^{p-1} = (1 - \beta_n^{p-1})x_n + \beta_n^{p-1} T x_n, \qquad i = 1, 2, \ldots, p-2, \ p \geq 2$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}, i = 1, 2, \ldots, p-1$ are real sequences in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called the modified multistep iteration scheme.

Motivated by the above results, we introduce the following modified multistep iteration scheme (1.11) and use it to approximate the fixed point of a general class of functions introduced by Bosede and Rhoades [4] in a complete metrisable locally convex space. As corollaries, some strong convergence results are obtained for SP and Mann iterative schemes for this general class of funtions (1.10). Our convergence results generalize and extend the results of Berinde [2], Bosede and Rhoades [4], Olaleru [11], Phuengrattana and Suantai [13] and Rafiq [14] among others.
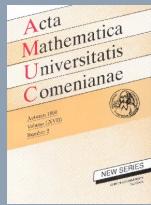
We shall need the following Lemma which appeared in [1] to prove our results.

Lemma 1.5 ([21]). *Let δ be a real number satisfying $0 \leq \delta < 1$ and $\{\varepsilon_n\}_{n=0}^{\infty}$ a sequence of positive numbers such that $\lim_{n \to \infty} \varepsilon_n = 0$. Then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying $u_{n+1} \leq \delta u_n + \varepsilon_n$, $n = 0, 1, 2, \ldots$, we have $\lim_{n \to \infty} u_n = 0$.*

2. MAIN RESULT

Theorem 2.1. *Let (X, f_c) be a complete metrisable locally convex space, K a closed convex subset of X and $T \colon K \to K$ be an operator with a fixed point p satisfying the condition*

$$(2.1) \qquad\qquad f_c(p - Ty) \quad \leq \quad \delta f_c(p - y)$$

for each $y \in K$ and $0 \leq \delta < 1$. For $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the modified multistep iteration scheme defined by (1.11) converging to p (that is, $Tp = p$), where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^i\}_{n=0}^{\infty}$ (for each i) are real sequences in $[0, 1]$. Then the modified multistep iteration scheme (1.11) converges strongly to p.

Proof. In view of (1.11) and (2.1), we have

$$(2.2) \qquad \begin{aligned} f_c(x_{n+1} - p) &\leq f_c((1 - \alpha_n)y_n^1 + \alpha_n Ty_n^1 - (1 - \alpha_n + \alpha_n)p) \\ &\leq (1 - \alpha_n)f_c(y_n^1 - p) + \alpha_n f_c(Ty_n^1 - p) \\ &\leq (1 - \alpha_n)f_c(y_n^1 - p) + \delta\alpha_n f_c(y_n^1 - p) \\ &= [1 - \alpha_n(1 - \delta)]f_c(y_n^1 - p). \end{aligned}$$
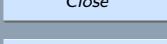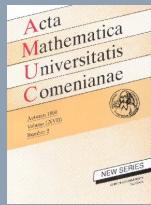
An application of (1.11) and (2.1) with $y = y_n^1$ gives

$$f_c(y_n^1 - p) = f_c((1 - \beta_n^1)y_n^2 + \beta_n^1 Ty_n^2 - (1 - \beta_n^1 + \beta_n^1)p\|$$

(2.3)
$$\leq (1 - \beta_n^1)f_c(y_n^2 - p) + \beta_n^1 f_c(Ty_n^2 - p)$$
$$\leq (1 - \beta_n^1)f_c(y_n^2 - p) + \delta\beta_n^1 f_c(y_n^2 - p)$$
$$= [1 - \beta_n^1(1 - \delta)]f_c(y_n^2 - p).$$

Also an application of (1.11) and (2.1) with $y = y_n^2$ gives

$$f_c(y_n^2 - p) = f_c((1 - \beta_n^2)y_n^3 + \beta_n^2 Ty_n^3 - (1 - \beta_n^2 + \beta_n^2)p$$

(2.4)
$$\leq (1 - \beta_n^2)f_c(y_n^3 - p) + \beta_n^2 f_c(Ty_n^3 - p)$$
$$\leq (1 - \beta_n^2)f_c(y_n^3 - p) + \delta\beta_n^2 f_c(y_n^3 - p)$$
$$= [1 - \beta_n^2(1 - \delta)]f_c(y_n^3 - p).$$

Similarly, an application of (1.11) and (2.1) with $y = y_n^3$ gives

$$f_c(y_n^3 - p) \leq (1 - \beta_n^3)f_c(y_n^4 - p) + \beta_n^3 f_c(Ty_n^4 - p)$$

(2.5)
$$\leq (1 - \beta_n^3)f_c(y_n^4 - p) + \delta\beta_n^3 f_c(y_n^4 - p)$$
$$= (1 - \beta_n^3)f_c(y_n^4 - p) + \delta\beta_n^3 f_c(y_n^4 - p)$$
$$= [1 - \beta_n^3(1 - \delta)](y_n^4 - p).$$

Continuing the above process, we have

$$f_c(x_{n+1} - p) \leq [1 - \alpha_n(1 - \delta)][1 - \beta_n^1(1 - \delta)][1 - \beta_n^2(1 - \delta)][1 - \beta_n^3(1 - \delta)]$$

(2.6)
$$\ldots [1 - \beta_n^{k-2}(1 - \delta)]f_c(y_n^{k-1} - p).$$
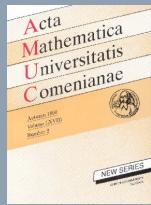
An application of (1.11) and (2.1) also gives

$$
\begin{aligned}
f_c(y_n^{k-1} - p) &\le (1 - \beta_n^{k-1})f_c(x_n - p) + \beta_n^{k-1}f_c(Tx_n - p) \\
&\le (1 - \beta_n^{k-1})f_c(x_n - p) + \delta\beta_n^{k-1}f_c(x_n - p) \\
&= (1 - \beta_n^{k-1})f_c(x_n - p) + \delta\beta_n^{k-1}f_c(x_n - p) \\
&= [1 - \beta_n^{k-1}(1 - \delta)]f_c(x_n - p).
\end{aligned}
$$
(2.7)

Substituting (2.7) in (2.6), we have

(2.8)
$$
\begin{aligned}
f_c(x_{n+1} - p) &\le [1 - \alpha_n(1 - \delta)][1 - \beta_n^1(1 - \delta)][1 - \beta_n^2(1 - \delta)][1 - \beta_n^3(1 - \delta)] \\
&\qquad \dots [1 - \beta_n^{k-2}(1 - \delta)][1 - \beta_n^{k-1}(1 - \delta)]f_c(x_n - p) \\
&\le [1 - \alpha_n(1 - \delta)]f_c(x_n - p) \\
&\le \prod_{j=0}^{n}[1 - \alpha_j(1 - \delta)]f_c(x_0 - p) \\
&\le \mathrm{e}^{-(1-\delta)\sum_{j=0}^{\infty}\alpha_j} f_c(x_0 - p).
\end{aligned}
$$

Since $0 \le \delta < 1$, $\alpha_j \in [0, 1)$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$, so $\mathrm{e}^{-(1-\delta)\sum_{j=0}^{\infty}\alpha_j} \to 0$ as $n \to \infty$. Thus $\lim_{n\to\infty} f_c(x_{n+1} - p) = 0$.

Therefore, $\{x_n\}_{n=0}^{\infty}$ converges strongly to p. □

Theorem 2.1 leads to the following corollaries:

Corollary 2.2. *Let (X, f_c) be a complete metrisable locally convex space, K a closed convex subset of X and $T\colon K \to K$ be an operator with a fixed point p satisfying the condition*

(2.9)
$$
f_c(p - Ty) \le \delta f_c(p - y)
$$
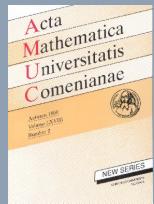
*for each $y \in K$ and $0 \leq \delta < 1$. For $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the SP-iteration scheme defined by* (1.4) *converging to p (that is, $Tp = p$), where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in* [0,1]. *Then the SP-iteration scheme converges strongly to p.*

Corollary 2.3. *Let (X, f_c) be a complete metrisable locally convex space, K a closed convex subset of X and $T \colon K \to K$ be an operator with a fixed point p satisfying the condition*

$$(2.10) \qquad\qquad f_c(p - Ty) \quad \leq \quad \delta f_c(p - y)$$

for each $y \in K$ and $0 \leq \delta < 1$. For $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the Mann iteration scheme defined by (1.1) *converging to p (that is, $Tp = p$), where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequences in* $[0, 1]$. *Then the Mann iteration scheme converges strongly to p.*

Remark 1. Our Theorem 2.1 improves several known results in literature including the results of Berinde [2] and Rhoades [18, Theorem 2] by extending the space to complete metrisable locally convex spaces. The fact that there are complete metrisable spaces including many useful function spaces that are not normable makes our Corollary 2.2 a useful generalization of Berinde's theorem [2].

1. Berinde V., *Iterative approximation of fixed points*, Baia Mare, Editura Efemeride (2002).
2. ———, *A convergence theorem for Mann iteration in the class of Zamfirescu operators*, Seria Mathematica Informatica **XLV(1)**, 33–41.
3. Bosede A. O., *Strong convergence of Noor iteration for a general class of functions*, Bulletin of Mathematical Analysis and Applications **3(4)** (2011), 140–145.
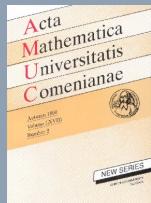
4. Bosede A. O. and Rhoades B. E., *Stability of Picard and Mann iteration for a general class of functions*, Journal of Advanced Mathematical Studies **3(2)** (2010), 1–3.

5. Chatterjea S. K., *Fixed point theorems*, R. Acad. Bulgare Sci. **25(6)** (1972), 727–730.

6. Imoru C. O. and Olatinwo M. O., *On the stability of Picard and Mann iteration*, Carpathian J. Math. **19** (2003), 155–160.

7. Ishikawa S., *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147–150.

8. Kannan R., *Some results on fixed points*, Bulletin Calcutta Mathematical Society **10** (1968), 71–76.

9. Mann W. R., *Mean value methods in iterations*, Proc. Amer. Math. Soc. **44** (1953), 506–510.

10. Noor M. A., *New approximation schemes for general variational inequalities*, Math. Anal. Appl. **251** (2000), 217–229.

11. Olaleru J. O., *On the convergence of the Mann iteration in locally convex spaces*, Carpathian J. Math. **22(1–2)** (2006), 115–120.

12. Osilike M. O., *Stability results for Ishikawa fixed point iteration procedure*, Indian J. Pure Appl. Math. **26(10)** (1995/96), 937–941.

13. Phuengrattana W. and Suantai S., *On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval*, Comput. Appl. Math. **235** (2011), 3006–3014.

14. Rafiq A., *A convergence theorem for Mann fixed point iteration procedure*, Applied Mathematics E-Notes, **6** (2006), 289–293.

15. _____ , *On the convergence of the three-step iteration process in the class of quasi-contractive operators*, Acta Math. Acad. Paedagog. Nyhazi. (N.S.) **22** (2006), 305–309.

16. Rhoades B. E., *Fixed point iteration using infinite matrices*, Trans. Amer. Math. Soc. **196** (1974), 161–176.

17. _____ , *A comparison of various definition of contractive mapping*, Trans. Amer. Math. Soc. **226** (1977), 257–290.

18. _____ , *Fixed point theorems and stability results for fixed point iteration procedure II*, Indian J. Pure Appl. Math. **24(11)** (1993), 691–703.

19. Rhoades B. E. and Soltuz S. M., *The equivalence between Mann-Ishikawa iterations and multi-step iteration*, Nonlinear Analysis, **58(2004)**, 219–228.

20. Schaffer H. H., *Topological Vector Spaces*, Springer-Verlag, 1999.

21. Weng X., *Fixed point iteration for local strictly pseudo-contractive mapping*, Proc. Amer. Math. Soc. **113(3)** (1991), 727–731.

22. Zamfirescu T., *Fixed point theorems in metric spaces*, Arch. Math. (Basel) **23(1972)**, 292–298.
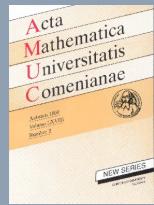
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