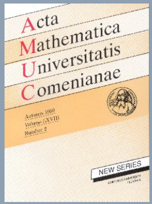

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CLOSED HEREDITARY ADDITIVE AND DIVISIBLE SUBCATEGORIES IN EPIREFLECTIVE SUBCATEGORIES OF TOP

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Abstract. The aim of this paper is to investigate closed hereditary additive and divisible (AD) subcategories of epireflective subcategories of the category **Top**. In quotient reflective subcategories of **Top**, AD subcategories are precisely the coreflective subcategories. We describe the closed hereditary AD hull and the closed hereditary AD kernel of AD subcategories and present some results concerning minimal non-trivial closed hereditary AD subcategories in epireflective subcategories of **Top**. We also show that some of the results obtained for AD subcategories are not valid in the case of coreflective subcategories in epireflective subcategories that are not quotient reflective, for instance, in the category of Tychonoff spaces.

1. Introduction

Herrlich and Hušek (see [8]) suggested the investigation of hereditary and closed hereditary coreflective subcategories in **Top** and **Haus**, e.g., hereditary (closed hereditary) coreflective kernels and hulls of coreflective subcategories of **Top** and **Haus**. Hereditary coreflective subcategories and hereditary additive and divisible (AD) subcategories in epireflective subcategories of the category

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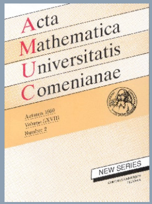

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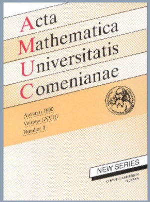
Top are studied in the papers of J. Činčura (e. g. [3], [4], [5]), V. Kannan (e. g. [9], [10], [11]) and M. Sleziak (e. g. [15], [16]).

In the category **Top**, the extremal subobjects are precisely the subspaces. However, in **Haus**, **Tych** and many other epireflective subcategories of the category **Top**, the extremal subobjects are precisely the closed subspaces. That is why the notion of closed subspace is important in these subcategories.

In this paper, we study closed hereditary AD subcategories and closed hereditary coreflective subcategories in epireflective subcategories of **Top**. In quotient reflective subcategories of **Top**, the notion of a coreflective subcategory is equivalent to the notion of an AD subcategory. In epireflective subcategories that are not quotient reflective each coreflective subcategory is also an AD subcategory. However, there exist AD subcategories that are not coreflective. First we investigate closed hereditary AD subcategories. We describe the closed hereditary AD hull and kernel of AD subcategories in epireflective subcategories of **Top**. As a consequence we obtain the description of the closed hereditary coreflective hull and kernel of coreflective subcategories in quotient reflective subcategories of **Top**. Also some interesting differences between closed hereditary AD subcategories and closed hereditary coreflective subcategories are studied. We provide some results on minimal closed hereditary AD subcategories containing a non-discrete space. Moreover, we show that in the category $\mathbf{Top_1}$, there exist no such minimal closed hereditary coreflective subcategories.

2. Preliminaries

The undefined terminology may be found in [1] and [6]. By $|A|$, we denote the cardinality of the set A and by $\mathcal{P}(A)$ the power set of A. By **Top**, we denote the category of all topological spaces and by $\mathbf{Top_0}$ ($\mathbf{Top_1}$, **Dis**, **Tych**, **ZD**), the category of all T_0-spaces (T_1-spaces, discrete spaces, Tychonoff spaces, zero-dimensional spaces). All subcategories are supposed to be full (so that they are completely described by the class of their objects), isomorphism-closed and to contain a



non-empty space. To avoid the trivial cases, all epireflective subcategories of **Top** are supposed to contain a non-indiscrete space. Such epireflective subcategories are called non-trivial. Note that if an epireflective subcategory of **Top** is non-trivial, then it contains all zero-dimensional spaces, in particular, all discrete spaces and coproducts in this subcategory are precisely the topological sums. It is known that a subcategory of **Top** is epireflective if and only if it is closed under topological products and subspaces, and it is quotient reflective if and only if it is closed under topological products, subspaces and spaces with finer topology. If $\mathbf{A}$ is a non-trivial epireflective subcategory of **Top**, then a subcategory of $\mathbf{A}$ is coreflective in $\mathbf{A}$ if and only if it is closed under topological sums and extremal quotient objects. If $\mathbf{A}$ is quotient reflective, then extremal quotient objects in $\mathbf{A}$ are precisely (usual) quotient maps, and a subcategory of $\mathbf{A}$ is coreflective if and only if it is closed under topological sums and (usual) quotient spaces. The coreflective hull $\mathrm{CH}_{\mathbf{A}}(\mathbf{B})$ of a subcategory $\mathbf{B}$ of $\mathbf{A}$ consists of all $X \in \mathbf{A}$ such that there exists a family $\{X_i\}_{i\in I} \subseteq \mathbf{B}$ and a quotient map $f\colon \coprod_{i\in I} X_i \to X$.

If $\mathbf{A}$ is a non-trivial epireflective subcategory of **Top**, then a subcategory of $\mathbf{A}$ is said to be additive if it is closed under topological sums. It is said to be divisible in $\mathbf{A}$ if it is closed under quotient spaces from $\mathbf{A}$. The AD hull $\mathrm{AD}_{\mathbf{A}}(\mathbf{B})$ of a subcategory $\mathbf{B}$ of $\mathbf{A}$ consists of all $X \in \mathbf{A}$ such that there exists a family $\{X_i\}_{i\in I} \subseteq \mathbf{B}$ and a quotient map $f\colon \coprod_{i\in I} X_i \to X$.

A category of topological spaces is said to be (closed) hereditary if it is closed under the formation of (closed) subspaces. For a subcategory $\mathbf{A}$ of topological spaces, $\mathrm{CS}\,\mathbf{A}$ denotes the subcategory consisting of all closed subspaces of spaces from $\mathbf{A}$. If $\mathbf{A}$ is a non-trivial epireflective subcategory of **Top** and $\mathbf{B}$ is a subcategory of $\mathbf{A}$, then the smallest closed hereditary AD subcategory of $\mathbf{A}$ containing $\mathbf{B}$ is called the closed hereditary AD hull of $\mathbf{B}$ in $\mathbf{A}$. The largest closed hereditary AD subcategory of $\mathbf{A}$ contained in $\mathbf{B}$ is called the closed hereditary AD kernel of $\mathbf{B}$ in $\mathbf{A}$. Similarly, we define the closed hereditary coreflective hull and kernel of a subcategory.

Let α be an (infinite) regular cardinal. Then $C(\alpha)$ is the topological space on the set $\alpha \cup \{\alpha\}$ such that a set $U \subseteq \alpha \cup \{\alpha\}$ is open in $C(\alpha)$ if and only if $U \subseteq \alpha$ or $\alpha \in U$ and $|\alpha \smallsetminus U| < \alpha$. A

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space X is said to be a P_α-space if for every family $\mathcal{S}$ of open subsets of X with $|\mathcal{S}| < \alpha$, $\bigcap_{U\in\mathcal{S}} U$ is also an open subset of X. The category of all P_α-spaces is denoted by $\mathbf{Top}(\alpha)$.

Recall that a prime space is a topological space with a unique accumulation point. Let X be a topological space and a be a point in X. The prime factor X_a of X at a is the space obtained by making all points of X other than a isolated and retaining the original neighborhoods of a.

3. Closed hereditary AD subcategories

The best known examples of (non-trivial) closed hereditary coreflective (AD) subcategories of the category $\mathbf{Top}$ are both the category $\mathbf{CG}$ of compactly generated spaces and the category $\mathbf{Seq}$ of sequential spaces. It is easy to see that if $\mathbf{A}$ is a non-trivial epireflective subcategory of $\mathbf{Top}$, then $\mathbf{CG} \cap \mathbf{A}$ and $\mathbf{Seq} \cap \mathbf{A}$ are closed hereditary AD subcategories of $\mathbf{A}$. If a non-trivial epireflective subcategory $\mathbf{A}$ of $\mathbf{Top}$ contains the space $\mathbb{R}$, then obviously, the coreflective hull $\mathrm{CH}_{\mathbf{A}}(\mathbb{R})$ of $\mathbb{R}$ in $\mathbf{A}$ is a coreflective subcategory of $\mathbf{A}$ which is not closed hereditary (the subspace $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ of $\mathbb{R}$ does not belong to $\mathrm{CH}_{\mathbf{A}}(\mathbb{R})$). Next we want to show that in any non-trivial epireflective subcategory of $\mathbf{Top}$, there exists a coreflective (and consequently, AD) subcategory that is not closed hereditary. We shall use the following notions and results. Let $D(2)$ be the discrete space on the set $\{0, 1\}$. Recall that a topological space X is called a $k_{\mathbb{R}}$-space ($k_{D(2)}$-space) provided that every map $X \to \mathbb{R}$ ($X \to D(2)$) which is continuous on compact subspaces of X is continuous. It is obvious that every $k_{\mathbb{R}}$-space is also a $k_{D(2)}$-space. Let us denote the category of all $k_{\mathbb{R}}$-spaces ($k_{D(2)}$-spaces) by $\mathbf{K}_{\mathbb{R}}$ ($\mathbf{K}_{D(2)}$). It is known (see, e.g., [12]) that $\mathbf{K}_{\mathbb{R}}$ is a coreflective subcategory of $\mathbf{Top}$.

Proposition 3.1. [5, Proposition 3] *Let* $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{Top}$ *and* $f\colon X \to Y$ *be an* $\mathbf{A}$*-morphism. Then* $f\colon X \to Y$ *is an extremal* $\mathbf{A}$*-epimorphism if and only if it is surjective and for any map* $g\colon Y \to Z$ *with* $Z \in \mathbf{A}$ *the map* g *is continuous whenever the map* $g \circ f$ *is continuous.*

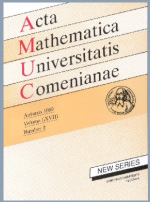
The following proposition is a consequence of [14, Theorem 5.6 (ii)].

Proposition 3.2. *Let $X = \prod_{a\in A} X_a$. If each X_a is a locally compact T_2-space, then X is a $k_{\mathbb{R}}$-space.*

Proposition 3.3. [2, Chapter 2, Problem 367] *Let X be a T_2-space. If $\{B_a\}_{a\in A}$ is a family of subspaces of X and $C = \bigcap_{a\in A} B_a$, then the subspace C of X is homeomorphic to a closed subspace of the space $\prod_{a\in A} B_a$.*

Proposition 3.4. *If $\mathbf{A}$ is a non-trivial epireflective subcategory of* **Top**, *then $\mathbf{K}_{D(2)} \cap \mathbf{A}$ is a coreflective subcategory of $\mathbf{A}$ which is not closed hereditary.*

Proof. It is easy to see that $\mathbf{K}_{D(2)} \cap \mathbf{A}$ is closed under topological sums. We show that it is also closed under extremal quotient objects. Let X be a $k_{D(2)}$-space from $\mathbf{A}$, $f\colon X \to Y$ be an extremal $\mathbf{A}$-epimorphism and the map $g\colon Y \to D(2)$ be continuous on compact subspaces. Then also the map $g \circ f\colon X \to D(2)$ is continuous on compact subspaces, hence it is continuous. According to Proposition 3.1, the map $g\colon Y \to D(2)$ is continuous and therefore Y is a $k_{D(2)}$-space. Thus $\mathbf{K}_{D(2)} \cap \mathbf{A}$ is coreflective in $\mathbf{A}$. It remains to show that $\mathbf{K}_{D(2)} \cap \mathbf{A}$ is not closed hereditary. Let $(\mathbb{R}, \mathcal{T})$ be the topological space on the set $\mathbb{R}$ with $\mathcal{T} = \mathcal{P}(\mathbb{R} \smallsetminus \{0\}) \cup \{U \subseteq \mathbb{R} : 0 \in U$ and $\mathbb{R} \smallsetminus U$ is countable$\}$. This space is zero-dimensional and every compact subspace of $(\mathbb{R}, \mathcal{T})$ is finite. The map $f\colon (\mathbb{R}, \mathcal{T}) \to D(2)$, defined by $f(0) = 0$ and $f(x) = 1$ otherwise, is continuous on every compact subspace of $(\mathbb{R}, \mathcal{T})$ without being continuous. Hence $(\mathbb{R}, \mathcal{T})$ is not a $k_{D(2)}$-space. Since $(\mathbb{R}, \mathcal{T})$ is zero-dimensional and the weight of $(\mathbb{R}, \mathcal{T})$ is equal to $\mathfrak{c} = |\mathbb{R}|$, the space $(\mathbb{R}, \mathcal{T})$ is homeomorphic to a subspace C of the space $D(2)^{\mathfrak{c}}$. By A denote the set $D(2)^{\mathfrak{c}} \smallsetminus C$ and for each $a \in A$, by B_a denote the subspace $D(2)^{\mathfrak{c}} \smallsetminus \{a\}$ of the space $D(2)^{\mathfrak{c}}$. Clearly, each B_a is a zero-dimensional locally compact T_2-space and $C = \bigcap_{a\in A} B_a$. According to Proposition 3.3, C is homeomorphic to a closed subspace D of the space $\prod_{a\in A} B_a$ which is (by Proposition 3.2) a



$k_{\mathbb{R}}$-space and consequently, a $k_D(2)$-space. Hence $(\mathbb{R}, \mathcal{T})$ is homeomorphic to a closed subspace of the space $\prod_{a\in A} B_a$ and the space $\prod_{a\in A} B_a$ belongs to $\mathbf{K}_{D(2)} \cap \mathbf{A}$. $\square$

Remark. Similarly, it can be shown that if $\mathbf{A}$ is an epireflective subcategory of **Top** containing the space $\mathbb{R}$, then $\mathbf{K}_{\mathbb{R}} \cap \mathbf{A}$ is a coreflective subcategory of $\mathbf{A}$ which is not closed hereditary.

Proposition 3.5. *Let* $\mathbf{A}$ *be a non-trivial epireflective subcategory of* **Top** *and* $\mathbf{B}$ *be a closed hereditary subcategory of* $\mathbf{A}$. *Then the subcategory* $\mathrm{AD}_{\mathbf{A}}(\mathbf{B})$ *is also closed hereditary.*

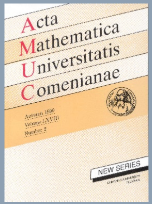
Proof. Let X be a space from $\mathrm{AD}_{\mathbf{A}}(\mathbf{B})$ and Y be a closed subspace of X. Then there exists a family of spaces $\{X_i\}_{i\in I}$ in $\mathbf{B}$ and a quotient map $f\colon \coprod_{i\in I} X_i \to X$. For every $i \in I$, the space $f^{-1}(Y) \cap X_i$ is a closed subspace of X_i, hence $f^{-1}(Y) \cap X_i \in \mathrm{AD}_{\mathbf{A}}(\mathbf{B})$. The space $f^{-1}(Y) = \coprod_{i\in I} f^{-1}(Y) \cap X_i$ also belongs to $\mathrm{AD}_{\mathbf{A}}(\mathbf{B})$. And since the map $f|_{f^{-1}(Y)}\colon f^{-1}(Y) \to Y$ is quotient, we obtain that $Y \in \mathrm{AD}_{\mathbf{A}}(\mathbf{B})$ and the subcategory $\mathrm{AD}_{\mathbf{A}}(\mathbf{B})$ is closed hereditary. $\square$

The preceding proposition does not hold in general if the AD hull is replaced by the coreflective hull (see Example 4.2).

Next we describe the closed hereditary AD hull of an AD subcategory in a non-trivial epireflective subcategory of **Top**.

Proposition 3.6. *Let* $\mathbf{A}$ *be a non-trivial epireflective subcategory of the category* **Top** *and* $\mathbf{B}$ *be an AD subcategory of* $\mathbf{A}$. *Let* $\mathbf{C}$ *denote the subcategory of* $\mathbf{A}$ *consisting of all spaces* $Y \in \mathbf{A}$ *such that there exists a quotient map* $f\colon X \to Y$ *from a closed subspace* X *of a space from* $\mathbf{B}$. *Then* $\mathbf{C}$ *is the closed hereditary AD hull of* $\mathbf{B}$ *in* $\mathbf{A}$. *Moreover, if* $\mathbf{Top_1} \subseteq \mathbf{A}$, *then the closed hereditary AD hull of* $\mathbf{B}$ *coincides with* $\mathrm{CS}\,\mathbf{B}$.

Proof. Obviously, $\mathbf{C}$ is an AD subcategory. We show that it is also closed hereditary. Let $X \in \mathbf{C}$ and Y be a closed subspace of X. Then there exists a space $X_2 \in \mathrm{CS}\,\mathbf{B}$ and a quotient



map $f\colon X_2 \to X$. The subspace $f^{-1}(Y)$ is closed in X_2, hence $f^{-1}(Y) \in \mathrm{CS}\,\mathbf{B}$ and the map $f|_{f^{-1}(Y)}\colon f^{-1}(Y) \to Y$ is quotient. Thus $Y \in \mathbf{C}$ and $\mathbf{C}$ is additive, divisible and closed hereditary.

Let $\mathbf{D}$ be an arbitrary additive, divisible and closed hereditary subcategory of $\mathbf{A}$ such that $\mathbf{B} \subseteq \mathbf{D}$. Let $X \in \mathbf{C}$. Then there exists a space $Y \in \mathrm{CS}\,\mathbf{B}$ and a quotient map $f\colon Y \to X$. Since $\mathbf{B} \subseteq \mathbf{D}$ and $\mathbf{D}$ is closed hereditary and divisible, we obtain that $X \in \mathbf{D}$. Hence $\mathbf{C} \subseteq \mathbf{D}$ and $\mathbf{C}$ is the closed hereditary AD hull of $\mathbf{B}$.

Finally let $\mathbf{Top_1} \subseteq \mathbf{A}$. Then $\mathbf{A} = \mathbf{Top}$, $\mathbf{A} = \mathbf{Top_0}$ or $\mathbf{A} = \mathbf{Top_1}$. The subcategory $\mathrm{CS}\,\mathbf{B}$ of $\mathbf{A}$ is obviously closed hereditary and additive. We show that $\mathrm{CS}\,\mathbf{B}$ is also divisible. Let $X \in \mathrm{CS}\,\mathbf{B}$ and $f\colon X \to Y$ be a quotient map with $Y \in \mathbf{A}$. There exists a space $X_2 \in \mathbf{B}$ such that X is a closed subspace of X_2. Suppose (without loss of generality) that $Y \cap X_2 = \emptyset$ and define the map $g\colon X_2 \to Y \cup X_2 \smallsetminus X$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in X, \\ x & \text{if } x \in X_2 \smallsetminus X. \end{cases}$$

By Z denote the space on the set $Y \cup X_2 \smallsetminus X$ for which g is a quotient map. Obviously, the space Y is a closed subspace of Z and the subspace $X_2 \smallsetminus X$ of X_2 is an open subspace of Z. It is easy to check that if X_2 is a T_0-space (T_1-space), then Z is also a T_0-space (T_1-space). Hence $Z \in \mathbf{A}$ and therefore $Z \in \mathbf{B}$. Consequently, $Y \in \mathrm{CS}\,\mathbf{B}$ and $\mathrm{CS}\,\mathbf{B}$ is the closed hereditary AD hull of $\mathbf{B}$. $\square$

It is obvious that if $\mathbf{B}$ is a subcategory of an epireflective subcategory $\mathbf{A}$ of $\mathbf{Top}$, $\mathbf{C}$ is the AD hull of $\mathbf{B}$ in $\mathbf{A}$ and $\mathbf{D}$ is the closed hereditary AD hull of $\mathbf{C}$ in $\mathbf{A}$, then $\mathbf{D}$ is the closed hereditary AD hull of $\mathbf{B}$ in $\mathbf{A}$. Hence, as a consequence of Proposition 3.6, we obtain the following.

Theorem 3.7. *Let* $\mathbf{A}$ *be a non-trivial epireflective subcategory of the category* $\mathbf{Top}$ *and* $\mathbf{B}$ *be a subcategory of* $\mathbf{A}$. *Let* $\mathbf{C}$ *denote the subcategory of* $\mathbf{A}$ *consisting of all spaces* $Y \in \mathbf{A}$ *such that there exists a quotient map* $f\colon X \to Y$ *from a closed subspace* X *of a topological sum of spaces belonging to* $\mathbf{B}$. *Then* $\mathbf{C}$ *is the closed hereditary AD hull of* $\mathbf{B}$ *in* $\mathbf{A}$.

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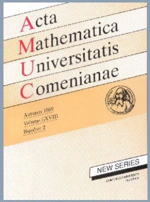


It is known (see [13]) that every Tychonoff space is homeomorphic to a closed subspace of a Tychonoff $k_{\mathbb{R}}$-space. From Theorem 3.7 we obtain the following proposition.

Proposition 3.8. *Each T_1-space (T_0-space, topological space) is homeomorphic to a closed subspace of a T_1-$k_{\mathbb{R}}$-space (T_0-$k_{\mathbb{R}}$-space, $k_{\mathbb{R}}$-space).*

Proof. The subcategory $\mathbf{K}_{\mathbb{R}} \cap \mathbf{Top_1}$ is coreflective in $\mathbf{Top_1}$. According to Theorem 3.7, the subcategory $\mathbf{A} = \mathrm{CS}(\mathbf{K}_{\mathbb{R}} \cap \mathbf{Top_1})$ is also coreflective in $\mathbf{Top_1}$. Let X be a T_1-space. Each prime factor X_a of X is a zero-dimensional space and similarly, to the case of the space $(\mathbb{R}, \mathcal{T})$ in the proof of Proposition 3.4, it can be shown that the space X_a is homeomorphic to a closed subspace of a zero-dimensional $k_{\mathbb{R}}$-space, and hence it belongs to $\mathbf{A}$. Each topological space is a quotient space of the sum of its prime factors, thus the space X belongs to $\mathbf{A}$ and consequently, it is homeomorphic to a closed subspace of some T_1-$k_{\mathbb{R}}$-space. The proof for T_0-spaces and topological spaces is analogous (using the fact that every space is a quotient space of a topological sum of prime T_1-spaces). $\square$

The closed hereditary AD hull of a subcategory $\mathbf{B}$ of $\mathbf{A}$ can be alternatively described as follows

Proposition 3.9. *Let $\mathbf{A}$ be a non-trivial epireflective subcategory of the category $\mathbf{Top}$ and $\mathbf{B}$ be a subcategory of $\mathbf{A}$. Then $\mathrm{CS}(\mathrm{AD}(\mathbf{B})) \cap \mathbf{A}$ is the closed hereditary AD hull of $\mathbf{B}$ in $\mathbf{A}$.*

Proof. Obviously, $\mathrm{CS}(\mathrm{AD}(\mathbf{B})) \cap \mathbf{A}$ is a closed hereditary AD subcategory of $\mathbf{A}$ that contains the subcategory $\mathbf{B}$. Let $\mathbf{C}$ be a closed hereditary AD subcategory of $\mathbf{A}$ containing $\mathbf{B}$. Then, according to Proposition 3.5, $\mathrm{AD}(\mathbf{C})$ is a closed hereditary AD subcategory of $\mathbf{Top}$ and, clearly, $\mathrm{AD}(\mathbf{C}) \cap \mathbf{A} = \mathbf{C}$. Obviously, $\mathrm{AD}(\mathbf{B}) \subseteq \mathrm{AD}(\mathbf{C})$ and since $\mathrm{AD}(\mathbf{C})$ is closed hereditary we get $\mathrm{CS}(\mathrm{AD}(\mathbf{B})) \subseteq \mathrm{AD}(\mathbf{C})$. Hence $\mathrm{CS}(\mathrm{AD}(\mathbf{B})) \cap \mathbf{A} \subseteq \mathrm{AD}(\mathbf{C}) \cap \mathbf{A} = \mathbf{C}$. $\square$

The following proposition describes the closed hereditary AD kernel of an AD subcategory of a non-trivial epireflective subcategory of $\mathbf{Top}$.

Proposition 3.10. *Let* **A** *be a non-trivial epireflective subcategory of the category* **Top** *and* **B** *be an AD subcategory of* **A**. *By* **C** *denote the subcategory of* **B** *consisting of all spaces* $X \in \mathbf{B}$ *such that each closed subspace* Y *of* X *belongs to* **B**. *Then* **C** *is the closed hereditary AD kernel of* **B** *in* **A**.

Proof. Obviously, **C** is closed hereditary and additive. Let $X \in \mathbf{C}$ and $f\colon X \to Y$ be a quotient map. If Y_2 is a closed subspace of Y, then $f^{-1}(Y_2)$ is a closed subspace of X and $f|_{f^{-1}(Y_2)}\colon f^{-1}(Y_2) \to Y_2$ is a quotient map. Therefore, $Y_2 \in \mathbf{B}$, $Y \in \mathbf{C}$ and **C** is divisible.

It remains to show that if **D** is a closed hereditary, additive and divisible subcategory of **A** such that $\mathbf{D} \subseteq \mathbf{B}$, then $\mathbf{D} \subseteq \mathbf{C}$. Let $X \in \mathbf{D}$. Every closed subspace of X belongs to **D**, thus it belongs to **B**. Hence $X \in \mathbf{C}$, therefore, $\mathbf{D} \subseteq \mathbf{C}$. □

Remark. If **B** is not an AD subcategory of an epireflective subcategory **A** of **Top**, then it does not need to have a closed hereditary AD kernel. For instance, if **C** and **D** are closed hereditary AD subcategories such that $\mathbf{C} \cup \mathbf{D}$ is not a closed hereditary AD subcategory, then $\mathbf{C} \cup \mathbf{D}$ does not have a closed hereditary AD kernel. For example, we may take $\mathbf{C} = \mathrm{AD}_{\mathbf{A}}(C(\alpha))$, $\mathbf{D} = \mathrm{AD}_{\mathbf{A}}(C(\beta))$, where α and β are distinct regular cardinals.

Let $\mathrm{K}(\mathbf{C})$ denote the closed hereditary AD kernel of an AD subcategory **C** in the category **Top**. The closed hereditary AD kernel of an AD subcategory **B** in **A** can be alternatively described in the following way.

Proposition 3.11. *Let* **A** *be a non-trivial epireflective subcategory of the category* **Top** *and* **B** *be an AD subcategory of* **A**. *Then* $\mathrm{K}(\mathrm{AD}(\mathbf{B})) \cap \mathbf{A}$ *is the closed hereditary AD kernel of* **B** *in* **A**.

Proof. Obviously, $\mathrm{K}(\mathrm{AD}(\mathbf{B})) \cap \mathbf{A}$ is a closed hereditary AD subcategory of **A** and $\mathrm{K}(\mathrm{AD}(\mathbf{B})) \cap \mathbf{A} \subseteq \mathrm{AD}(\mathbf{B}) \cap \mathbf{A} = \mathbf{B}$. Let **C** be a closed hereditary AD subcategory of **A** such that **B** contains **C**. According to Proposition 3.5, $\mathrm{AD}(\mathbf{C})$ is a closed hereditary AD subcategory of **Top**. Clearly,

$\mathrm{AD}(\mathbf{C}) \subseteq \mathrm{AD}(\mathbf{B})$ and $\mathrm{AD}(\mathbf{C}) \subseteq \mathrm{K}(\mathrm{AD}(\mathbf{B}))$ ($\mathrm{AD}(\mathbf{C})$ is closed hereditary). Therefore, $\mathbf{C} = \mathrm{AD}(\mathbf{C}) \cap \mathbf{A} \subseteq \mathrm{K}(\mathrm{AD}(\mathbf{B}) \cap \mathbf{A}$. □

4. Closed hereditary coreflective subcategories

Every coreflective subcategory of a non-trivial epireflective subcategory of **Top** is also an AD subcategory. However, in epireflective subcategories of **Top** that are not quotient reflective, the converse is not true. There are AD subcategories that are not coreflective. For example, in the category **Tych** (**ZD**) the subcategory of all Tychonoff (zero-dimensional) k-spaces is additive and divisible without being coreflective. The categories $\mathbf{Top}(\alpha) \cap \mathbf{Tych}$ ($\mathbf{Top}(\alpha) \cap \mathbf{ZD}$) are hereditary coreflective subcategories of the category **Tych** (**ZD**), (see [5]) and consequently, they are closed hereditary. The subcategory of all Tychonoff (zero-dimensional) $k_{D(2)}$-spaces is coreflective in **Tych** (**ZD**) but not closed hereditary.

From Proposition 3.10 we obtain the following result for quotient reflective subcategories of the category **Top**.

Corollary 4.1. *Let* **A** *be a quotient reflective subcategory of the category* **Top** *and* **B** *be coreflective in* **A**. *By* **C** *denote the subcategory of* **B** *consisting of all spaces $X \in \mathbf{B}$ such that each closed subspace Y of X belongs to* **B**. *Then* **C** *is the closed hereditary coreflective kernel of* **B** *in* **A**.

The following example shows that Corollary 4.1 does not hold in general if **A** is epireflective but not quotient reflective. Note that every compact space is supposed to be a T_2-space (see [6]).

Example 4.2. The category **Tych** is epireflective in **Top** but not quotient reflective. The subcategory $\mathbf{K}_{\mathbb{R}} \cap \mathbf{Tych}$ is coreflective in **Tych** and it is not closed hereditary (see the remark after Proposition 3.4). Then the category **C** defined as above for $\mathbf{B} = \mathbf{K}_{\mathbb{R}} \cap \mathbf{Tych}$ contains all compact T_2-spaces. Suppose that it is the closed hereditary coreflective kernel of $\mathbf{K}_{\mathbb{R}} \cap \mathbf{Tych}$. Then

$\mathbf{C}$ contains the coreflective hull $\mathbf{L}$ of the subcategory of all compact T_2-spaces in $\mathbf{Tych}$. According to [16], $\mathbf{L} = \mathbf{K}_{\mathbb{R}} \cap \mathbf{Tych}$, and therefore, $\mathbf{C} = \mathbf{K}_{\mathbb{R}} \cap \mathbf{Tych}$. But $\mathbf{K}_{\mathbb{R}} \cap \mathbf{Tych}$ is not closed hereditary, a contradiction.

As a consequence of Theorem 3.7, we obtain the following result for quotient reflective subcategories of the category $\mathbf{Top}$.

Corollary 4.3. *Let $\mathbf{A}$ be a quotient reflective subcategory of the category $\mathbf{Top}$ and $\mathbf{B}$ be a subcategory of $\mathbf{A}$. Let $\mathbf{C}$ denote the subcategory of $\mathbf{A}$ consisting of all spaces $Y \in \mathbf{A}$ such that there exists a quotient map $f\colon X \to Y$ from a closed subspace X of a topological sum of spaces belonging to $\mathbf{B}$. Then $\mathbf{C}$ is the closed hereditary coreflective hull of $\mathbf{B}$ in $\mathbf{A}$.*

The following proposition describes the construction of the closed hereditary coreflective hull of a subcategory in an arbitrary non-trivial epireflective subcategory of the category $\mathbf{Top}$.

Proposition 4.4. *Let $\mathbf{A}$ be a non-trivial epireflective subcategory of the category $\mathbf{Top}$ and $\mathbf{B}$ be a subcategory of $\mathbf{A}$. Let $\mathbf{B_1} = \mathbf{B}$, $\mathbf{B}_{\alpha+\mathbf{1}} = \mathrm{CH}_{\mathbf{A}}(\mathrm{CS}\mathbf{B}_\alpha)$ and $\mathbf{B}_\alpha = \bigcup_{\beta<\alpha} \mathbf{B}_\beta$ if α is a limit ordinal. Then the subcategory $\mathbf{B}^* = \bigcup_{\alpha\in\mathrm{On}} \mathbf{B}_\alpha$ is the closed hereditary coreflective hull of $\mathbf{B}$ in $\mathbf{A}$.*

Proof. We first show that $\mathbf{B}^*$ is closed hereditary. Let $X \in \mathbf{B}^*$ and Y be a closed subspace of X. There exists an ordinal α such that $X \in \mathbf{B}_\alpha$. Then $Y \in \mathbf{B}_{\alpha+\mathbf{1}} \subseteq \mathbf{B}^*$ and $\mathbf{B}^*$ is closed hereditary. Now, let $X \in \mathbf{B}^*$ and $f\colon X \to Y$ be an extremal epimorphism. There exists an ordinal α such that $X \in \mathbf{B}_\alpha$. Then also $Y \in \mathbf{B}_\alpha \subseteq \mathbf{B}^*$. Therefore, $\mathbf{B}^*$ is closed under extremal quotient objects. Finally, let $\{X_i\}_{i\in I} \subseteq \mathbf{B}^*$. There exists an ordinal α such that $\{X_i\}_{i\in I} \subseteq \mathbf{B}_\alpha$. Then $\coprod_{i\in I} X_i \in \mathbf{B}_{\alpha+\mathbf{1}} \subseteq \mathbf{B}^*$. Hence $\mathbf{B}^*$ is coreflective.

Let $\mathbf{C}$ be a closed hereditary coreflective subcategory of $\mathbf{A}$ such that $\mathbf{B} = \mathbf{B_1} \subseteq \mathbf{C}$. We show by transfinite induction that $\mathbf{B}_\alpha \subseteq \mathbf{C}$ for every ordinal α, thus $\mathbf{B}^* \subseteq \mathbf{C}$. Let $\mathbf{B}_\beta \subseteq \mathbf{C}$ for every $\beta < \alpha$. If α is a limit ordinal, then $\mathbf{B}_\alpha = \bigcup_{\beta<\alpha} \mathbf{B}_\beta$, therefore, $\mathbf{B}_\alpha \subseteq \mathbf{C}$. If α is not a limit ordinal, then

$\mathrm{CS}\mathbf{B}_{\alpha-\mathbf{1}} \subseteq \mathbf{C}$, because $\mathbf{B}_{\alpha-\mathbf{1}} \subseteq \mathbf{C}$ and $\mathbf{C}$ is closed hereditary. $\mathbf{C}$ is also coreflective, therefore, $\mathrm{CH}_{\mathbf{A}}(\mathrm{CS}\mathbf{B}_{\alpha-\mathbf{1}}) = \mathbf{B}_\alpha \subseteq \mathbf{C}$. Thus for every ordinal α, we have $\mathbf{B}_\alpha \subseteq \mathbf{C}$, consequently $\mathbf{B}^* \subseteq \mathbf{C}$. Hence $\mathbf{B}^*$ is the closed hereditary coreflective hull of $\mathbf{B}$ in $\mathbf{A}$. □

5. Minimal closed hereditary AD subcategories containing a non-discrete space

Let $\mathbf{A}$ be a non-trivial epireflective subcategory of **Top**. An AD subcategory $\mathbf{B}$ of $\mathbf{A}$ is called non-trivial if it contains a non-discrete space. In this section we present some results related to minimal non-trivial closed hereditary AD subcategories in non-trivial epireflective subcategories of **Top**. Recall that in quotient reflective subcategories of **Top**, the AD subcategories are precisely the coreflective subcategories.

It is well known (see, e.g., [7]) that the category of all sums of indiscrete spaces is the smallest non-trivial AD subcategory of **Top** and the category of all finitely generated T_0-spaces is the smallest non-trivial AD subcategory of $\mathbf{Top_0}$. Moreover, both of them are hereditary (and, consequently, closed hereditary). Recall that a topological space is finitely generated if every intersection of open subsets is again an open subset.

We prove that in the category $\mathbf{Top_1}$, there exist no minimal non-trivial closed hereditary AD subcategories. We need the following proposition.

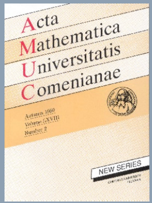
Proposition 5.1. *Let $\mathbf{A}$ be a non-trivial AD subcategory of the category $\mathbf{Top_1}$ and X be a non-discrete space from $\mathbf{A}$. Let α be the smallest cardinal such that there exists a non-closed subset A of X with $|A| = \alpha$. Then there exists a space $Y \in \mathbf{A}$ with $|Y| = |X|$ such that a subset B of Y is closed if and only if $|B| < \alpha$ or $B = Y$. Moreover, the category $\mathrm{AD}_{\mathbf{Top_1}}(Y)$ is closed hereditary.*

Proof. Let $\mathbf{A}$ be an AD subcategory of $\mathbf{Top_1}$ and $(X, \mathcal{T}) \in \mathbf{A}$ be a non-discrete space. Let $M = \{f_i : i \in I\}$ be the set of all bijections on the set X and $\mathcal{T}_i$ be the topology on the set X for which the map $f_i \colon (X, \mathcal{T}) \to (X, \mathcal{T}_i)$ is a homeomorphism. Let $g \colon \coprod_{i\in I}(X, \mathcal{T}_i) \to X$ be the map defined by $g \circ m_i = id_X$ for all natural embeddings $m_i \colon (X, \mathcal{T}_i) \to \coprod_{j \in I}(X, \mathcal{T}_j)$ and $\mathcal{T}'$ be the quotient topology on X with respect to g. Obviously, $\mathcal{T}' = \bigcap_{i \in I} \mathcal{T}_i$, $(X, \mathcal{T}')$ is a T_1-space and $(X, \mathcal{T}') \in \mathbf{A}$. Let α be the smallest cardinal such that there exists a non-closed subset A of $(X, \mathcal{T})$ with $|A| = \alpha$ (clearly, α is infinite). Finally, let B be a subset of $(X, \mathcal{T}')$ such that $|B| \geq \alpha$ and $B \neq X$. Then there exists a subset C of $(X, \mathcal{T})$ with cardinality $|B|$ that is not closed. Suppose the contrary and take subsets B_1, B_2 of X such that $|B_1| = |B_2| = |B|$ and $B_1 \cap B_2 = \emptyset$. The space $(X, \mathcal{T})$ has a subset A that is not closed and $|A| = \alpha$. Then $|B_1 \cup A| = |B_2 \cup A| = |B|$, therefore, the subsets $B_1 \cup A$ and $B_2 \cup A$ are closed in $(X, \mathcal{T})$. But then also $(B_1 \cup A) \cap (B_2 \cup A) = A$ is closed, a contradiction. Clearly, there exists a bijection f_{i_0} such that $f_{i_0}[C] = B$. Since the subset C is not closed in $(X, \mathcal{T})$, the subset B is not closed in $(X, \mathcal{T}_{i_0})$, and therefore, (because $\mathcal{T}' \subseteq \mathcal{T}_{i_0}$) B is not closed in $(X, \mathcal{T}')$. It remains to show that the subcategory $\mathrm{AD}_{\mathbf{Top_1}}((X, \mathcal{T}'))$ is closed hereditary. Let Z be a closed subspace of $(X, \mathcal{T}')$. Then $Z = X$ or $|Z| < \alpha$ and in this case Z is a discrete space. Hence $Z \in \mathrm{AD}_{\mathbf{Top_1}}((X, \mathcal{T}'))$ and $\mathrm{AD}_{\mathbf{Top_1}}((X, \mathcal{T}'))$ is closed hereditary. □

Corollary 5.2. *Let* $\mathbf{A}$ *be an AD subcategory of the category* $\mathbf{Top_1}$*. The category* $\mathbf{Dis}$ *is the closed hereditary AD kernel of* $\mathbf{A}$ *in* $\mathbf{Top_1}$ *only if* $\mathbf{A} = \mathbf{Dis}$*.*

Proposition 5.3. *In the category* $\mathbf{Top_1}$*, there exist no minimal non-trivial closed hereditary AD subcategories.*

Proof. Assume that $\mathbf{A}$ is a minimal non-trivial closed hereditary AD subcategory of $\mathbf{Top_1}$. By Proposition 5.1, there exists an (infinite) cardinal α and a space $X \in \mathbf{A}$ with $\alpha \leq |X|$ such that a subspace A of X is closed if and only if $|A| < \alpha$ or $A = X$. The subcategory $\mathrm{AD}_{\mathbf{Top_1}}(X)$ is closed hereditary and it is a subcategory of $\mathbf{A}$. Since $\mathbf{A}$ is minimal, $\mathrm{AD}_{\mathbf{Top_1}}(X) = \mathbf{A}$ holds. Let I be a set such that $|I| > |X|$ and $Y = \coprod_{a \in I} X_a$, where $X_a = X$ for every $a \in I$. Clearly, Y is non-discrete and α is the smallest cardinal such that Y has a subset of cardinality α that is not closed. By Proposition 5.1, there exists a space $Y' \in \mathrm{AD}_{\mathbf{Top_1}}(X)$ such that $|Y'| = |Y| > |X|$ and



a subset A of Y' is closed if and only if $|A| < \alpha$ or $A = Y'$. The category $\mathrm{AD}_{\mathbf{Top_1}}(Y')$ is closed hereditary and it is a subcategory of $\mathrm{AD}_{\mathbf{Top_1}}(X)$. Therefore, $\mathrm{AD}_{\mathbf{Top_1}}(Y') = \mathrm{AD}_{\mathbf{Top_1}}(X)$. Then $X \in \mathrm{AD}_{\mathbf{Top_1}}(Y')$, thus there exists a quotient map $f\colon \coprod_{i\in J} Y_i \to X$, where $Y_i = Y'$ for every $i \in J$. For each $j \in J$, by $m_j\colon Y' \to \coprod_{i\in I} Y_i$ denote the corresponding natural embedding. Then for some $i_0 \in J$, the set $f[m_{i_0}[Y']]$ is not a singleton (otherwise, X would be a discrete space). Singletons are closed in X, therefore, for every $c \in X$, the set $Z_c = f^{-1}(c) \cap m_{i_0}[Y']$ is closed in $m_{i_0}[Y']$. Since $Z_c \neq m_{i_0}[Y']$, we get $|Z_c| < \alpha$. Then $m_{i_0}[Y'] = \bigcup_{c\in X} Z_c$ and we obtain that $|X| < |Y| = |Y'| = |m_{i_0}[Y']| = |\bigcup_{c\in X} Z_c| \leq |X| \cdot \alpha = |X|$, a contradiction. $\square$

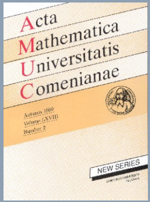
Next we investigate minimal non-trivial closed hereditary AD subcategories in epireflective subcategories $\mathbf{A}$ of $\mathbf{Top}$ such that $\mathbf{A} \subseteq \mathbf{Haus}$. A topological space X is called pseudoradial if a subspace A of X is closed whenever every limit of every transfinite sequence of elements of A belongs to A. It is known that the category of all pseudoradial spaces is the AD hull of the subcategory $\{C(\alpha) : \alpha \text{ is a regular cardinal}\}$ in $\mathbf{Top}$.

Proposition 5.4. *Let $\mathbf{A}$ be a non-trivial epireflective subcategory of $\mathbf{Top}$ such that $\mathbf{A} \subseteq \mathbf{Haus}$ and $\mathbf{B}$ be a closed hereditary AD subcategory of $\mathbf{A}$ which contains a non-discrete pseudoradial space. Then there exists a regular cardinal α such that $\mathrm{AD}_{\mathbf{A}}(C(\alpha)) \subseteq \mathbf{B}$.*

First we prove the following lemma.

Lemma 5.5. *Let α be a regular cardinal, X be a Hausdorff P_α-space and $f\colon C(\alpha) \to X$ be a continuous map. Then the following holds:*

a) *The subspace $f[C(\alpha)]$ is closed in X.*
b) *The map $f\colon C(\alpha) \to X$ is closed.*
c) *If $\{f(\alpha)\}$ is an open subset in the subspace $f[C(\alpha)]$ of X, then $f[C(\alpha)]$ is a discrete space.*
d) *If $f[C(\alpha)]$ is not discrete, then there exists a set $A \subseteq \alpha$ such that $|A| = \alpha$ and the map $f|_{A\cup\{\alpha\}}\colon A \cup \{\alpha\} \to f[C(\alpha)]$ is a homeomorphism. Thus $f[C(\alpha)]$ is homeomorphic to $C(\alpha)$.*



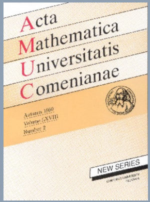
Proof. a) Let $c \in X \setminus f[C(\alpha)]$. Since X is a T_2-space, there are neighbourhoods U of $f(\alpha)$ and V of c in X such that $U \cap V = \emptyset$. The subset $U_1 = f^{-1}(U)$ is open in $C(\alpha)$ and $\alpha \in U_1$, therefore, $|C(\alpha) \setminus U_1| < \alpha$. Let B denote $f[C(\alpha) \setminus U_1]$. Then $|B| < \alpha$. For every $x \in B$, we have $x \neq c$, thus there exist neighbourhoods U_x of x and V_x of c in X such that $U_x \cap V_x = \emptyset$. Obviously, $f[C(\alpha)] \subseteq U \cup \bigcup_{x \in B} U_x = W_1$. The set $W_2 = V \cap \bigcap_{x \in B} V_x$ is a neighbourhood of c and $W_1 \cap W_2 = \emptyset$. Hence $W_2 \cap f[C(\alpha)] = \emptyset$ and therefore, the subspace $f[C(\alpha)]$ is closed in X.

b) Let F be a closed subspace of $C(\alpha)$. Then $|F| < \alpha$ or $|F| = \alpha$ and $\alpha \in F$. If $|F| < \alpha$, then also $|f[F]| < \alpha$, thus $f[F]$ is closed in X. If $|F| = \alpha$ and $\alpha \in F$, then F is homeomorphic to $C(\alpha)$ and the map $f|_F \colon F \to X$ is continuous. By part a), $f|_F[F] = f[F]$ is a closed subspace of X.

c) Let $\{f(\alpha)\}$ be an open subset of $f[C(\alpha)]$. Then $f^{-1}(f(\alpha)) = U$ is open in $C(\alpha)$, $\alpha \in U$ and $|C(\alpha) \setminus U| < \alpha$. For every $x \in f[C(\alpha)] \setminus \{f(\alpha)\}$, we have $f^{-1}(x) \subseteq C(\alpha) \setminus U$, therefore, it is an open subset of $C(\alpha)$. Since the map $f \colon C(\alpha) \to X$ is closed, also $f \colon C(\alpha) \to f[C(\alpha)]$ is closed, thus it is a quotient map. Since the subset $f^{-1}(x)$ is open in $C(\alpha)$, also $\{x\}$ is open in $f[C(\alpha)]$.

d) Let $f^{-1}(f(\alpha)) = C$. Then $\alpha \in C$ and C is not open in $C(\alpha)$ (otherwise, $\{f(\alpha)\}$ would be open in $f[C(\alpha)]$). Therefore, for $D = C(\alpha) \setminus C$, we have $|D| = \alpha$. For every $x \in f[C(\alpha)] \setminus \{f(\alpha)\}$, let $D_x = f^{-1}(x)$. The subset D_x is closed in $C(\alpha)$ and $\alpha \notin D_x$, hence $|D_x| < \alpha$. Then obviously, $|f[C(\alpha)] \setminus \{f(\alpha)\}| = \alpha$ because α is regular. Let A be a subset of α such that for every $x \in f[C(\alpha)] \setminus \{f(\alpha)\}$, $A \cap D_x$ is a singleton. The subspace $A \cup \{\alpha\}$ of $C(\alpha)$ is homeomorphic to $C(\alpha)$ because $|A| = \alpha$. Then the map $f|_{A \cup \{\alpha\}} : A \cup \{\alpha\} \to f[C(\alpha)]$ is continuous, bijective and closed, therefore it is a homeomorphism. □

Proof of Proposition 5.4. Let X be a non-discrete pseudoradial space from $\mathbf{B}$. Then there exists a family of regular cardinals $\{\alpha_i\}_{i \in I}$ and a quotient map $f \colon \coprod_{j \in I} C(\alpha_j) \to X$. Let $m_i \colon C(\alpha_i) \to \coprod_{j \in I} C(\alpha_j)$ be the natural embedding of $C(\alpha_i)$ to $\coprod_{j \in I} C(\alpha_j)$, $J = \{i \in I : f \circ m_i[C(\alpha_i)]$ is not a discrete subspace of $X\}$ and $m'_i \colon C(\alpha_i) \to \coprod_{j \in J} C(\alpha_j)$ be the natural embedding corresponding to $\coprod_{j \in J} C(\alpha_j)$. Then also the map $g \colon \coprod_{j \in J} C(\alpha_j) \to X$ such that $g \circ m'_i = f \circ m_i$ for every $i \in J$, is



quotient. Choose $i_0 \in J$ such that $\alpha_{i_0} \leq \alpha_i$, for every $i \in J$. Then $\coprod_{j\in J} C(\alpha_j)$ is a $P_{\alpha_{i_0}}$-space and consequently, X is a Hausdorff $P_{\alpha_{i_0}}$-space. The map $g \circ m'_{i_0} = h\colon C(\alpha_{i_0}) \to X$ is continuous and the subspace $h[C(\alpha_{i_0})]$ of X is not discrete. Hence by Lemma 5.5 a) and d), $h[C(\alpha_{i_0})]$ is a closed subspace of X and it is homeomorphic to $C(\alpha_{i_0})$. Thus $C(\alpha_{i_0}) \in \mathbf{B}$ and $\mathrm{AD}_{\mathbf{A}}(C(\alpha_{i_0})) \subseteq \mathbf{B}$. □

Corollary 5.6. *Let* $\mathbf{A}$ *be a non-trivial epireflective subcategory of* $\mathbf{Top}$ *such that* $\mathbf{A} \subseteq \mathbf{Haus}$. *Then* $\mathrm{AD}_{\mathbf{A}}(C(\alpha))$ *are minimal non-trivial closed hereditary AD subcategories of* $\mathbf{A}$.

We do not know whether the collection of all subcategories $\mathrm{AD}_{\mathbf{A}}(C(\alpha))$, α being a regular cardinal, contains all minimal non-trivial closed hereditary AD subcategories.

Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces. Then $X \prec Y$ means that $X = Y$ and $\mathcal{T}_Y \subseteq \mathcal{T}_X$.

Lemma 5.7. [16, Lema 6.3] *Let* α *be any infinite cardinal. If* $Y \prec C(\alpha)$ *is a prime space (with the accumulation point* α*), then there exists a regular cardinal* β *with* $C(\beta) \in \mathrm{CH}(Y)$.

Proposition 5.8. [16, Proposition 6.5] *If* X *is not a sum of connected spaces, then there exists a cardinal* α *and a quotient map* $f\colon X \to P$*, where* P *is a prime* T_2*-space and* $P \prec C(\alpha)$.

In the following let $\mathbf{SC}$ denote the category of all topological sums of connected spaces. From Lemma 5.7 and Proposition 5.8, we obtain the following result.

Proposition 5.9. *Let* $\mathbf{A}$ *be a non-trivial epireflective subcategory of the category* $\mathbf{Top}$ *such that* $\mathbf{A} \subseteq \mathbf{Haus}$ *and* $\mathbf{B}$ *be an AD subcategory of* $\mathbf{A}$ *such that* $\mathbf{B} \not\subseteq \mathbf{SC}$. *Then there exists a regular cardinal* α *such that* $\mathrm{AD}_{\mathbf{A}}(C(\alpha)) \subseteq \mathbf{B}$.

Recall that a topological space is said to be totally disconnected if all its components are singletons. The category $\mathbf{TD}$ of all totally disconnected spaces is a quotient reflective subcategory of $\mathbf{Top}$. If a totally disconnected space X is a topological sum of connected spaces, then it is a discrete space. As a consequence of Proposition 5.9 we obtain the following result.

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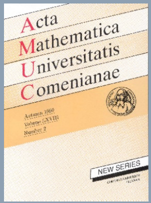

Proposition 5.10. *Let* $\mathbf{A}$ *be a non-trivial epireflective subcategory of the category* $\mathbf{Top}$ *such that* $\mathbf{A} \subseteq \mathbf{TD} \cap \mathbf{Haus}$ *and* $\mathbf{B} \neq \mathbf{Dis}$ *be an AD subcategory of* $\mathbf{A}$. *Then there exists a regular cardinal* α *such that* $\mathrm{AD}_{\mathbf{A}}(C(\alpha)) \subseteq \mathbf{B}$.

In this case for every minimal non-trivial AD subcategory $\mathbf{B}$ of $\mathbf{A}$, there exists a regular cardinal α such that $\mathbf{B} = \mathrm{AD}_{\mathbf{A}}(C(\alpha))$.

Proposition 5.9 yields also the following result.

Proposition 5.11. *Let* $\mathbf{A}$ *be a non-trivial epireflective subcategory of the category* $\mathbf{Top}$ *such that* $\mathbf{A} \subseteq \mathbf{Haus}$ *and* $\mathbf{B}$ *be an AD subcategory of* $\mathbf{A}$ *such that its closed hereditary AD kernel is the subcategory of all discrete spaces. Then* $\mathbf{B} \subseteq \mathbf{SC}$.

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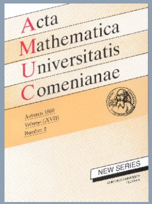

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