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DUALS OF VECTOR VALUED FUNCTION SPACES $c_0(X,U,M)$, $c(X,U,M)$ AND $l_\infty(X,U,M)$ DEFINED BY ORLICZ FUNCTION

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Abstract. In this paper we obtain the Köthe-Toeplitz duals of $c_0(X,U,M)$, $c(X,U,M)$ and $l_\infty(X,U,M)$. We extend the definition of Maddox and study of function spaces and sequence spaces defined also by Orlicz function. Further we characterize the continuous dual of $c_0(X,U,M)$ and $c(X,U,M)$.

1. Introduction and Preliminaries

We recall that an Orlicz function is a function $M\colon [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(u) > 0$ for all $u > 0$ and $M(u) \to \infty$ as $u \to \infty$ see [**4**]. The Theory of function spaces and sequence spaces using Orlicz function was extended by several authors [**1, 3, 5, 6, 8, 9, 10**]. Some of them characterized their topological properties and some their duals also.

Let U and V be Banach spaces over the field of complex number $\mathbb{C}$ and U^* be the continuous dual of U. $L(U,V)$ is the linear space of all linear operators $T\colon U \to V$. $B(U,V) \subset L(U,V)$ denotes the Banach space of all bounded linear operators T with a usual operator norm $\|T\| = \{\|Tu\| \mid u \in S\}$, where $S = \{u \in U \mid \|u\| \le 1\}$. θ denotes the zero of all these spaces.

Let X be an arbitrary set (not necessarily countable) and $\mathcal{F}(X)$ be the collection of all finite subsets of X directed by inclusion relation.

We now introduce the following classes of U-valued functions using Orlicz function M.

$$\begin{aligned} c_0(X,U,M) = \{f\colon X \to U \mid{} & \text{there exists } \rho > 0 \text{ such that for every } \varepsilon > 0 \\ & \text{there exists } J \in \mathcal{F}(X) \text{ satisfying } M\left(\|f(x)\|/\rho\right) < \varepsilon \\ & \text{for all } x \in X/J\ \}; \end{aligned} \tag{1.1}$$

$$\begin{aligned} c(X,U,M) = \{f\colon X \to U| \ & \text{there exists } \rho > 0 \text{ such that } \varepsilon > 0 \\ & \text{for every } l \in U\colon \text{there exists } J \in \mathcal{F}(X) \text{ satisfying} \\ & M\left(\|f(x) - l\|/\rho\right) < \varepsilon \text{ for all } x \in X/J\ \}; \end{aligned} \tag{1.2}$$

$$l_\infty(X,U,M) = \{f\colon X \to U \mid \sup_{x\in X} M\left(\|f(x)\|/\rho\right) < \infty \text{ for some } \rho > 0\}. \tag{1.3}$$

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It is obvious that $c_0(X,U,M) \subset c(X,U,M) \subset l_\infty(X,U,M)$.

For $\phi\colon X \to U$, we say that $\sum_{x\in X}\phi(x)$ is summable to $u \in U$ written as $\sum_{x\in X}\phi(x) = u$ if the directed system $(s_J)_{J\in\mathcal{F}(X)}$ with respect to set theoretic inclusion converges to $u \in U$, where $s_J = \sum_{x\in J}\phi(x)$. Of course, if such u exists, then it is unique. Similarly, replacing U by $\mathbb{C}$, we get the case of scalars (see [**2**, p. 32]).

Theorem 1.1. ([**2**, p. 32]) *If $\phi\colon X \to U$ then $\sum_{x\in X}\phi(x)$ is summable if and only if for every $\varepsilon > 0$, there exists a $J \in \mathcal{F}(X)$ such that $\|\sum_{x\in J_1}\phi(x)\| < \varepsilon$ for every $J_1 \in \mathcal{F}(X)$ satisfying $J_1 \cap J = \phi$.*

In 1980 I. J. Maddox examined generalized Köthe-Toeplitz duals of X-termed sequence classes, where X is a Banach space. Analogous by to the definition of Köthe-Toeplitz duals and group norm defined by I. J. Maddox for X-valued sequence, for function spaces definitions are as follows.

Definition 1.2. ([**7**]) Let $A\colon X \to L(U,V)$ not necessarily all $A(x)$ be bounded. Suppose $E(X,U)$ is a non-empty set of U-valued functions on X. Then the generalized Köthe-Toeplitz duals, i.e., generalized α- and β-duals of $E(X,U)$ are defined by

$$E^\alpha(X,U) = \{A\colon X \to L(U,V) : \sum_{x\in X}\|A(x)\phi(x)\| \text{ is summable for all } \phi \in E(X,U)\}$$

$$E^\beta(X,U) = \{A\colon X \to L(U,V) : \sum_{x\in X} A(x)\phi(x) \text{ is summable in } V \text{ for all } \phi \in E(X,U)\},$$

respectively, [**7**].

Definition 1.3. [**7**] The group norm of the family of operators $\{A(x) : x \in X\} \subset B(U,V)$ is defined by

$$\|A(x) : x \in X\| = \sup \Big\| \sum_{x\in J} A(x)\, u(x) \Big\|$$

where the supremum is taken over all $J \in \mathcal{F}(X)$ and all $u(x) \in S$.

The property of group norm for function spaces [**7**] analogous to sequences are as follows:

Lemma 1.4. *If $\{A(x) : x \in X\}$ is a family of operators in $B(U,V)$ then*

(i) *for any $J \in \mathcal{F}(X)$, $\|A(x)\| \le \|\{A(x) : x \in X|J\}\|$ for all $x \in X \setminus J$,*

(ii) *for any $J_1, J_2 \in \mathcal{F}(X)$ with $J_1 \subset J_2$,*

$$\|\{A(x) : x \in X \setminus J_2\}\| \le \|\{A(x) : x \in X \setminus J_1\}\|,$$

(iii) *for any $J, J_1 \in \mathcal{F}(X)$, with $J_1 \cap J = \phi$ and $u(x) \in U$, $x \in J_1$,*

$$\Big\| \sum_{x\in J_1} A(x)u(x) \Big\| \le \|\{A(x) : x \in X \setminus J\}\| \max\{\|u(x)\| : x \in J_1\}.$$

Lemma 1.5. *For any family $\{A(x) : x \in X\}$ of operators in $B(U, V)$ exactly only one of the following is true.*

(i) $\|\{A(x) : x \in X \setminus J\}\| = \infty$ *for all* $J \in \mathcal{F}(X)$,
(ii) $\|\{A(x) : x \in X \setminus J\}\| < \infty$ *for all* $J \in \mathcal{F}(X)$.

2. Topological Structure and Köthe-Toeplitz Duals

We easily see that $c_0(X, U, M)$, $c(X, U, M)$ and $l_\infty(X, U, M)$ form linear spaces over the field $\mathbb{C}$ with respect to point wise vector operations. Clearly the function $\theta\colon X \to U$ where $\theta(x) = 0$ for all $x \in X$, is the zero (functions) of these linear spaces. We can easily show that $c_0(X, U, M)$, $c(X, U, M)$ and $l_\infty(X, U, M)$ turn out to be a Banach space under the norm

$$\|x\|_\infty = \inf \left\{ \rho > 0 : \sum_{x \in X} M\left(\frac{\|f(x)\|}{\rho}\right) \le 1 \right\}$$

for $f \in c_0(X, U, M)$, $c(X, U, M)$ and $l_\infty(X, U, M)$, (see [**8**]).

Theorem 2.1. *Let $A\colon X \to L(U, V)$. Then $A \in c_0^\alpha(X, U, M)$, $c^\alpha(X, U, M)$ and $l_\infty^\alpha(X, U, M)$ if and only if*

(i) *there exists $J \in \mathcal{F}(X)$ such that $A(x) \in B(U, V)$ for all $x \in X \setminus J$, and*
(ii) $\sum_{x \in X \setminus J} \|A(x)\| < \infty$.

Proof. We give the proof for $c_0(X, U, M)$ only and the rest follows. For sufficiency of the conditions, take $f \in c_0(X, U, M)$ and $\rho > 0$ arbitrary. Then for given $\varepsilon > 0$, we can find $J_1 \in \mathcal{F}(X)$, $J_1 \supset J$, satisfying $M\left(\frac{\|f(x)\|}{\rho}\right) < \varepsilon$ for all $x \in X \setminus J_1$. Further we can choose $r > \varepsilon$ and $t_0 > 0$ to be fixed positive real number such that $r\frac{t_0}{2}\, q\left(\frac{t_0}{2}\right) > \varepsilon$, where q is the kernel associated with M. Hence, $M\left(\frac{\|f(x)\|}{\rho}\right) < r\frac{t_0}{2}\, q\left(\frac{t_0}{2}\right)$ for all $x \in X \setminus J_1$. Using the integral representation of Orlicz function, we easily get $\|f(x)\| \le \rho t_0 r$ for all $x \in X \setminus J_1$ and so

$$\sum_{x \in X} ||A(x) f(x)\| \le \sum_{x \in J_1} \|A(x) f(x)\| + \rho t_0 r \sum_{x \in X \setminus J_1} \|A(x)\| < \infty,$$

which clearly implies that $A \in c_0^\alpha(X, U, M)$.

For necessity of the conditions let $A \in c_0^\alpha(X, U, M)$. If (i) fails, then there exists a sequence (x_k) of distinct terms in X such that $A(x_k) \notin B(U, V)$ and so for each $k \ge 1$, we can find $u(x_k) \in S$, where S is the closed unit sphere $S[0, 1]$ in U such that

$$\|A(x_k)\, u(x_k)\| > k.$$

Let $\rho > 0$ and consider the function $f\colon X \to U$ defined by

$$f(x) = \begin{cases} k^{-1} u(x_k), & \text{for } x = x_k,\ k \ge 1, \\ \theta, & \text{otherwise} \end{cases}$$

is in $c_0(X,U,M)$ because for $x \neq x_k$, $k \geq 1$, $M\left(\frac{\|f(x)\|}{\rho}\right) = 0$ and for each $x = x_k$, we have $M\left(\frac{\|f(x)\|}{\rho}\right) = M\left(\frac{1}{k\rho}\right) \leq \frac{1}{k} M\left(\frac{1}{\rho}\right)$. But $\frac{1}{k} M\left(\frac{1}{\rho}\right) \to 0$ as $k \to \infty$, therefore, we can find K such that $\frac{1}{k} M\left(\frac{1}{\rho}\right) < \varepsilon$ for every $k \geq K$. Now if we take $J = \{x_1, x_2, \ldots, x_{k-1}\}$, then we have $M\left(\frac{\|f(x)\|}{\rho}\right) < \varepsilon$ for all $x \in X \setminus J$. On the other hand, we have

$$\|A(x_k)\, f(x_k)\| = \|k^{-1} A(x_k)\, u(x_k)\| \geq 1 \quad \text{for each } k \geq 1,$$

which implies that $A \notin c_0^\alpha(X,U,M)$, a contradiction.

Similarly if (ii) fails then there exists a sequence of pairwise disjoint sets $(J(N))_{N \geq 2}$, $J(N) \in \mathcal{F}(X)$ with $J(1) = J$ such that

$$\sum_{x \in J(N)} \|A(x)\| > 2N, \qquad N = 2, 3, 4, \ldots$$

Now for each $x \in X$, we choose $u(x) \in S$ such that $\|A(x)\| < 2\|A(x)u(x)\|$. Further take $\rho > 0$. It is straightforward to verify that the function $f\colon X \to U$ defined by

$$f(x) = \begin{cases} N^{-1} u(x), & x \in J(N),\ N = 2, 3, \ldots, \\ \theta, & \text{otherwise}, \end{cases}$$

is in $c_0(X,U,M)$, but

$$\sum_{x \in X} \|A(x) f(x)\| \geq \sum_{N=2}^{\infty} \frac{1}{2} \sum_{x \in J(N)} \|A(x)\| N^{-1} > \sum_{N=2}^{\infty} 1$$

contradicts that $A \in c_0^\alpha(X,U,M)$. Hence, it follows the necessity of (i) and (ii). This completes the proof. □

If we take $A\colon X \to B(U,V)$ in the above Theorem 2.1, then we have the following.

Theorem 2.2. *If $A\colon X \to B(U,V)$, then each one form $c_0^\alpha(X,U,M)$, $c^\alpha(X,U,M)$ and $l_\infty^\alpha(X,U,M)$ equals $H_0(X, B(U,V))$, where*

$$H_0(X, B(U,V)) = \big\{A\colon X \to B(U,V) \mid \sum_{x \in X} \|A(x)\| < \infty\big\}.$$

Theorem 2.3. *Let $A\colon X \to L(U,V)$. Then $A \in c_0^\beta(X,U,M)$ if and only if*

(i) *there exists $J \in \mathcal{F}(X)$ such that $A(x) \in B(U,V)$ for all $x \in X \setminus J$,*

(ii) $\|\{A(x) \ : \ x \in X \setminus J\}\| = L < \infty$.

Proof. Suppose that (i) and (ii) hold, $f \in c_0(X,U,M)$, $\rho > 0$ associated with f and $\varepsilon > 0$. Now for $M\left(\frac{\varepsilon}{\rho}\right) > 0$ we can find $J_1 \supset J$ such that $M\left(\frac{\|f(x)\|}{\rho}\right) < M\left(\frac{\varepsilon}{\rho}\right)$ for all $x \in X \setminus J_1$. Since M is non-decreasing, we have $\|f(x)\| < \varepsilon$ for all $x \in X \setminus J_1$.

Now for any $J_2 \in \mathcal{F}(X)$ with $J_2 \cap J_1 = \phi$, by Lemma 1.4 (iii), we get

$$\|\sum_{x\in J_2} A(x)f(x)\| \le \|\{A(x) : x \in X \smallsetminus J\}\| \max_{x\in J_2} \|f(x)\| < L\varepsilon.$$

Hence by Theorem 1.1, $\sum_{x\in X} A(x)\, f(x)$ is summable and consequently, $A \in c_0^\beta(X, U, M)$.

The necessity of (i) can be established on the lines of Theorem 2.1. For necessity of (ii), suppose that $\|\{A(x) : x \in X \smallsetminus J\}\| = \infty$.

Then by Lemma 1.5, there exists a sequence $(J(N))$ in $\mathcal{F}(X)$ with $J(1) = J$ of pairwise disjoint sets such that for each $N \ge 2$, $\|\sum_{x\in J(N)} A(x)u(x)\| > N$, where $u(x) \in S$ for $x \in J(N)$. Let $\rho > 0$. Then we easily see that $f\colon X \to U$ defined by

$$f(x) = \begin{cases} N^{-1}u(x), & x \in J(N),\ N \ge 2, \\ \theta, & \text{otherwise}, \end{cases}$$

is in $c_0(X, U, M)$, but for each $N \ge 2$, $\|\sum_{x\in J(N)} A(x)f(x)\| > 1$ shows that $\sum_{x\in X} A(x)f(x)$ is not summable. Hence $A \notin c_0^\beta(X, U, M)$. This completes the proof. □

Theorem 2.4. *If $V = C$, i.e., $B(U, V) = U^*$, then we have*

$$c_0^\alpha(X, U, M) = c_0^\beta(X, U, M) = H_0(X, U^*).$$

Proof. $H_0(X, U^*) = c_0^\alpha(X, U, M) \subset c_0^\beta(X, U, M)$ follows immediately from Theorem 2.2 and completeness of U. Now suppose that $F\colon X \to U^*$ belongs to $c_0^\beta(X, U, M)$, but $F \notin H_0(X, U^*)$. Then we can find a sequence $(J(N))$, $N \ge 2$, of pairwise disjoint sets in $\mathcal{F}(X)$ such that

$$\sum_{x\in J(N)} \|F(x)\| > 2N, \qquad N = 2, 3, 4, \ldots$$

Further we take $\rho > 0$ and for each $x \in J(N)$, we choose $u(x) \in S$ such that $|F(x)| < 2|F(x)u(x)|$ and define $f\colon X \to U$ by

$$f(x) = \begin{cases} \operatorname{sgn}(F(x)u(x))\, N^{-1}u(x), & x \in J(N),\ N \ge 2, \\ \theta, & \text{otherwise}. \end{cases}$$

We easily see that $f \in c_0(X, U, M)$, but for each $N \ge 2$,

$$\sum_{x\in J(N)} F(x)f(x) = \sum_{x\in J(N)} |F(x)u(x)|N^{-1} > \sum_{x\in J(N)} \frac{1}{2}\|F(x)\|\, N^{-1} > 1$$

shows that $\sum_{x\in X} F(x)f(x)$ is not summable, which contradicts that $F \in c_0^\beta(X, U, M)$ (see Theorem 1.1). This completes the proof. □

Theorem 2.5. *Let $A\colon X \to L(U, V)$. Then $A \in c^\beta(X, U, M)$ if and only if*

(i) *there exists $J \in \mathcal{F}(X)$ such that $A(x) \in B(U, V)$ for all $x \in X \smallsetminus J$,*
(ii) *$\|\{A(x) : x \in X \smallsetminus J\}\| < \infty$, and*
(iii) *$\sum_{x\in X\smallsetminus J} A(x)u$ is summable in V for every $u \in U$.*

Proof. To show (iii), it is necessary we take $u \in U$ and consider $f_u\colon X \to U$ defined by $f_u(x) = u$ for each $x \in X$. Then $f_u \in c(X, U, M)$, and so $\sum_{x\in X} A(x)u$ is summable in V.

Further since $c^\beta(X, U, M) \subset c_0^\beta(X, U, M)$, the necessity of (i) and (ii) follows from the Theorem 2.3.

For the sufficiency let $f \in c(X, U, M)$. Then there exist $\rho > 0$ and $l \in U$ such that for every given $\varepsilon > 0$ we can find $J_1 \in \mathcal{F}(X)$, $J \supset J_1$ satisfying $M\left(\frac{\|f(x)-l\|}{\rho}\right) < \varepsilon$ for all $x \in X \setminus J_1$. Now consider $\psi\colon X \to U$ and $f_l\colon X \to U$ defined by $\psi(x) = f(x) - l$ and $f_l(x) = l$ for all $x \in X$ respectively. Obviously, $f_l \in c(X, U, M)$ and $\psi \in c_0(X, U, M)$ and hence $\psi \in c(X, U, M)$. Clearly $f = \psi + f_l$. Moreover by Theorem 2.3, $\sum_{x\in X} A(x)\psi(x)$ is summable in U. Similarly by (iii), $\sum_{x\in X} A(x)l$ is summable in U. Thus we note that

$$\sum_{x\in X} A(x)f(x) = \sum_{x\in X} A(x)(f(x) - l) + \sum_{x\in X} A(x)f_l(x) = \sum_{x\in X} A(x)\psi(x) + \sum_{x\in X} A(x)l$$

is summable in U. Hence $A \in c^\beta(X, U, M)$. This completes the proof. □

In the special case when $V = C$, i.e., $B(U, V) = U^*$, proof of the Theorem 2.6 given below follows easily from making use of Theorem 2.1.

Theorem 2.6. *If $V = C$, i.e., $B(U, V) = U^*$, then we have*

$$c^\alpha(X, U, M) = c^\beta(X, U, M) = H_0(X, U^*).$$

Proof. By Theorem 2.1, we have $H_0(X, U^*) = c^\alpha(X, U, M)$ and by completeness of $\mathbb{C}$ we immediately get

$$c^\alpha(X, U, M) \subset c^\beta(X, U, M).$$

Since $c^\beta(X, U, M) \subset c_0^\beta(X, U, M)$ is always true, but $c_0^\beta(X, U, M) = H_0(X, U^*)$ follows from Theorem 2.4. Thus we get $H_0(X, U^*) = c^\alpha(X, U, M) \subset c^\beta(X, U, M) \subset H_0(X, U^*)$. Hence $c^\alpha(X, U, M) = c^\beta(X, U, M) = H_0(X, U^*)$. □

Theorem 2.7. *Let $A\colon X \to L(U, V)$. Then $A \in l_\infty^\beta(X, U, M)$ if and only if*

(i) *there exists $J \in \mathcal{F}(X)$ such that $A(x) \in B(U, V)$ for all $x \in X \setminus J$; and*
(ii) *for each $\varepsilon > 0$, there exist $K = K(\varepsilon) \in \mathcal{F}(X)$, $J \subset K$ such that $R_H = \|\{A(x) : x \in X \setminus H\}\| < \varepsilon$ for all $H \in \mathcal{F}(X)$ with $K \subset H$.*

Proof. Suppose (i) and (ii) hold, $f \in l_\infty(X, U, M)$ and $\rho > 0$ is associated with f. Then we have $L > 0$ such that $\sup_{x\in X} M\left(\frac{\|f(x)\|}{\rho}\right) = L$, i.e., $M\left(\frac{\|f(x)\|}{\rho}\right) < L$ for all $x \in X$.

Further we can choose $r > L$ and $t_0 > 0$ a fixed positive real number such that $r\frac{t_0}{2}\, q\left(\frac{t_0}{2}\right) > L$, where q is the kernel associated with M. Hence for each $x \in X$, $M\left(\frac{\|f(x)\|}{\rho}\right) < r\frac{t_0}{2}\, q\left(\frac{t_0}{2}\right)$.

Thus using the integral representation of Orlicz function M, we get

$$\|f(x)\| \le \rho r t_0 \qquad \text{for all } x \in X.$$

Then by Lemma 1.5 (iii), for any $G \in \mathcal{F}(X)$ with $G \cap H = \phi$, we have

$$\Big\|\sum_{x\in G} A(x)f(x)\Big\| \le R_H \max\{\|f(x)\| : x \in G\} \le \varepsilon\rho r t_0.$$

Thus, $\| \sum_{x\in G} A(x)f(x)\| < \varepsilon\rho r t_0$ for all $G \in \mathcal{F}(X)$ with $G \cap H = \phi$, and so by Theorem 1.1, $\sum_{x\in X\smallsetminus H} A(x)f(x)$ is summable. Hence $\sum_{x\in X} A(x)f(x)$ is summable.

Conversely (i) can be established on the lines of Theorem 2.1. Now to prove the necessity of (ii), we first show that $R_H < \infty$. Suppose on contrary that $R_H = \infty$. Then we can find a sequence of pairwise disjoint sets (F_n) in $\mathcal{F}(X \smallsetminus J)$ and sets $\{u_n(x) : x \in F_n\} \subset S$ such that

$$\Big\|\sum_{x\in F_n} A(x)u_n(x)\Big\| > 1 \qquad \text{for each } n \ge 1. \tag{2.1}$$

Let $\rho > 0$. Then $f\colon X \to U$ defined by

$$f(x) = \begin{cases} u(x), & x \in F_n, u(x) = u_n(x), n \ge 1,\\ 0, & \text{otherwise}\end{cases}$$

is in $l_\infty(X, U, M)$ but by (2.1) for each $n \ge 1$, $\|\sum_{x\in F_n} A(x)f(x)\| > 1$ shows that $\sum_{x\in X} A(x)f(x)$ is not summable (see Theorem 1.1). This proves our assertion, i.e., $R_H < \infty$.

Now suppose (ii) does not hold, i.e., there exists $\varepsilon > 0$ such that for every given $K \supset J$, $K \in \mathcal{F}(X)$, we can find $H \in \mathcal{F}(X)$, $H \supset K$ such that $R_H > \varepsilon$. For $n = 1$, take $K_1 \supset J$ such that $R_{K_1} > \varepsilon$. So there exist $F_1 \in \mathcal{F}(X \smallsetminus K_1)$ and $G_1 = \{u_1(x) : x \in F_1\} \subset S$ such that $\|\sum_{x\in F_1} A(x)u_1(x)\| > \varepsilon$.

Next take $K_2 = K_1 \cup F_1$ then there exist $F_2 \in \mathcal{F}(X \smallsetminus K_2)$ and $G_2 = \{u_2(x) : x \in F_2\} \subset S$ such that $\|\sum_{x\in F_2} A(x)\, u_2(x)\| > \varepsilon$.

If we continue this process, then we get sequences (F_n) and (G_n) such that for $K_n = K_1 \cup F_1 \cup \cdots \cup F_{n-1}$, there exist $F_n \in \mathcal{F}(X \smallsetminus K_n)$ and $G_n = \{u_n(x) : x \in F_n\} \subset S$ for which

$$\Big\|\sum_{x\in F_n} A(x)u_n(x)\Big\| > \varepsilon. \tag{2.2}$$

Let $\rho > 0$. Then the function $f\colon X \to U$ defined by

$$f(x) = \begin{cases} u(x), & x \in F_n,\ u(x) = u_n(x) \in G_n,\ n \ge 1\\ \theta, & \text{otherwise,}\end{cases}$$

is in $l_\infty(X, U, M)$, where as due to (2.2) for each $n \ge 1$, $\|\sum_{x\in F_n} A(x)f(x)\| > \varepsilon$ shows that $\sum_{x\in X} A(x)f(x)$ is not summable, i.e., $A \notin l^\beta_\infty(X, U, M)$. This completes the proof. □

Theorem 2.8. *If $V = C$, i.e., $B(U, C) = U^*$, then we have*

$$l^\alpha_\infty(X, U, M) = l^\beta_\infty(X, U, M) = H_0(X, U^*).$$

Proof. In view of Theorem 2.7 and completeness of $\mathbb{C}$, we have

$$H_0(X,U^*) = l^\alpha_\infty(X,U,M) \subset l^\beta_\infty(X,U,M).$$

Now suppose $F \in l^\beta_\infty(X,U,M)$, but $F \notin H_0(X,U^*)$, then $\sum_{x\in X} ||F(x)\| = \infty$.

So we get a pairwise disjoint sequence $(J_n) \in \mathcal{F}(X)$ such that $\sum_{x\in J_n} \|F(x)\| > 1$ for each $n \geq 1$. For each $x \in X$, let $u(x) \in S$ be such that $\|F(x)\| \leq 2|F(x)u(x)|$. Let $\rho > 0$. Then the function $f\colon X \to U$ is defined by

$$f(x) = \begin{cases} \operatorname{sgn}(F(x)u(x))\, u(x), & x \in J_n,\ n \geq 1 \\ \theta, & \text{otherwise.} \end{cases}$$

We note that $\|f(x)\| = 0$ for $x \in X \setminus \cup_{n=1}^\infty J_n$ and $\|f(x)\| \leq 1$ if $x \in J_n$, $n \geq 1$. This shows that $\sup_{x\in X} M\left(\frac{\|f(x)\|}{\rho}\right) < \infty$ and hence $f \in l_\infty(X,U,M)$. But on the other hand, we have

$$\sum_{x\in X} \|F(x)f(x)\| = \sum_{n=1}^\infty \sum_{x\in J_n} \|F(x)f(x)\| \geq \sum_{n=1}^\infty \frac{1}{2} \sum_{x\in J_n} \|F(x)\| > \sum_{n=1}^\infty \frac{1}{2} = \infty,$$

this contradicts that $F \in l^\beta_\infty(X,U,M)$. Hence $l^\beta_\infty(X,U,M) \subset H_0(X,U^*)$. This completes the proof of the theorem. □

3. Continuous Dual of $c_0(X,U,M)$ and $c(X,U,M)$

In the following theorems continuous duals $c_0^*(X,U,M)$ and $c^*(X,U,M)$ of the topological linear spaces $(c_0(X,U,M), \|\cdot||_\infty)$ and $(c(X,U,M), \|\cdot||_\infty)$, respectively, are investigated.

Theorem 3.1. *$c_0^*(X,U,M)$, the continuous dual of $(c_0(X,U,M), \|\cdot||_\infty)$, is isomorphic to $H_0(X,U^*)$.*

Proof. Let $F \in c_0^*(X,U,M)$ and for each $x \in X$, define $\phi(x)\colon U \to C$ by $\phi(x)u = F(\delta^u_x)$. Each $\phi(x)$ is linear on U. Further if (u_n) is a sequence in U which converges to $u \in U$, then $M\left(\frac{\|u_n-u\|}{\rho}\right) \to 0$ as $n \to \infty$ for each $\rho > 0$. So for given $0 < \varepsilon < 1$, we can find ρ_ε, $0 < \rho_\varepsilon < \varepsilon$ and $N \geq 1$ such that $M\left(\frac{\|u_n-u\|}{\rho_\varepsilon}\right) < \varepsilon < 1$ for all $n \geq N$. Let $y \in X$ and fix it. Now

$$\begin{aligned}\|\delta^{u_n}_y - \delta^u_y\| &= \inf\left\{\rho > 0 : \sup_{x\in X} M\left(\frac{\|\delta^{u_n}_y(x) - \delta^u_y(x)\|}{\rho}\right) \leq 1\right\} \\ &= \inf\left\{\rho > 0 : M\left(\frac{\|u_n - u\|}{\rho}\right) \leq 1\right\} < \rho_\varepsilon < \varepsilon.\end{aligned}$$

Thus for each $x \in X$, $\delta^{u_n}_x \to \delta^u_x$ in $c_0(X,U,M)$ as $n \to \infty$. So we have $F(\delta^{u_n}_x) \to F(\delta^u_x)$ as $n \to \infty$ which clearly implies that $\phi(x)u_n \to \phi(x)u$ as $n \to \infty$. Hence $\phi(x) \in U^*$ for each $x \in X$. Moreover, since $c_0(X,U,M)$ is an AK-function space, for each $f \in c_0(X,U,M)$, we have $s_J(f) \to f$, and so

$$F(f) = F(\lim s_J(f)) = \lim F\left(\sum_{x\in J} \delta^{f(x)}_x\right) = \sum_{x\in X} \phi(x)f(x).$$

Thus $\sum_{x\in X}\phi(x)\,f(x)$ is summable for every $f\in c_0(X,U,M)$ and therefore by Theorem 2.4 we get $\phi\in H_0(X,U^*)$.

Conversely, if $\phi\in H_0(X,U^*)$, then by Theorem 2.4 $\sum_{x\in X}\phi(x)f(x)$ is summable for every $f\in c_0(X,U,M)$. Now F defined by $F(f)=\sum_{x\in X}\phi(x)\,f(x)$ is clearly a linear functional on $c_0(X,U,M)$. Since $\phi\in H_0(X,U^*)$, $\sum_{x\in X}\|\phi(x)\|=L<\infty$.

Let $\varepsilon>0$ be given. Suppose that for $f\in c_0(X,U,M)$, $\|f\|_\infty<\varepsilon$. Then we get $|F(f)|\le\sum_{x\in X}\|\phi(x)\|\,\|f(x)\|<\varepsilon H$ which shows that F is continuous. Hence $F\in c_0^*(X,U,M)$.

Thus for each $F\in c_0^*(X,U,M)$, there exists $\phi\in H_0(X,U^*)$ and vice-versa. Hence the correspondence $F\to\phi$ clearly determines an isomorphism of $c_0^*(X,U,M)$ onto $H_0(X,U^*)$. This completes the proof. □

Theorem 3.2. *$F\in c^*(X,U,M)$ is the continuous dual of $(c(X,U,M),\|\cdot\|_\infty)$ if and only if there exist $\phi\in H_0(X,U^*)$ and $g\in U^*$ such that $F(f)=g(l)+\sum_{x\in X}\phi(x)f(x)$ for every $f\in c(X,U,M)$, where $l\in U$ satisfies that for every $\varepsilon>0$, there exists $J\in\mathcal{F}(X)$ such that for all $x\in X\setminus J$, $M\left(\frac{\|f(x)-l\|}{\rho}\right)<\varepsilon$ for some $\rho>0$.*

Proof. Let $F\in c^*(X,U,M)$ and $f\in c(X,U,M)$. Let $l\in U$ be as in the statement of the theorem. Clearly $F\in c_0^*(X,U,M)$ and the function $\psi\colon X\to U$, $\psi(x)=f(x)-l$ is in $c_0(X,U,M)$. Thus by Theorem 3.1, we have $\phi\in H_0(X,U^*)$ such that $F(\psi)=\sum_{x\in X}\phi(x)(f(x)-l)$ and right hand side is summable. Since $\phi\in H_0(X,U^*)$, $\sum_{x\in X}\phi(x)u$ is summable. Clearly, $\xi_l\colon X\to U$ defined by $\xi_l(x)=l$ for each $x\in X$ is in $c(X,U,M)$ and $f=\psi+\xi_l$. Thus

$$\begin{aligned}F(f)=F(\psi)+F(\xi_l)&=\sum_{x\in X}\phi(x)(f(x)-l)+F(\xi_l)\\&=F(\xi_l)-\sum_{x\in X}\phi(x)l+\sum_{x\in X}\phi(x)f(x)=g(l)+\sum_{x\in X}\phi(x)f(x)\end{aligned}$$

where we write $g(l)=F(\xi_l)-\sum_{x\in X}\phi(x)l$. g is linear on U. For continuity of g, suppose that the sequence (u_n) in U converges to 0 as $n\to\infty$. Clearly (ξ_{u_n}) converges to function θ in $c(X,U,M)$ and hence $F(\xi_{u_n})$ converges to 0 as $n\to\infty$. Further

$$\left|\sum_{x\in X}\phi(x)u_n\right|\le\|u_n\|\sum_{x\in X}||\phi(x)||,\qquad n\ge1,$$

shows that $\sum_{x\in X}\phi(x)u_n$ converges to 0 as $n\to\infty$. Hence $g(u_n)\to0$ as $n\to\infty$, i.e., g is continuous on U. For converse part, suppose that $\phi\in H_0(X,U^*)$ and $g\in U^*$. Then

$$\sum_{x\in X}\|\phi(x)\|=L<\infty.$$

We now define F on $c(X,U,M)$ by $F(f)=g(l)+\sum_{x\in X}\phi(x)f(x)$, $f\in c(X,U,M)$. Clearly, F is well defined and linear (see Theorem 2.1) on $c(X,U,M)$.

Let $0<\varepsilon<1$. Suppose that for $f\in c(X,U,M)$, $\|f\|_\infty<\varepsilon$ and ρ is associated with f. Now consider $M\left(\frac{\varepsilon}{\rho}\right)>0$. Since $f\in c(X,U,M)$, there exists $l\in U$ such that for $M\left(\frac{\varepsilon}{\rho}\right) > 0$ there exists $J \in \mathcal{F}(x)$ satisfying $M\left(\frac{\|f(x)-l\|}{\rho}\right) < M\left(\frac{\varepsilon}{\rho}\right)$, for all $x \in X \setminus J$. M is non-decreasing, therefore also by the definition of norm, we have $\sup_{x\in X} M(\frac{\|f(x)\|}{\|f\|}) \leq 1$, i.e., $M(\frac{\|f(x)\|}{\|f\|}) \leq 1$.

Now we can find $r > 1$ and $t_0 > 0$ which is a fixed real number such that $r\frac{t_0}{2}q\left(\frac{t_0}{2}\right) \geq 1$.

Hence, $M\left(\frac{\|f(x)\|}{\|f\|}\right) < r\frac{t_0}{2}q\left(\frac{t_0}{2}\right)$ which gives us that for each $x \in X$, $\|f(x)\| \leq \varepsilon r t_0$. Then we have

$$\|l\| \leq \|f(x) - l\| + \|f(x)\| < \varepsilon + \varepsilon r t_0 = \varepsilon(1 + rt_0)$$

and so we clearly get that $\|l\| < \varepsilon$. Now the continuity of F easily follows from

$$\begin{aligned}|F(\phi)| &= \left|g(l) + \sum_{x\in X}\phi(x)f(x)\right| \leq \|g\|\,\|l\| + \sup_{x\in X}\|f(x)\|\sum_{x\in X}\|\phi(x)\| \\ &< \varepsilon[\|g\|\,(1+rt_0) + rt_0 L].\end{aligned}$$

Hence, $F \in c^*(X, U, M)$. This completes the proof. □

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