

RANDOM CHORDS AND POINT DISTANCES IN REGULAR POLYGONS

U. BÄSEL

ABSTRACT. In this paper we obtain the chord length distribution function for any regular polygon. From this function we conclude the density function and the distribution function of the distance between two uniformly and independently distributed random points in the regular polygon. The method calculating the chord length distribution function is quite different from those of Harutyunyan and Ohanyan. It uses only elementary methods and provides the result with only a few natural case distinctions.

1. INTRODUCTION

A random line g intersecting a convex set \mathcal{K} in the plane produces a chord of \mathcal{K} . The length s of this chord is a random variable. If the motion invariant line measure (see below) is used for the definition of the line, the expectation of the chord length is equal to $\pi A/u$ where A is the area of \mathcal{K} and u is the length of its perimeter [19, p. 30]. The chord length distribution function of a regular triangle was calculated by Sulanke [20, p. 57]. Harutyunyan and Ohanyan [13] calculated the chord length distribution function for regular polygons using Dirac's δ -function in Pleijel's identity. Bertrand's paradox associated with the chord length distribution of a circle is well-known [8, pp. 116–118], [16, pp. 172–179].

The distance t between two points chosen independently and uniformly at random from \mathcal{K} is also a random variable. Borel [4] considered this distance in elementary geometric figures such as triangles, squares and so on (see [17, p. 163]). The expectations for the distance between two random points for an equilateral triangle and a rectangle are to be found in [19, p. 49]. Ghosh [11] derived the distance distribution for a rectangle. Bailey, Borwein & Crandall [3] studied the expected distance between two random points in the unit n -cube giving closed form expressions for the cases $n = 1, \dots, 5$ (see also [5], especially Example 14). There are a lot of results concerning the distance t within a convex set or in two convex sets (see Chapter 2 in [16], and [7]).

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The moments of s and t , resp., are closely connected by a simple formula [19, pp. 46–47]. The second moments of the chord length for regular polygons were obtained by Heinrich [14].

For practical applications of chord lengths and point distances of convex sets in physics, material sciences, operations research and other fields see [12] and [15].

The first aim of the present paper is to derive the chord length distribution function for any regular polygon in a simple form with only a few natural case distinctions using a method that requires only elementary geometric considerations and elementary integrations (especially not using Dirac's δ -function in Pleijel's identity as done in [13]). Our method is also suitable for irregular and even (with slight modifications) non-convex polygons as shown in [2]. The second aim is to conclude the density function and the distribution function of the distance between two random points in every regular polygon. This result is new to the author's knowledge. By $\mathcal{P}_{n,r}$ we denote the regular polygon with n sides and circumscribed

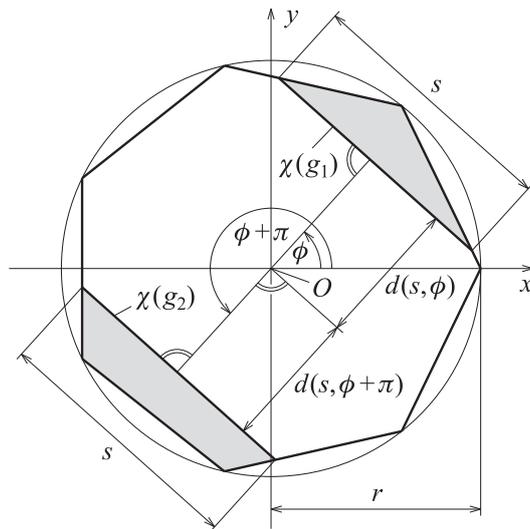


Figure 1. The polygon $\mathcal{P}_{n,r}$ (example $n = 7$).

circle with radius r and centre point in the origin O (see Fig. 1). A straight line g in the plane is determined by the angle ϕ , $0 \leq \phi < 2\pi$, between the direction perpendicular to g and a fixed direction (e.g., the x -axis), and by its distance p , $0 \leq p < \infty$, from the origin O

$$g = g(p, \phi) = \{(x, y) \in \mathbb{R}^2 : x \cos \phi + y \sin \phi = p\}.$$

The measure μ of a set of lines $g(p, \phi)$ is defined by the integral, over the set, of the differential form $dg = dp d\phi$. Up to a constant factor, this measure is the only one that is invariant under motions in the Euclidean plan [19, p. 28].

The chord length distribution function of $\mathcal{P}_{n,r}$ is usually defined as

$$F(s) = \frac{1}{u} \mu(\{g : g \cap \mathcal{P}_{n,r} \neq \emptyset, |\chi(g)| \leq s\}),$$

where $\chi(g) = g \cap \mathcal{P}_{n,r}$ is the chord of $\mathcal{P}_{n,r}$, produced by the line g , $|\chi(g)|$ is the length of $\chi(g)$, and u is the length of the perimeter of $\mathcal{P}_{n,r}$. (The measure of all lines g that intersect a convex set is equal to its perimeter [19, p. 30].) We use the distribution function in the form

$$(1) \quad F(s) = 1 - \frac{1}{u} \mu(\{g : g \cap \mathcal{P}_{n,r} \neq \emptyset, |\chi(g)| > s\})$$

(cf. [1, p. 161]). So it remains to calculate the measure of all lines that produce a chord of length $|\chi(g)| > s$. Using the abbreviation

$$\mathcal{S}(g:s) := \{g : g \cap \mathcal{P}_{n,r} \neq \emptyset, |\chi(g)| > s\},$$

we have

$$\mu(\mathcal{S}(g:s)) = \int_{\mathcal{S}(g:s)} dg = \int_{\mathcal{S}(g:s)} dp d\phi.$$

We consider all lines g , having a direction perpendicular to a fixed angle $\phi \in [0, \pi)$ with $g \cap \mathcal{P}_{n,r} \neq \emptyset$. In almost all cases among these lines there are two lines g_1 and g_2 with chords of equal length s (see Fig. 1). All parallel lines g lying in the strip between g_1 and g_2 have a chord with length $|\chi(g)| > s$. The breadth of this strip is equal to $d(s, \phi) + d(s, \phi + \pi)$, where $d(s, \phi)$ and $d(s, \phi + \pi)$ are the distances between O and g_1 and O and g_2 , respectively. So we have

$$(2) \quad \mu(\mathcal{S}(g:s)) = \int_0^\pi [d(s, \phi) + d(s, \phi + \pi)] d\phi.$$

2. THE DISTANCE FUNCTION

In the following we determine the distance function in formula (2)

$$(3) \quad d: [0, \max(s)] \times [0, \infty) \rightarrow [0, r], \quad (s, \phi) \mapsto d(s, \phi),$$

where $\max(s)$ is the maximum chord length s in $\mathcal{P}_{n,r}$. If no chord of length s in the direction perpendicular to ϕ exists, we put $d(s, \phi) = 0$. Of course for fixed value of s , $d(s, \cdot)$ is a $2\pi/n$ -periodic function.

We put

$$K = \left\lfloor \frac{n-2}{2} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ is the integer part of \cdot , and define the function $m: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ by

$$m(k, n) = \begin{cases} k \bmod n & \text{if } k \bmod n \neq 0, \\ n & \text{if } k \bmod n = 0. \end{cases}$$

The angle δ_k (see Fig. 2) between the lines i and $m(i+k, n)$ containing the sides i , $i = 1, \dots, n$, and $m(i+k, n)$ of $\mathcal{P}_{n,r}$ is given by

$$\delta_k = \left(1 - \frac{2k}{n}\right) \pi, \quad k = 1, \dots, K^*,$$

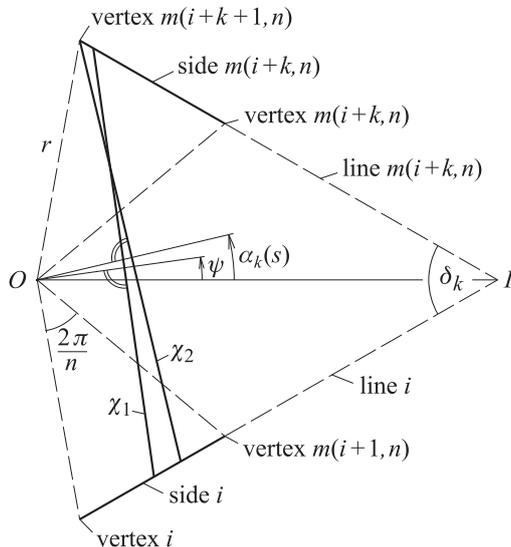


Figure 2. Chords χ between side i and side $m(i+k, n)$.

where

$$K^* = \begin{cases} K+1 & \text{if } n \text{ is odd,} \\ K & \text{if } n \text{ is even.} \end{cases}$$

The distance ℓ_k between the vertices i and $m(i+k, n)$ is for $k = 0, \dots, K+1$ given by

$$\ell_k = 2r \sin \frac{k\pi}{n}.$$

The maximum chord length in $\mathcal{P}_{n,r}$ is equal to ℓ_{K+1} . For the distance x between one point of side i and one point of side $m(i+k, n)$, $k = 1, \dots, K^*$, we have $\ell_{k-1} \leq x \leq \ell_{k+1}$, and $\ell_k \leq x \leq \ell_{k+2}$ for the analogous distance of the sides i and $m(i+k+1, n)$. Therefore, a chord of length s , $\ell_k \leq s \leq \ell_{k+1}$, is a chord between two sides i and $m(i+k, n)$ or two sides i and $m(i+k+1, n)$.

In the first step we derive formulas for the distance $d_k^*(s, \psi)$ between O and a chord χ of length s , $\ell_k \leq s \leq \ell_{k+1}$, $k = 0, \dots, K$, where ψ denotes the oriented angle between the segment from O to the intersection point I of the lines i and $m(i+k, n)$ and the line perpendicular to χ (Fig. 2). We only consider the interval $0 \leq \psi \leq \pi/n$. It is necessary to distinguish the following cases:

Case 1

$$(4) \ell_k \leq s < \ell_{k+1} \quad \text{with} \quad \begin{cases} k \in \{1, \dots, K-1\} \text{ if } n \text{ is even,} \\ k \in \{1, \dots, K\} \text{ if } n \text{ is odd and } s \leq 2r \cos^2 \frac{\pi}{2n}. \end{cases}$$

For $0 \leq \psi \leq \alpha_k(s)$ (e.g., for position χ_1 of χ in Fig. 2) the distance $d_k^*(s, \psi)$ between O and χ is equal to

$$(5) \quad q_k(s, \psi) := r \cos \frac{\pi}{n} \sec \frac{k\pi}{n} \cos \psi - \frac{s}{2} \left(\tan \frac{k\pi}{n} \cos^2 \psi - \cot \frac{k\pi}{n} \sin^2 \psi \right).$$

The angle α_k is determined by the position χ_2 of χ with the upper end-point in the vertex $m(i+k+1, n)$

$$(6) \quad \alpha_k(s) = \arcsin \left(\frac{2r}{s} \sin \frac{k\pi}{n} \sin \frac{(k+1)\pi}{n} \right) - \frac{k\pi}{n}.$$

For $\alpha_k(s) \leq \psi \leq \pi/n$, χ is a chord between the sides i and $m(i+k+1, n)$. So we find

$$(7) \quad d_k^*(s, \psi) = \begin{cases} q_k(s, \psi) & \text{if } 0 \leq \psi \leq \alpha_k(s), \\ q_{k+1}(s, \psi - \pi/n) & \text{if } \alpha_k(s) < \psi \leq \pi/n. \end{cases}$$

Case 1a

$$(8) \quad 0 = \ell_0 \leq s < \ell_1 \quad \text{and} \quad s < 2r \cos^2 \frac{\pi}{2n}$$

We have $\alpha_0(s) = 0$ if $s \neq 0$, and the limit of $\alpha_0(s)$ at $s = 0$ is 0. Therefore, we get

$$(9) \quad d_0^*(s, \psi) = q_1(s, \psi - \pi/n) \quad \text{for } 0 \leq \psi \leq \pi/n$$

as special case of case 1 with $\alpha_0(s) = 0$ in formula (7).

Case 2

$$(10) \quad n \text{ is even and } \ell_K \leq s \leq \ell_{K+1}$$

A chord χ in the direction perpendicular to ψ does not exist if $\alpha_K(s) < \psi \leq \pi/n$, therefore

$$(11) \quad d_K^*(s, \psi) = \begin{cases} q_K(s, \psi) & \text{if } 0 \leq \psi \leq \alpha_K(s), \\ 0 & \text{if } \alpha_K(s) < \psi \leq \pi/n. \end{cases}$$

Case 3

$$(12) \quad n \text{ is odd and } \ell_K \leq s \leq \ell_{K+1} \quad \text{and} \quad s \geq 2r \cos^2 \frac{\pi}{2n}$$

A chord χ in the direction perpendicular to ψ does not exist if

$$\beta(s) < \psi < \frac{\pi}{n} - \beta(s),$$

with

$$(13) \quad \beta(s) = \frac{\pi}{2n} - \arccos \left(\frac{2r}{s} \cos^2 \frac{\pi}{2n} \right),$$

therefore,

$$(14) \quad d_K^*(s, \psi) = \begin{cases} q_K(s, \psi) & \text{if } 0 \leq \psi < \alpha_K(s), \\ q_{K+1}(s, \psi - \pi/n) & \text{if } \alpha_K(s) \leq \psi \leq \beta(s), \\ 0 & \text{if } \beta(s) < \psi < \pi/n - \beta(s), \\ q_{K+1}(s, \psi - \pi/n) & \text{if } \pi/n - \beta(s) \leq \psi \leq \pi/n. \end{cases}$$

Due to the symmetry of the graph of $d_k^*(s, \psi)$ with respect to the line $\psi = \pi/n$, the values in the interval $\pi/n < \psi \leq 2\pi/n$ can be easily calculated from (7), (9), (11) and (14) with

$$(15) \quad d_k^*(s, \psi) = d_k^*(s, 2\pi/n - \psi).$$

Since $d_k^*(s, \psi)$ is a $2\pi/n$ -periodic function, we get the values for $2\pi/n < \psi < \infty$ with the translation

$$\psi \mapsto \psi - \delta(\psi) \quad \text{with} \quad \delta(\psi) = \left\lfloor \frac{n\psi}{2\pi} \right\rfloor \frac{2\pi}{n}.$$

In the case of even n , the substitution $\psi = \phi + \pi/n$ yields the distances for angle ϕ starting from a vertex as shown in Fig. 1. In the case of odd n we have $\psi = \phi$. So we get the following lemma.

Lemma 1. *The restrictions $d_k(s, \phi) = d(s, \phi)|_{\ell_k \leq s < \ell_{k+1}}$ of the distance function d are given by*

$$d_k(s, \phi) = \begin{cases} d_k^*(s, \phi - \delta(\phi)) & \text{if } n \text{ is odd,} \\ d_k^*\left(s, \phi + \frac{\pi}{n} - \delta\left(\phi + \frac{\pi}{n}\right)\right) & \text{if } n \text{ is even,} \end{cases}$$

for $k = 0, \dots, K$, where

$$\delta(\cdot) = \left\lfloor \frac{n \cdot}{2\pi} \right\rfloor \frac{2\pi}{n}$$

and d_k^* according to the formulas (7), (11), (14) and (15) with α_k and β according to (6) and (13), respectively.

3. CHORD LENGTH DISTRIBUTION FUNCTION

So we can write the chord length distribution function (1) in the form

$$F(s) = \begin{cases} 0 & \text{if } -\infty < s < \ell_0 = 0, \\ H_k(s) & \text{if } \ell_k \leq s < \ell_{k+1} \text{ for } k = 0, \dots, K, \\ 1 & \text{if } \ell_{K+1} \leq s < \infty, \end{cases}$$

where

$$H_k(s) = 1 - \frac{\mu_k(s)}{2nr \sin(\pi/n)} \quad \text{with} \quad \mu_k(s) := \int_0^\pi [d_k(s, \phi) + d_k(s, \phi + \pi)] d\phi.$$

With (\star) the $2\pi/n$ -periodicity of $d_k(s, \phi)$ and (\diamond) the symmetry of $d_k(s, \phi)$ with respect to the line $\phi = \pi/n$, for odd and even n , we find

$$\begin{aligned} \mu_k(s) &= \int_0^\pi [d_k(s, \phi) + d_k(s, \phi + \pi)] d\phi \\ &\stackrel{\star}{=} \frac{n}{2} \int_0^{2\pi/n} [d_k(s, \phi) + d_k(s, \phi + \pi)] d\phi \\ &= \frac{n}{2} \int_0^{2\pi/n} \left[d_k(s, \phi) + d_k\left(s, \phi + n \frac{2\pi}{n}\right) \right] d\phi \\ &\stackrel{\star}{=} n \int_0^{2\pi/n} d_k(s, \phi) d\phi \stackrel{\diamond}{=} 2n \int_0^{\pi/n} d_k(s, \phi) d\phi. \end{aligned}$$

Note that this integral formula (together with the piecewise definition of the distance function) allows us to calculate the distribution function of every regular polygon in a rather simple way.

The indefinite integral of the function q_k (see (5)) is given by

$$(16) \quad \begin{aligned} J_k(s, \phi) &= \int q_k(s, \phi) d\phi = r \cos \frac{\pi}{n} \sec \frac{k\pi}{n} \sin \phi \\ &\quad - \frac{s}{8} \left[(2\phi + \sin 2\phi) \tan \frac{k\pi}{n} - (2\phi - \sin 2\phi) \cot \frac{k\pi}{n} \right]. \end{aligned}$$

In case 1 (see (4)) we get with $\alpha_k = \alpha_k(s)$

$$\begin{aligned} \frac{\mu_k(s)}{2n} &= \int_0^{\alpha_k} d_k(s, \phi) d\phi + \int_{\alpha_k}^{\pi/n} d_k(s, \phi) d\phi \\ &= \int_0^{\alpha_k} q_k(s, \phi) d\phi + \int_{\alpha_k}^{\pi/n} q_{k+1}\left(s, \phi - \frac{\pi}{n}\right) d\phi \end{aligned}$$

and with the substitution $\phi^* = \phi - \pi/n$

$$\begin{aligned} \frac{\mu_k(s)}{2n} &= \int_0^{\alpha_k} q_k(s, \phi) d\phi + \int_{\alpha_k - \pi/n}^0 q_{k+1}(s, \phi^*) d\phi^* \\ &= J_k(s, \alpha_k) - J_k(s, 0) + J_{k+1}(s, 0) - J_{k+1}\left(s, \alpha_k - \frac{\pi}{n}\right) \\ &= J_k(s, \alpha_k) - J_{k+1}\left(s, \alpha_k - \frac{\pi}{n}\right), \end{aligned}$$

since $J_k(s, 0) = 0$. In case 1a (see (8)), one finds with $\alpha_0(s) = 0$ and $J_k(s, -\phi) = -J_k(s, \phi)$

$$\frac{\mu_0(s)}{2n} = -J_1\left(s, -\frac{\pi}{n}\right) = J_1\left(s, \frac{\pi}{n}\right).$$

Putting $J_0(s, 0) = 0$, this formula can be considered as a special case of the formula for case 1. In case 2 (see (10)), one easily finds

$$\frac{\mu_K(s)}{2n} = J_K(s, \alpha_K)$$

and in case 3 (see (12)) with $\beta = \beta(s)$, we have

$$\begin{aligned}
\frac{\mu_K(s)}{2n} &= \left[\int_0^{\alpha_K} + \int_{\alpha_K}^{\beta} + \int_{\beta}^{\pi/n-\beta} + \int_{\pi/n-\beta}^{\pi/n} \right] d_K(s, \phi) d\phi \\
&= \left[\int_0^{\alpha_K} + \int_{\alpha_K}^{\beta} + \int_{\pi/n-\beta}^{\pi/n} \right] d_K(s, \phi) d\phi \\
&= \int_0^{\alpha_K} q_K(s, \phi) d\phi + \left[\int_{\alpha_K}^{\beta} + \int_{\pi/n-\beta}^{\pi/n} \right] q_{K+1}(s, \phi - \pi/n) d\phi \\
&= \int_0^{\alpha_K} q_K(s, \phi) d\phi + \left[\int_{\alpha_K-\pi/n}^{\beta-\pi/n} + \int_{-\beta}^0 \right] q_{K+1}(s, \phi^*) d\phi^* \\
&= J_K(s, \alpha_K) - J_{K+1}\left(s, \alpha_K - \frac{\pi}{n}\right) + J_{K+1}\left(s, \beta - \frac{\pi}{n}\right) + J_{K+1}(s, \beta).
\end{aligned}$$

The function J_k (see (16)) can be written as

$$J_k(s, \phi) = r \cos \frac{\pi}{n} \sec \frac{k\pi}{n} \sin \phi + \frac{s}{4} \left(2\phi \cot \frac{2k\pi}{n} - \sin(2\phi) \csc \frac{2k\pi}{n} \right).$$

Furthermore, we write both functions α_k (see (6)) and β (see (13)) in the form

$$\arcsin \frac{a}{s} - b$$

with

$$(17) \quad a = A_1(k) = 2r \sin \frac{k\pi}{n} \sin \frac{(k+1)\pi}{n}, \quad b = B_1(k) = \frac{k\pi}{n}$$

for α_k , and

$$(18) \quad a = A_2 = 2r \cos^2 \frac{\pi}{2n}, \quad b = B_2 = \frac{\pi}{2} \left(1 - \frac{1}{n} \right)$$

for β . Using some easy algebraic manipulations, one finds

$$\begin{aligned}
\frac{1}{r} J_k \left(s, \arcsin \frac{a}{s} - b \right) &= \Theta_1(k, a, b) s + \Theta_2(k, a, b) \frac{1}{s} + \Theta_3(k, a, b) \frac{\sqrt{s^2 - a^2}}{s} \\
&\quad + \Theta_4(k, a, b) s \arcsin \frac{a}{s} =: h_k(s, a, b),
\end{aligned}$$

where

$$(19) \quad \left\{ \begin{array}{l} \Theta_1(k, a, b) = \frac{1}{4r} \csc \frac{\pi}{n} \left(\sin(2b) \csc \frac{2k\pi}{n} - 2b \cot \frac{2k\pi}{n} \right), \\ \Theta_2(k, a, b) = a \left(\cos b \cot \frac{\pi}{n} \sec \frac{k\pi}{n} - \frac{a}{2r} \sin(2b) \csc \frac{\pi}{n} \csc \frac{2k\pi}{n} \right), \\ \Theta_3(k, a, b) = - \left(\sin b \cot \frac{\pi}{n} \sec \frac{k\pi}{n} + \frac{a}{2r} \cos(2b) \csc \frac{\pi}{n} \csc \frac{2k\pi}{n} \right), \\ \Theta_4(k, a, b) = \frac{1}{2r} \csc \frac{\pi}{n} \cot \frac{2k\pi}{n}. \end{array} \right.$$

In summary, we have proved the following theorem.

Theorem 1. *The chord length distribution function F of the regular polygon $\mathcal{P}_{n,r}$ is given by*

$$F(s) = \begin{cases} 0 & \text{if } -\infty < s < \ell_0 = 0, \\ H_k(s) & \text{if } \ell_k \leq s < \ell_{k+1} \text{ for } k = 0, \dots, K, \\ 1 & \text{if } \ell_{K+1} \leq s < \infty, \end{cases}$$

where

$$\ell_k = 2r \sin \frac{k\pi}{n}, \quad K = \left\lfloor \frac{n-2}{2} \right\rfloor$$

and

$$H_k(s) = \begin{cases} 1 - h_k(s, A_1(k), B_1(k)) + h_{k+1}(s, A_1(k), B_1(k) + \pi/n) \\ \quad \text{if } (n \text{ is even} \wedge k \in \{0, \dots, K-1\}) \vee (n \text{ is odd} \wedge s < \lambda), \\ 1 - h_K(s, A_1(K), B_1(K)) + h_{K+1}(s, A_1(K), B_1(K) + \pi/n) \\ - h_{K+1}(s, A_2, B_2 + \pi/n) - h_{K+1}(s, A_2, B_2) \\ \quad \text{if } n \text{ is odd} \wedge s \geq \lambda, \\ 1 - h_K(s, A_1(K), B_1(K)) \quad \text{if } n \text{ is even} \wedge k = K \end{cases}$$

with

$$\lambda = 2r \cos^2 \frac{\pi}{2n},$$

$A_1(k)$ and $B_1(k)$ according to (17), A_2 and B_2 according to (18), and

$$h_k(s, a, b) = \begin{cases} 0 & \text{if } k = 0, \\ \sum_{i=1}^4 \Theta_i(k, a, b) L_i(s, a) & \text{if } k = 1, 2, \dots, \end{cases}$$

with $\Theta_i(k, a, b)$ according to (19), and

$$L_1(s, a) = s, \quad L_2(s, a) = \frac{1}{s}, \quad L_3(s, a) = \frac{\sqrt{s^2 - a^2}}{s}, \quad L_4(s, a) = s \arcsin \frac{a}{s}.$$

F can be written in the form

$$F(s) = H_0(s) = \left[\left(1 - \frac{\pi}{n} \cot \frac{\pi}{n} \right) \csc \frac{\pi}{n} + \frac{\pi}{n} \sec \frac{\pi}{n} \right] \frac{s}{4r}$$

for $0 \leq s \leq \lambda$ if $n = 3$, and $0 \leq s \leq \ell_1$ if $n = 4, 5, \dots$. Note that this is a linear equation of s (cf. [10, pp. 866–867]).

From [20, p. 55, Satz 2], it follows that the chord length distribution function of a regular polygon is a continuous function.

4. POINT DISTANCES

In the following, we consider the distance between two uniformly and independently distributed random points within the polygon $\mathcal{P}_{n,r}$ with perimeter u and area A

$$u = 2nr \sin \frac{\pi}{n}, \quad A = \frac{1}{2} nr^2 \sin \frac{2\pi}{n}.$$

Theorem 2. *The density function g of the distance t between two random points in $\mathcal{P}_{n,r}$ is given by*

$$g(t) = \begin{cases} \frac{2t}{A} \left[\pi + \frac{u}{A} (\phi^*(t) - t) \right] & \text{if } t \in [0, \ell_{K+1}), \\ 0 & \text{if } t \in \mathbb{R} \setminus [0, \ell_{K+1}), \end{cases}$$

where

$$\phi^*(t) = \sum_{\nu=0}^{k-1} J_{\nu}^*(\ell_{\nu}, \ell_{\nu+1}) + J_k^*(\ell_k, t) \quad \text{if } \ell_k \leq t < \ell_{k+1}, \quad k = 0, \dots, K$$

with

$$J_k^*(s, t) = H_k^*(t) - H_k^*(s),$$

where

$$H_k^*(t) = \begin{cases} t - h_k^*(t, A_1(k), B_1(k)) + h_{k+1}^*(t, A_1(k), B_1(k) + \pi/n) \\ \quad \text{if } (n \text{ is even} \wedge k \in \{0, \dots, K-1\}) \vee (n \text{ is odd} \wedge t < \lambda), \\ t - h_K^*(t, A_1(K), B_1(K)) + h_{K+1}^*(t, A_1(K), B_1(K) + \pi/n) \\ \quad - h_{K+1}^*(t, A_2, B_2 + \pi/n) - h_{K+1}^*(t, A_2, B_2) \\ \quad + h_{K+1}^*(\lambda, A_2, B_2 + \pi/n) + h_{K+1}^*(\lambda, A_2, B_2) \\ \quad \text{if } n \text{ is odd} \wedge t \geq \lambda, \\ t - h_K^*(t, A_1(K), B_1(K)) \quad \text{if } n \text{ is even} \wedge k = K \end{cases}$$

with

$$h_k^*(t, a, b) = \begin{cases} 0 & \text{if } k = 0 \vee (k \neq 0 \wedge t = 0), \\ \sum_{i=1}^4 \Theta_i(k, a, b) L_i^*(t, a) & \text{if } k \neq 0 \wedge t > 0, \end{cases}$$

$$L_1^*(t, a) = \frac{t^2}{2}, \quad L_2^*(t, a) = \ln t, \quad L_3^*(t, a) = \sqrt{t^2 - a^2} + a \arcsin \frac{a}{t},$$

$$L_4^*(t, a) = \frac{1}{2} \left(a \sqrt{t^2 - a^2} + t^2 \arcsin \frac{a}{t} \right).$$

Proof. According to Piefke [18, p. 130], the density function of the distance is given by

$$g(t) = \frac{2ut}{A^2} \int_t^{\ell_{K+1}} (s-t)f(s) ds,$$

where f is the density function of the chord length. From integral geometry it is well-known that

$$\int_0^{\ell_{K+1}} sf(s) ds = \frac{\pi A}{u}$$

(see [19, p. 47], [16, p. 94]), hence

$$\int_t^{\ell_{K+1}} sf(s) ds = \frac{\pi A}{u} - \int_0^t sf(s) ds.$$

Using integration by parts, we have

$$\int_0^t s f(s) ds = sF(s) \Big|_0^t - \int_0^t F(s) ds = tF(t) - \int_0^t F(s) ds.$$

Therefore, we obtain

$$g(t) = \frac{2t}{A} \left[\pi - \frac{u}{A} \left(t - \int_0^t F(s) ds \right) \right] = \frac{2t}{A} \left[\pi + \frac{u}{A} (\phi^*(t) - t) \right]$$

with

$$\phi^*(t) := \int_0^t F(s) ds.$$

For $\ell_k \leq t < \ell_{k+1}$, $k = 0, \dots, K$, this yields

$$\phi^*(t) = \sum_{\nu=0}^{k-1} \int_{\ell_\nu}^{\ell_{\nu+1}} H_\nu(s) ds + \int_{\ell_k}^t H_k(s) ds$$

(in case $k = 0$, the sum is empty). With

$$H_k^*(t) := \int H_k(t) dt \quad \text{und} \quad J_k^*(s, t) = H_k^*(t) - H_k^*(s)$$

it follows that

$$\begin{aligned} \phi^*(t) &= \sum_{\nu=0}^{k-1} [H_\nu^*(\ell_{\nu+1}) - H_\nu^*(\ell_\nu)] + H_k^*(t) - H_k^*(\ell_k) \\ &= \sum_{\nu=0}^{k-1} J_\nu^*(\ell_\nu, \ell_{\nu+1}) + J_k^*(\ell_k, t). \end{aligned}$$

Furthermore, if $k \neq 0$ and $t > 0$,

$$\begin{aligned} h_k^*(t, a, b) &:= \int h_k(t, a, b) dt = \sum_{i=1}^4 \Theta_i(k, a, b) \int L_i(t, a) dt \\ &= \sum_{i=1}^4 \Theta_i(k, a, b) L_i^*(t, a) \end{aligned}$$

with the indefinite integrals

$$\begin{aligned} L_1^*(t, a) &= \int t dt = \frac{t^2}{2}, \quad L_2^*(t, a) = \int \frac{1}{t} dt = \ln t, \\ L_3^*(t, a) &= \int \frac{\sqrt{t^2 - a^2}}{t} dt = \sqrt{t^2 - a^2} + a \arcsin \frac{a}{t} \end{aligned}$$

(see [6, p. 48, Eq. 217]) and using integration by parts,

$$\begin{aligned} L_4^*(t, a) &= \int L_4(t, a) dt = \int t \arcsin \frac{a}{t} dt \\ &= \frac{1}{2} \left(t^2 \arcsin \frac{a}{t} + a \int \frac{t}{\sqrt{t^2 - a^2}} dt \right) \\ &= \frac{1}{2} \left(t^2 \arcsin \frac{a}{t} + a \sqrt{t^2 - a^2} \right). \end{aligned}$$

For odd n , the function

$$H_K^*(t) = \begin{cases} H_{K,1}^*(t) & \text{if } t < \lambda, \\ H_{K,2}^*(t) & \text{if } t \geq \lambda \end{cases}$$

with

$$\begin{aligned} H_{K,1}^*(t) &:= t - h_K^*(t, A_1(k), B_1(k)) + h_{K+1}^*(t, A_1(K), B_1(K) + \pi/n), \\ H_{K,2}^*(t) &:= t - h_K^*(t, A_1(K), B_1(K)) + h_{K+1}^*(t, A_1(K), B_1(K) + \pi/n) \\ &\quad - h_{K+1}^*(t, A_2, B_2 + \pi/n) - h_{K+1}^*(t, A_2, B_2) \end{aligned}$$

is not continuous in $t = \lambda$. This causes a false result when calculating the integral

$$J_K^*(\ell_K, \lambda) = H_K^*(\lambda) - H_K^*(\ell_K).$$

In order to avoid this problem (and unnecessary case distinctions), we define

$$\begin{aligned} \tilde{H}_{K,2}^*(t) &= H_{K,2}^*(t) - H_{K,2}^*(\lambda) + H_{K,1}^*(\lambda) \\ &= H_{K,2}^*(t) + h_{K+1}^*(\lambda, A_2, B_2 + \pi/n) + h_{K+1}^*(\lambda, A_2, B_2) \end{aligned}$$

and put

$$H_K^*(t) := \begin{cases} H_{K,1}^*(t) & \text{if } t < \lambda, \\ \tilde{H}_{K,2}^*(t) & \text{if } t \geq \lambda, \end{cases}$$

so that H_K^* is now a continuous function. This completes the proof. \square

Corollary 1. *The distribution function G of the distance t between two random points in $\mathcal{P}_{n,r}$ is given by*

$$G(t) = \begin{cases} 0 & \text{if } -\infty < s < 0, \\ \frac{1}{A} \left[t^2 \left(\pi - \frac{2u}{3A} t \right) + \frac{2u}{A} \phi^{\natural}(t) \right] & \text{if } 0 \leq t < \ell_{K+1}, \\ 1 & \text{if } t \geq \ell_{K+1} \end{cases}$$

with

$$\phi^{\natural}(t) = \sum_{\nu=0}^{k-1} K_{\nu}(\ell_{\nu+1}) + K_k(t) \quad \text{if } \ell_k \leq t < \ell_{k+1}, \quad k = 0, \dots, K,$$

where

$$K_k(t) = \frac{1}{2} (t^2 - \ell_k^2) \left(\sum_{\nu=0}^{k-1} J_{\nu}^*(\ell_{\nu}, \ell_{\nu+1}) - H_k^*(\ell_k) \right) + J_k^{\natural}(\ell_k, t)$$

with J_k^* and H_k^* according to Theorem 2 and

$$J_k^{\natural}(s, t) = H_k^{\natural}(t) - H_k^{\natural}(s),$$

where

$$H_k^{\natural}(t) = \begin{cases} \frac{t^3}{3} - h_k^{\natural}(t, A_1(k), B_1(k)) + h_{k+1}^{\natural}(t, A_1(k), B_1(k) + \pi/n) \\ \quad \text{if } (n \text{ is even} \wedge k \in \{0, \dots, K-1\}) \vee (n \text{ is odd} \wedge t < \lambda), \\ \frac{t^3}{3} - h_K^{\natural}(t, A_1(K), B_1(K)) + h_{K+1}^{\natural}(t, A_1(K), B_1(K) + \pi/n) \\ \quad - h_{K+1}^{\natural}(t, A_2, B_2 + \pi/n) - h_{K+1}^{\natural}(t, A_2, B_2) \\ \quad + \frac{t^2}{2} [h_{K+1}^*(\lambda, A_2, B_2 + \pi/n) + h_{K+1}^*(\lambda, A_2, B_2)] \\ \quad \text{if } n \text{ is odd} \wedge t \geq \lambda, \\ \frac{t^3}{3} - h_K^{\natural}(t, A_1(K), B_1(K)) \quad \text{if } n \text{ is even} \wedge k = K \end{cases}$$

with h_{K+1}^* from Theorem 2, and

$$h_k^{\natural}(t, a, b) = \begin{cases} 0 & \text{if } k = 0 \vee (k \neq 0 \wedge t = 0), \\ \sum_{i=1}^4 \Theta_i(k, a, b) L_i^{\natural}(t, a) & \text{if } k \neq 0 \wedge t > 0, \end{cases}$$

$$L_1^{\natural}(t, a) = \frac{t^4}{8}, \quad L_2^{\natural}(t, a) = \frac{t^2}{4} (2 \ln t - 1),$$

$$L_3^{\natural}(t, a) = \frac{1}{3} (t^2 - a^2)^{3/2} + \frac{a}{2} (a \sqrt{t^2 - a^2} + t^2 \arcsin \frac{a}{t}),$$

$$L_4^{\natural}(t, a) = \frac{1}{8} \left[\frac{5a}{3} (t^2 - a^2)^{3/2} + a^3 \sqrt{t^2 - a^2} + t^4 \arcsin \frac{a}{t} \right].$$

Proof. For $0 \leq t < \ell_{K+1}$, one gets

$$\begin{aligned} G(t) &= \int_0^t g(\tau) d\tau = \int_0^t \left(\frac{2\pi\tau}{A} - \frac{2u\tau^2}{A^2} + \frac{2u\tau}{A^2} \int_0^\tau F(s) ds \right) d\tau \\ &= \frac{\pi t^2}{A} - \frac{2ut^3}{3A^2} + \frac{2u}{A^2} \int_0^t \tau \left(\int_0^\tau F(s) ds \right) d\tau \\ &= \frac{\pi t^2}{A} - \frac{2ut^3}{3A^2} + \frac{2u}{A^2} \int_0^t \tau \phi^*(\tau) d\tau = \frac{1}{A} \left[t^2 \left(\pi - \frac{2u}{3A} t \right) + \frac{2u}{A} \phi^{\natural}(t) \right] \end{aligned}$$

with

$$\phi^{\natural}(t) := \int_0^t s \phi^*(s) ds.$$

It remains to calculate $\phi^{\natural}(t)$. For $\ell_k \leq t < \ell_{k+1}$, $k = 0, \dots, K$, we have

$$\begin{aligned}\phi^{\natural}(t) &= \int_0^{\ell_k} s\phi^*(s) \, ds + \int_{\ell_k}^t s\phi^*(s) \, ds \\ &= \sum_{\nu=0}^{k-1} \int_{\ell_{\nu}}^{\ell_{\nu+1}} s\phi^*(s) \, ds + \int_{\ell_k}^t s\phi^*(s) \, ds\end{aligned}$$

with

$$\begin{aligned}\int_{\ell_k}^t s\phi^*(s) \, ds &= \int_{\ell_k}^t s \left(\sum_{\nu=0}^{k-1} J_{\nu}^*(\ell_{\nu}, \ell_{\nu+1}) + J_k^*(\ell_k, s) \right) \, ds \\ &= \sum_{\nu=0}^{k-1} J_{\nu}^*(\ell_{\nu}, \ell_{\nu+1}) \int_{\ell_k}^t s \, ds + \int_{\ell_k}^t s J_k^*(\ell_k, s) \, ds \\ &= \sum_{\nu=0}^{k-1} J_{\nu}^*(\ell_{\nu}, \ell_{\nu+1}) \int_{\ell_k}^t s \, ds + \int_{\ell_k}^t s [H_k^*(s) - H_k^*(\ell_k)] \, ds \\ &= \left(\sum_{\nu=0}^{k-1} J_{\nu}^*(\ell_{\nu}, \ell_{\nu+1}) - H_k^*(\ell_k) \right) \int_{\ell_k}^t s \, ds + \int_{\ell_k}^t s H_k^*(s) \, ds.\end{aligned}$$

Putting

$$H_k^{\natural}(t) := \int_{\ell_k}^t s H_k^*(s) \, ds \quad \text{and} \quad J_k^{\natural}(s, t) := H_k^{\natural}(t) - H_k^{\natural}(s),$$

it follows that

$$\begin{aligned}\int_{\ell_k}^t s\phi^*(s) \, ds &= \frac{1}{2}(t^2 - \ell_k^2) \left(\sum_{\nu=0}^{k-1} J_{\nu}^*(\ell_{\nu}, \ell_{\nu+1}) - H_k^*(\ell_k) \right) + H_k^{\natural}(t) - H_k^{\natural}(\ell_k) \\ &= \frac{1}{2}(t^2 - \ell_k^2) \left(\sum_{\nu=0}^{k-1} J_{\nu}^*(\ell_{\nu}, \ell_{\nu+1}) - H_k^*(\ell_k) \right) + J_k^{\natural}(\ell_k, t) =: K_k(t)\end{aligned}$$

and hence

$$\phi^{\natural}(t) = \sum_{\nu=0}^{k-1} K_{\nu}(\ell_{\nu+1}) + K_k(t), \quad \ell_k \leq t < \ell_{k+1}, \quad k = 0, \dots, K.$$

If $k \neq 0$ and $t > 0$, one finds

$$\begin{aligned}h_k^{\natural}(t, a, b) &:= \int t h_k^*(t, a, b) \, dt = \sum_{i=1}^4 \Theta_i(k, a, b) \int t L_i^*(t, a) \, dt \\ &= \sum_{i=1}^4 \Theta_i(k, a, b) L_i^{\natural}(t, a)\end{aligned}$$

with the indefinite integrals

$$L_1^{\natural}(t, a) = \int \frac{t^3}{2} \, dt = \frac{t^4}{8}, \quad L_2^{\natural}(t, a) = \int t \ln t \, dt = \frac{t^2}{4} (2 \ln t - 1)$$

and

$$\begin{aligned} L_4^{\natural}(t, a) &= \int t L_4^*(t, a) dt = \frac{1}{2} \int t \left(a\sqrt{t^2 - a^2} + t^2 \arcsin \frac{a}{t} \right) dt \\ &= \frac{1}{2} \left(a \int t \sqrt{t^2 - a^2} dt + \int t^3 \arcsin \frac{a}{t} dt \right) \end{aligned}$$

with

$$\int t \sqrt{t^2 - a^2} dt = \frac{1}{3} (t^2 - a^2)^{3/2} \quad [\mathbf{6}, \text{p. 47, Eq. 214}].$$

Using integration by parts, we find

$$\int t^3 \arcsin \frac{a}{t} dt = \frac{1}{4} \left(t^4 \arcsin \frac{a}{t} + a \int \frac{t^2}{\sqrt{1 - (a/t)^2}} dt \right).$$

Since $t \geq a > 0$ in the present cases,

$$\int \frac{t^2}{\sqrt{1 - (a/t)^2}} dt = \int \frac{t^3}{\sqrt{t^2 - a^2}} dt,$$

and

$$\int \frac{t^3}{\sqrt{t^2 - a^2}} dt = \frac{1}{3} (t^2 - a^2)^{3/2} + a^2 \sqrt{t^2 - a^2} \quad [\mathbf{6}, \text{p. 48, Eq. 223}].$$

This yields

$$L_4^{\natural}(t, a) = \frac{1}{8} \left[\frac{5a}{3} (t^2 - a^2)^{3/2} + a^3 \sqrt{t^2 - a^2} + t^4 \arcsin \frac{a}{t} \right].$$

Furthermore,

$$\begin{aligned} L_3^{\natural}(t, a) &= \int t \sqrt{t^2 - a^2} dt + a \int t \arcsin \frac{a}{t} dt \\ &= \frac{1}{3} (t^2 - a^2)^{3/2} + \frac{a}{2} \left(a\sqrt{t^2 - a^2} + t^2 \arcsin \frac{a}{t} \right) \end{aligned}$$

(see the calculations of $L_4^{\natural}(t, a)$ and $L_4^*(t, a)$). □

5. EXAMPLES

Fig. 3 shows examples of chord length distribution functions F .

As special case of Theorem 2, the distance density function for an equilateral triangle $\mathcal{P}_{3,r}$ with circumscribed circle of radius r is given by

$$g(t) = \begin{cases} \frac{2t}{A} \left[\pi + \frac{u}{A} (\phi(t) - t) \right] & \text{if } t \in [0, \sqrt{3}r), \\ 0 & \text{if } t \in \mathbb{R} \setminus [0, \sqrt{3}r) \end{cases}$$

with $u = 3\sqrt{3}r$, $A = \frac{3}{4}\sqrt{3}r^2$ and

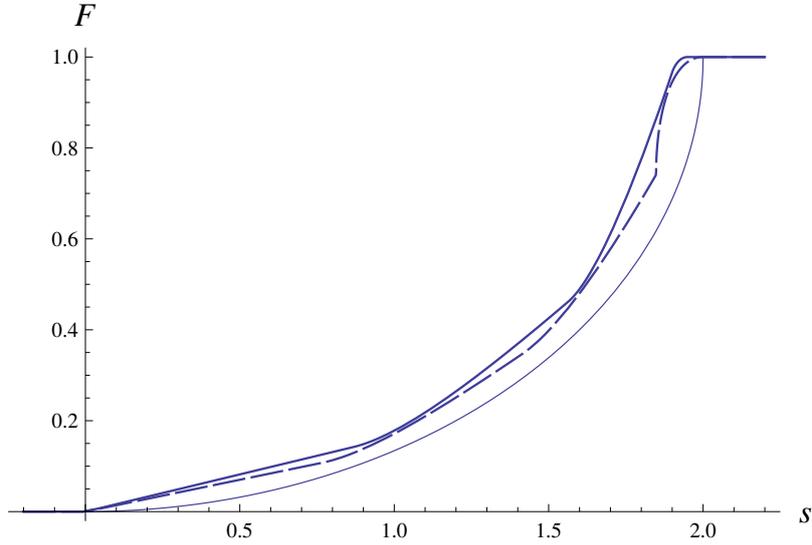


Figure 3. F for $\mathcal{P}_{7,1}$ (thick), $\mathcal{P}_{8,1}$ (dashed) and circle of radius $r = 1$ (thin).

$$\phi(t) = \begin{cases} \frac{(3\sqrt{3} + 2\pi)t^2}{36r} & \text{if } 0 \leq t < \frac{3r}{2}, \\ \frac{3}{2} \left[t \sqrt{1 - \left(\frac{3r}{2t}\right)^2} - \frac{\pi r}{2} \right] + \left(\frac{1}{4\sqrt{3}} - \frac{\pi}{9} \right) \frac{t^2}{r} \\ \quad + \left(\frac{3r}{2} + \frac{t^2}{3r} \right) \arcsin \frac{3r}{2t} & \text{if } \frac{3r}{2} \leq t < \sqrt{3}r. \end{cases}$$

Fig. 4 shows the function $r \times g(t)$ for $\mathcal{P}_{3,r}$ and some other examples.

For the expectation of the distance for $\mathcal{P}_{3,r}$ one finds

$$\begin{aligned} \mathbb{E}[t] &= \int_0^{\sqrt{3}r} t g(t) dt = \left(\int_0^{3r/2} + \int_{3r/2}^{\sqrt{3}r} \right) t g(t) dt \\ &= \frac{r}{20} (27 - 90\sqrt{3} + 26\sqrt{3}\pi) + \frac{r}{20} (-27 + 94\sqrt{3} - 26\sqrt{3}\pi \\ &\quad + \sqrt{3} \ln 27) = \frac{\sqrt{3}r}{20} (4 + 3 \ln 3). \end{aligned}$$

Since the side length a of $\mathcal{P}_{3,r}$ is equal to $\sqrt{3}r$, we get

$$\mathbb{E}[t] = \frac{a}{20} (4 + 3 \ln 3) = \frac{3a}{5} \left(\frac{1}{3} + \frac{1}{4} \ln 3 \right).$$

This is the result from [19, p. 49].

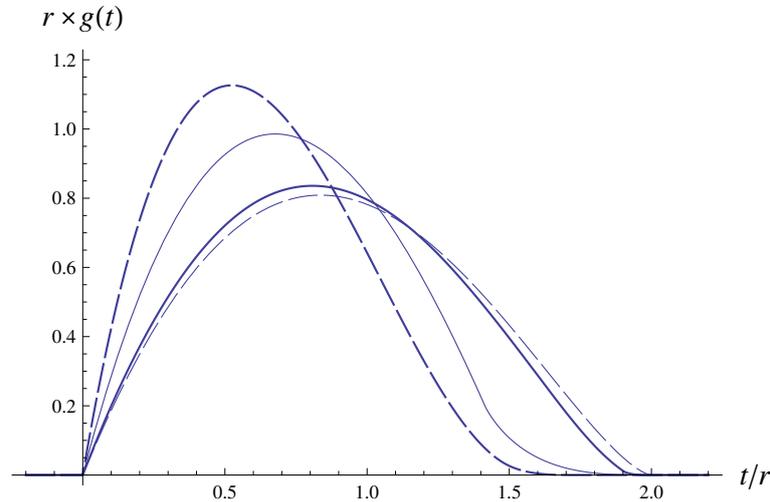


Figure 4. $r \times g(t)$ for $\mathcal{P}_{3,r}$ (thick and dashed), $\mathcal{P}_{4,r}$ (thin), $\mathcal{P}_{10,r}$ (thick) and circle of radius r (thin and dashed).

For the square $\mathcal{P}_{4,r}$, the side length is equal to $\sqrt{2}r$. For $a = 1$, a calculation using Mathematica yields the expected distance $E[t] = 0.5214054331\dots$. This result can be found in [9, p. 479]. It also follows from [3, p. 13] and [7].

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U. Basel, HTWK Leipzig, Faculty of Mechanical and Energy Engineering, PF 30 11 66, 04251 Leipzig, Germany, *e-mail*: uwe.baesel@htwk-leipzig.de