Projective parameterized linear codes

Manuel González Sarabia, Carlos Rentería Márquez and Eliseo Sarmiento Rosales

Abstract

In this paper we estimate the main parameters of some evaluation codes which are known as projective parameterized codes. We find the length of these codes and we give a formula for the dimension in terms of the Hilbert function associated to two ideals, one of them being the vanishing ideal of the projective torus. Also we find an upper bound for the minimum distance and, in some cases, we give some lower bounds for the regularity index and the minimum distance. These lower bounds work in several cases, particularly for any projective parameterized code associated to the incidence matrix of uniform clutters and then they work in the case of graphs.

1 Introduction

Let $K = \mathbb{F}_q$ be a finite field with q elements and $L = K[Z_1, \ldots, Z_n]$ be a polynomial ring over the field K. Let $Z^{a_1}, \ldots, Z^{a_m}$ be a finite set of monomials. As usual if $a_i = (a_{i1}, \ldots, a_{in}) \in \mathbb{N}^n$, where $\mathbb{N}$ stands for the non-negative integers, then we set

$$Z^{a_i} = Z_1^{a_{i1}} \cdots Z_n^{a_{in}} \text{ for all } i = 1, \ldots, m.$$

Consider the following set parameterized by these monomials

$$X = \left\{ [(t_1^{a_{11}} \cdots t_n^{a_{1n}}, \ldots, t_1^{a_{m1}} \cdots t_n^{a_{mn}})] \in \mathbb{P}^{m-1} \,|\, t_i \in K^* \right\}, \tag{1}$$

where $K^* = K \setminus \{0\}$ and $\mathbb{P}^{m-1}$ is a projective space over the field K. Following [17] we call X *an algebraic toric set* parameterized by $Z^{a_1}, \ldots, Z^{a_m}$. The set X is a multiplicative group under componentwise multiplication.

In the same way, let A be the $n \times m$ matrix given by

$$\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}. \tag{2}$$

We say that the set defined in (1) is an algebraic toric set associated to the matrix A. We note that

$$[(t_1^{a_{11}} \cdots t_n^{a_{1n}}, t_1^{a_{21}} \cdots t_n^{a_{2n}}, \ldots, t_1^{a_{m1}} \cdots t_n^{a_{mn}})] =$$

$$[(1, t_1^{a_{21}-a_{11}} \cdots t_n^{a_{2n}-a_{1n}}, \ldots, t_1^{a_{m1}-a_{11}} \cdots t_n^{a_{mn}-a_{1n}})].$$

By taking $b_{ij} = a_{ij} - a_{1j}$ for all $i = 2, \ldots, m$, $j = 1, \ldots, n$, we obtain

$$X = \left\{ \left[\left(1, t_1^{b_{21}} \cdots t_n^{b_{2n}}, \ldots, t_1^{b_{m1}} \cdots t_n^{b_{mn}} \right) \right] \in \mathbb{P}^{m-1} : t_i \in K^* \right\}. \tag{3}$$

From now on we will use any of the representations (1) or (3) to mean an algebraic toric set parameterized by the monomials $Z^{a_1}, \ldots, Z^{a_m}$ or, in an equivalent way, to represent an algebraic toric set associated to the matrix (2).

Let $S = K[X_1, \ldots, X_m] = \oplus_{d=0}^{\infty} S_d$ be a polynomial ring over the field K with the standard grading, let $[P_1], \ldots, [P_{|X|}]$ be the points of X, and let $f_0(X_1, \ldots, X_m) = X_1^d$. The evaluation map

$$\mathrm{ev}_d \colon S_d = K[X_1, \ldots, X_m]_d \to K^{|X|},$$

$$f \mapsto \left(\frac{f(P_1)}{f_0(P_1)}, \ldots, \frac{f(P_{|X|})}{f_0(P_{|X|})} \right) \tag{4}$$

defines a linear map of K-vector spaces. The image of ev_d, denoted by $C_X(d)$, defines a linear code. We will call $C_X(d)$ *a projective parameterized code* of order d arising from the toric set X or associated to the matrix A. As usual by a linear code we mean a linear subspace of $K^{|X|}$.

In this paper we will only deal with projective parameterized codes arising from the set X, defined in (1) or (3), over finite fields and we will describe their main characteristics.

The *dimension* and *length* of the code $C_X(d)$ are given by $\dim_K C_X(d)$ and $|X|$ respectively. The dimension and length are two of the *basic parameters* of a linear code. A third basic parameter is the *minimum distance* which is given by

$$\delta_X(d) = \min\{\|v\| : 0 \neq v \in C_X(d)\},$$

where $\|v\|$ is the number of non-zero entries of v. The basic parameters of $C_X(d)$ are related by the *Singleton bound* which is an upper bound for the minimum distance

$$\delta_X(d) \leq |X| - \dim_K C_X(d) + 1.$$

Projective parameterized codes are important because in some cases their main parameters have the best behavior. For example in [7] the resulting codes are MDS.

The parameters of evaluation codes over finite fields have been computed in several cases. Our approximation, when we consider the evaluation codes as associated to the matrix (2), generalizes many cases studied previously. For example if $A = I_m$, the projective parameterized codes associated to A become the Generalized Reed-Solomon codes [8]. If $X = \mathbb{P}^{m-1}$, the parameters of $C_X(d)$ are described in [20, Theorem 1]. If X is the image of the affine space $\mathbb{A}^{m-1}$ under the map $\mathbb{A}^{m-1} \to \mathbb{P}^{m-1}$, $x \mapsto [(1,x)]$, the parameters of $C_X(d)$ are described in [2, Theorem 2.6.2]. Also if we consider the matrix (2) as the incidence matrix of a graph G, we obtain the projective parameterized codes associated to G. In the following sections when we write graph we mean a simple graph, i.e., an undirected graph that has no loops and no more than one edge between any two different vertices. The main characteristics of evaluation codes associated to complete bipartite graphs were found in [6]. Some general results over projective parameterized codes were described in [16].

It is worth saying that projective parameterized codes are, in general, strictly different to toric codes which were defined in [11] and generalized for example in [13] and [19]. They evaluate over the complete torus, meanwhile we do it over specific subsets of the projective space.

In this work we will analyze the case where the parameterized codes of order d, $C_X(d)$, come from the general matrix (2) and we will estimate their main parameters.

The *vanishing ideal* of X, denoted by I_X, is the ideal of S generated by the homogeneous polynomials of S that vanish on X.

For all unexplained terminology and additional information we refer to [1, 21] (for the theory of polynomial ideals and Hilbert functions), and [14, 22, 24] (for the theory of error-correcting codes and algebraic geometric codes).

2   Preliminaries

In the following we will use the notation and definitions given in the introduction. In this section we introduce the basic algebraic invariants of S/I_X and their connection with the basic parameters of projective parameterized linear codes. Then we present some of the results that we are going to use later.

Recall that *the projective space* of dimension $m-1$ over K, denoted by $\mathbb{P}^{m-1}$, is the quotient space

$$(K^m \setminus \{0\})/\sim$$

where two points α, β in $K^m \setminus \{0\}$ are equivalent if $\alpha = \lambda\beta$ for some $\lambda \in K$. We denote the equivalence class of α by $[\alpha]$. Let $X \subset \mathbb{P}^{m-1}$ be an algebraic toric set parameterized by $Z^{a_1}, \ldots, Z^{a_m}$ and let $C_X(d)$ be a projective parameterized code of order d. The kernel of the evaluation map ev_d, defined in (4), is precisely $I_X(d)$, the degree d piece of I_X. Therefore there is an isomorphism of K-vector spaces

$$S_d/I_X(d) \simeq C_X(d).$$

Two of the basic parameters of $C_X(d)$ can be expressed using Hilbert functions of standard graded algebras [21], as we now explain. Recall that the *Hilbert function* of S/I_X is given by

$$H_X(d) = \dim_K(S/I_X)_d = \dim_K S_d/I_X(d) = \dim_K C_X(d).$$

The unique polynomial $h_X(t) = \sum_{i=0}^{k-1} c_i t^i \in \mathbb{Z}[t]$ of degree $k-1 = \dim(S/I_X) - 1$ such that $h_X(d) = H_X(d)$ for $d \gg 0$ is called the *Hilbert polynomial* of S/I_X. The integer $c_{k-1}(k-1)!$, denoted by $\deg(S/I_X)$, is called the *degree* or *multiplicity* of S/I_X. In our situation $h_X(t)$ is a non-zero constant because S/I_X has dimension 1. Furthermore $h_X(d) = |X|$ for $d \geq |X| - 1$, see [12, Lecture 13]. This means that $|X|$ equals the degree of S/I_X. Thus $H_X(d)$ and $\deg(S/I_X)$ equal the dimension and the length of $C_X(d)$ respectively. There are algebraic methods, based on elimination theory and Gröbner bases, to compute the dimension and the length of $C_X(d)$ [16].

The *regularity index* of S/I_X, denoted by $\mathrm{reg}(S/I_X)$, is the least integer $p \geq 0$ such that $h_X(d) = H_X(d)$ for $d \geq p$. The degree and the regularity index can be read off the Hilbert series as we now explain. The *Hilbert series* of S/I_X can be written as

$$F_X(t) = \sum_{i=0}^{\infty} H_X(i)t^i = \sum_{i=0}^{\infty} \dim_K(S/I_X)_i t^i = \frac{h_0 + h_1 t + \cdots + h_r t^r}{1-t},$$

where $h_0, \ldots, h_r$ are positive integers. In fact we have that

$$h_i = \dim_K(S/(I_X, X_m))_i$$

for $0 \leq i \leq r$ and $\dim_K(S/(I_X, X_m))_i = 0$ for $i > r$. This follows from the fact that I_X is a Cohen-Macaulay lattice ideal [16] and by observing that $\{X_m\}$ is a regular system of parameters for S/I_X (see [21]). The number r equals the regularity index of S/I_X and the degree of S/I_X equals $h_0 + \cdots + h_r$ (see [21] or [25, Corollary 4.1.12]).

The regularity index plays a very important role in the study of evaluation codes arising from a set X because in the case $d \geq \operatorname{reg}(S/I_X)$ we obtain that $H_X(d) = |X|$ and then $C_X(d) = K^{|X|}$, which is a trivial case. Therefore we always work with $0 \leq d < \operatorname{reg}(S/I_X)$. Another motivation to study the regularity index comes from commutative algebra because, in this case, $\operatorname{reg}(S/I_X)$ is equal to the *Castelnuovo–Mumford regularity* which is an algebraic invariant of central importance [4].

3 Main Results

From now on we will work with the toric set X defined in (1) or (3) and our goal is to describe the main parameters of the projective parameterized codes of order d, $C_X(d)$, which were defined as the image of the evaluation map ev_d introduced in (4).

3.1 Length

In order to cumpute the length of the projective parameterized codes arising from the toric set X, we introduce the following multiplicative subgroups of X for all $i = 1, \ldots, n$.

$$Y_i := \{[(1, t_i^{b_{2i}}, \ldots, t_i^{b_{mi}})] \in \mathbb{P}^{m-1} : t_i \in K^* \text{ for all } i\}.$$

It is easy to see that $|Y_i| = \frac{q-1}{(q-1, b_{2i}, \ldots, b_{mi})}$ for all $i = 1, \ldots, n$, where $(q-1, b_{2i}, \ldots, b_{mi})$ means the greatest common divisor of the corresponding integers. With this information we are able to prove the main result of this section.

Theorem 3.1. *The length of the projective parameterized codes of order d, $C_X(d)$, is given by*

$$|X| = \frac{1}{|M|} \prod_{i=1}^{n} |Y_i| \tag{5}$$

where M is the set of n-tuples $(i_1, \ldots, i_n)$ such that

$$1 \leq i_j \leq \frac{q-1}{(q-1, b_{2j}, \ldots, b_{mj})} \quad \text{for all} \quad j = 1, \ldots, n,$$

and

$$i_1 b_{21} + i_2 b_{22} + \cdots + i_n b_{2n} \equiv 0 \bmod (q-1)$$
$$i_1 b_{31} + i_2 b_{32} + \cdots + i_n b_{3n} \equiv 0 \bmod (q-1)$$
$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \qquad (6)$$
$$i_1 b_{m1} + i_2 b_{m2} + \cdots + i_n b_{mn} \equiv 0 \bmod (q-1)$$

Proof. Let ϕ be the following map

$$\phi : Y_1 \times \cdots \times Y_n \to X,$$

$$\phi([(1, t_1^{b_{21}}, \ldots, t_1^{b_{m1}})], \ldots, [(1, t_n^{b_{2n}}, \ldots, t_n^{b_{mn}})]) =$$
$$[(1, t_1^{b_{21}} \cdots t_n^{b_{2n}}, \ldots, t_1^{b_{m1}} \cdots t_n^{b_{mn}})].$$

It is immediate that ϕ is an epimorphism between multiplicative groups. Thus

$$|X| = \frac{|Y_1 \times \cdots \times Y_n|}{|\ker \phi|} = \frac{1}{|\ker \phi|} \prod_{i=1}^{n} |Y_i|.$$

Let β a generator of $(K^*, \cdot)$. Therefore

$$\ker \phi = \{([(1, \beta^{i_1 b_{21}}, \ldots, \beta^{i_1 b_{m1}})], \ldots, [(\beta^{i_n b_{2n}}, \ldots, \beta^{i_n b_{mn}})]) \in Y_1 \times \cdots \times Y_n :$$
$$[(1, \beta^{i_1 b_{21} + \cdots + i_n b_{2n}}, \ldots, \beta^{i_1 b_{m1} + \cdots + i_n b_{mn}})] = [(1, 1, \ldots, 1)]\}.$$

In this case $\beta^{i_1 b_{21} + \cdots + i_n b_{2n}} = 1, \ldots, \beta^{i_1 b_{m1} + \cdots + i_n b_{mn}} = 1$. These equalities imply the system of congruences (6). Then there is a bijection between $\ker \phi$ and the set of n−tuples $(i_1, \ldots, i_n)$ such that $1 \le i_j \le |Y_j|$ for all $j = 1, \ldots, n$ and satisfy (6).

The equation (5) follows immediately from last results. $\square$

We define the *projective torus* of dimension $m - 1$ as

$$\mathbb{T}_{m-1} = \{[(c_1, \ldots, c_m)] \in \mathbb{P}^{m-1} : c_i \in K^* \text{ for all } i\}. \qquad (7)$$

Obviously, $X \subseteq \mathbb{T}_{m-1}$. The following corollary is an easy consequence of Theorem 3.1. It gives the conditions under which last inclusion becomes an equality.

Corollary 3.2. *If $n = m$ then X is the projective torus of dimension $m - 1$ if and only if $|M| = 1$ and $(q-1, b_{2j}, \ldots, b_{mj}) = 1$ for all $j = 1, \ldots, n$.*

On the other hand if we consider the case where the monomials that parameterize the toric set X are all of them of the same degree then we obtain another corollary.

Corollary 3.3. *If the sum of the elements of each column of the matrix A defined in (2) is a constant or, equivalently, the monomials that parameterize the toric set X are all of them of the same degree, then $|X| \leq (q-1)^{n-1}$.*

Proof. Let $\sum_{j=1}^{n} a_{ij} = \alpha$ (a positive integer) for all $i = 1, \ldots, m$. We note that $|Y_i| \leq q - 1$ for all $i = 1, \ldots, n$. Moreover

$$\sum_{j=1}^{n} b_{ij} = \sum_{j=1}^{n} a_{ij} - \sum_{j=1}^{n} a_{1j} = \alpha - \alpha = 0,$$

and then $(1, \ldots, 1) \in M$. Let $\gamma = \min\{|Y_i| : i = 1, \ldots, n\}$. Therefore $(j, \ldots, j) \in M$ for all $1 \leq j \leq \gamma$ and it implies that $|M| \geq \gamma$. Thus

$$|X| = \tfrac{1}{|M|} \prod_{i=1}^{n} |Y_i| \leq \tfrac{\gamma (q-1)^{n-1}}{\gamma} = (q-1)^{n-1}$$

and the claim follows. $\square$

Remark 3.4. *If G is a graph and X is the algebraic toric set associated to the incidence matrix of G, then the sum of the elements of each column of this matrix is $\alpha = 2$ and the result of the last corollary follows. Actually in this situation $|Y_i| = q - 1$ for all $i = 1, \ldots, n$ and then we get that in any graph $|X| = \frac{(q-1)^n}{|M|}$. Moreover in [16, Corollary 3.8] it was found the exact value of $|X|$ if G is a connected graph. By using this result we obtain that*

$$|M| = \begin{cases} (q-1)^2 & \text{if} \quad G \text{ is bipartite} \\ q - 1 & \text{if} \quad G \text{ is non-bipartite.} \end{cases}$$

On the other hand if we consider disconnected graphs, in [15, Theorem 3.2] it was found the exact value of $|X|$, which is a generalization of [16, Corollary 3.8].

3.2   Dimension

In the following theorem we give the dimension of the projective parameterized codes arising from the algebraic toric set X in terms of the dimension of the projective parameterized codes arising from the projective torus $\mathbb{T}_{m-1}$, which is well known (see [8]).

Theorem 3.5. *The dimension of the projective parameterized codes of order d, $C_X(d)$, is given by*

$$H_X(d) = H_{\mathbb{T}_m - 1}(d) - \overline{H}(d) \tag{8}$$

for all $d \geq 0$ and where $\overline{H}$ is the Hilbert function of $I_X / I_{\mathbb{T}_{m-1}}$, i.e.,

$$\overline{H}(d) = \dim_K I_X(d)/I_{\mathbb{T}_{m-1}}(d).$$

Proof. We know that $X \subseteq \mathbb{T}_{m-1}$ and then $I_{\mathbb{T}_{m-1}} \subseteq I_X$. Let ψ be the following linear transformation.

$$\psi : S_d/I_{\mathbb{T}_{m-1}}(d) \to S_d/I_X(d),$$
$$f + I_{\mathbb{T}_{m-1}}(d) \to f + I_X(d). \tag{9}$$

This a well defined function and in fact it is a surjective linear map. Moreover $\ker \psi = I_X(d)/I_{\mathbb{T}_{m-1}}(d)$. Thus

$$\dim_K S_d/I_{\mathbb{T}_{m-1}}(d) =$$

$$\dim_K S_d/I_X(d) + \dim_K I_X(d)/I_{\mathbb{T}_{m-1}}(d),$$

and the equality (8) follows immediately. $\qquad\square$

For the following corollary we will use $r_X, r_{\mathbb{T}_{m-1}}$ and $r_{\overline{H}}$ as the regularity indexes of S/I_X, $S/I_{\mathbb{T}_{m-1}}$ and $I_X/I_{\mathbb{T}_{m-1}}$, respectively.

Corollary 3.6. $r_{\mathbb{T}_{m-1}} = \max\{r_X, r_{\overline{H}}\}$.

Proof. Let

$$\theta : I_X(d)/I_{\mathbb{T}_{m-1}}(d) \to I_X(d+1)/I_{\mathbb{T}_{m-1}}(d+1),$$

$$\theta(f + I_{\mathbb{T}_{m-1}}(d)) = X_1 f + I_{\mathbb{T}_{m-1}}(d+1).$$

It is easy to see that θ is a well defined map, moreover it is a linear transformation. If $f + I_{\mathbb{T}_{m-1}}(d) \in \ker\theta$ then $X_1 f \in I_{\mathbb{T}_{m-1}}(d+1)$. Let $[P] = [(1, t_1^{b_{21}} \cdots t_n^{b_{2n}}, \ldots, t_1^{b_{m1}} \cdots t_n^{b_{mn}})] \in X$. Thus $(X_1 f)(P) = 0$ and then $f(P) = 0$. Therefore $f \in I_{\mathbb{T}_{m-1}}(d)$ and $\ker\theta = I_{\mathbb{T}_{m-1}}(d)$. It implies that $\overline{H}(d) \leq \overline{H}(d+1)$ for all $d \geq 0$. By the last inequality and (8) the claim follows. $\qquad\square$

Remark 3.7. *By the Corollary 3.6 we obtain that $r_X \leq r_{\mathbb{T}_{m-1}}$. But in [8] it was proved that $r_{\mathbb{T}_{m-1}} = (m-1)(q-2)$. Therefore*

$$r_X \leq (m-1)(q-2). \tag{10}$$

As we observed in section 2, if $d \geq (m-1)(q-2)$ then $H_X(d) = |X|$ and thus $C_X(d) = K^{|X|}$. Therefore from now on we will use $d < (m-1)(q-2)$.

3.3   Minimum distance

The minimum distance has been computed in several cases associated to evaluation codes. In particular in [18] it was computed when we consider projective parameterized codes arising from the projective torus. Moreover in [23] some lower bounds on the minimum distance were found coming from syzigies. In this section we are going to find an upper bound for the minimum distance of any projective parameterized code and we will find a lower bound for this kind of codes when the sum of the elements of each column of the matrix (2) becomes a constant. We consider that $X \subset \mathbb{T}_{m-1}$ because the case $X = \mathbb{T}_{m-1}$ is well known (see [8]). Let $Y := \mathbb{T}_{m-1} \setminus X$ and $\delta_X(d)$, $\delta_Y(d)$ and $\delta_{\mathbb{T}_{m-1}}(d)$ be the minimum distances of the parameterized codes of order d, $C_X(d)$, $C_Y(d)$ and $C_{\mathbb{T}_{m-1}}(d)$, respectively. The following theorem relates them.

Theorem 3.8. *Let $0 \leq d < (m-1)(q-2)$. Then*

$$\delta_X(d) \leq \delta_{\mathbb{T}_{m-1}}(d) - \delta_Y(d). \tag{11}$$

Proof. Let $X = \{[P_1], \ldots, [P_{|X|}]\}$. We can write

$$\mathbb{T}_{m-1} = \{[P_1], \ldots, [P_{|X|}], [Q_1], \ldots, [Q_{|Y|}]\},$$

where of course $Y = \{[Q_1], \ldots, [Q_{|Y|}]\}$. If

$$\Lambda = \left(\frac{f(P_1)}{X_1^d(P_1)}, \ldots, \frac{f(P_{|X|})}{X_1^d(P_{|X|})}, \frac{f(Q_1)}{X_1^d(Q_1)}, \ldots, \frac{f(Q_{|Y|})}{X_1^d(Q_{|Y|})} \right) \in C_{\mathbb{T}_{m-1}}(d)$$

with $w(\Lambda) = \delta_{\mathbb{T}_{m-1}}(d)$, where $w(\Lambda)$ is the Hamming weight of the codeword Λ, then

$$\Lambda_1 := \left(\frac{f(P_1)}{X_1^d(P_1)}, \ldots, \frac{f(P_{|X|})}{X_1^d(P_{|X|})} \right) \in C_X(d),$$

and

$$\Lambda_2 := \left(\frac{f(Q_1)}{X_1^d(Q_1)}, \ldots, \frac{f(Q_{|Y|})}{X_1^d(Q_{|Y|})} \right) \in C_Y(d).$$

Moreover

$$\delta_{\mathbb{T}_{m-1}}(d) = w(\Lambda) = w(\Lambda_1) + w(\Lambda_2) \geq \delta_X(d) + \delta_Y(d). \tag{12}$$

Therefore the inequality (11) follows from (12).   □

Remark 3.9. *From the inequality (11) we obtain that $\delta_X(d) \leq \delta_{\mathbb{T}_{m-1}}(d) - 1$ for all $0 \leq d < (m-1)(q-2)$. But $\delta_{\mathbb{T}_{m-1}}(d)$ was computed in [18]. Thus in this case*

$$\delta_X(d) \leq (q-1)^{m-(k+2)}(q-1-\ell) - 1, \tag{13}$$

where k and ℓ are the unique integers such that $k \geq 0$, $1 \leq \ell \leq q-2$ and $d = k(q-2) + \ell$.

From now on we will consider the case worked in section 3.1, where the sum of the elements of each column of the matrix A defined in (2) is a constant, i.e., $\sum_{j=1}^{n} a_{ij} = \alpha$ (a positive integer) for all $i = 1, \ldots, m$. The following map will help us to find a lower bound for the minimum distance of the corresponding projective parameterized codes.

$$\mu : \mathbb{T}_{n-1} \to X,$$

$$[(t_1, \ldots, t_n)] \to [(t_1^{a_{11}} \cdots t_n^{a_{1n}}, \ldots, t_1^{a_{m1}} \cdots t_n^{a_{mn}})].$$

μ is a well defined map and in fact it is an epimorphism of multiplicative groups. Let $N := \ker \mu$. Thus $|N| = \frac{|\mathbb{T}_{n-1}|}{|X|} = \frac{(q-1)^{n-1}}{|X|}$. Moreover $\mathbb{T}_{n-1} = \cup_{i=1}^{|X|} N \cdot [Q_i]$ (disjoint union of the corresponding cosets) for some $[Q_i] \in \mathbb{T}_{n-1}$. Let $[P_i] = \mu([Q_i])$ for all $i = 1, \ldots, |X|$ and $N = \{[R_1], \ldots, [R_{|N|}]\}$. Thus $X = \{[P_1], \ldots, [P_{|X|}]\}$ and

$$\mathbb{T}_{n-1} = \{[R_1 Q_1], \ldots, [R_{|N|} Q_1], \ldots, [R_1 Q_{|X|}], \ldots, [R_{|N|} Q_{|X|}]\}.$$

As in the introduction let $L = K[Z_1, \ldots, Z_n]$. We define another map that will be useful later on.

$$\tau : S_d \to L_{\alpha d},$$

$$f(X_1, \ldots, X_m) \to f(Z_1^{a_{11}} \cdots Z_n^{a_{1n}}, \ldots, Z_1^{a_{m1}} \cdots Z_n^{a_{mn}}).$$

τ is a linear map between the vector spaces S_d and $L_{\alpha d}$. Now we are able to prove the following theorem. In this result we are going to find a lower bound for the minimum distance of the corresponding projective parameterized codes.

Theorem 3.10. *If the sum of the elements of each column of the matrix A defined in (2) is a constant α, then*

$$\delta_X(d) \geq \left\lceil \frac{|X| \cdot \delta_{\mathbb{T}_{n-1}}(\alpha d)}{(q-1)^{n-1}} \right\rceil, \tag{14}$$

where $\delta_{\mathbb{T}_{n-1}}(\alpha d)$ is the minimum distance of the paremetererized code of order αd arising from the projective torus $\mathbb{T}_{n-1}$ and $\delta_X(d)$ is the minimum distance of the projective parameterized code associated to the toric set X defined in (1). Also $\lceil x \rceil$ is the ceiling function of x, and it means that $\lceil x \rceil = \min\{y \in \mathbb{Z} : y \geq x\}$.

Proof. Let

$$\Gamma = \left(\frac{f(P_1)}{X_1^d(P_1)}, \ldots, \frac{f(P_{|X|})}{X_1^d(P_{|X|})} \right) \in C_X(d).$$

We choose Γ in such a way that $w(\Gamma) = \delta_X(d)$. On the other hand let

$$\Omega = \left(\frac{\tau(f)(R_1 Q_1)}{Z_1^{\alpha d}(R_1 Q_1)}, \ldots, \frac{\tau(f)(R_{|N|} Q_1)}{Z_1^{\alpha d}(R_{|N|} Q_1)}, \ldots, \frac{\tau(f)(R_1 Q_{|X|})}{Z_1^{\alpha d}(R_1 Q_{|X|})}, \ldots, \right.$$

$$\left. \frac{\tau(f)(R_{|N|} Q_{|X|})}{Z_1^{\alpha d}(R_{|N|} Q_{|X|})} \right).$$

We have that $\Omega \in C_{\mathbb{T}_{n-1}}(\alpha d)$ and if $f(P_i) \neq 0$ for some $[P_i] \in X$, then due to the fact that $\mu([R_j Q_i]) = [P_i]$, we obtain that $\tau(f)(R_j Q_i) = f(P_i) \neq 0$ for all $j = 1, \ldots, n$. Thus $w(\Omega) = |N| \cdot w(\Gamma) = |N| \cdot \delta_X(d)$ and thererefore $\delta_{\mathbb{T}_{n-1}}(\alpha d) \leq w(\Omega) = |N| \cdot \delta_X(d)$. Then

$$\delta_X(d) \geq \frac{\delta_{\mathbb{T}_{n-1}}(\alpha d)}{|N|}. \tag{15}$$

The inequality (14) follows from (15) and the fact that $|N| = \frac{(q-1)^{n-1}}{|X|}$. $\quad\square$

If X is the algebraic toric set arising from the incidence matrix of any graph then $\alpha = 2$ and we can apply Theorem 3.10. Moreover if we have a connected graph, by using [16, Corollary 3.8] we obtain the following general result.

Corollary 3.11. *Let X be the algebraic toric set arising from the incidence matrix of any connected graph G. Then*

$$\delta_X(d) \geq \begin{cases} \left\lceil \frac{\delta_{\mathbb{T}_{n-1}}(2d)}{q-1} \right\rceil & \text{if} \quad G \text{ is bipartite} \\[3mm] \delta_{\mathbb{T}_{n-1}}(2d) & \text{if} \quad G \text{ is non-bipartite.} \end{cases} \tag{16}$$

Corollary 3.12. *If the sum of the elements of each column of the matrix A defined in (2) is a constant α, then*

$$r_X \geq \left\lceil \frac{|X|(q-2)(n-1)}{\alpha(q-1)^{n-1}} \right\rceil, \tag{17}$$

where r_X is the regularity index of S/I_X.

Moreover if G is a connected graph and X is the algebraic toric set arising from its incidence matrix, then

$$r_X \geq \begin{cases} \left\lceil \frac{(q-2)(n-1)}{2(q-1)} \right\rceil & \text{if} \quad G \text{ is bipartite} \\[3mm] \left\lceil \frac{(q-2)(n-1)}{2} \right\rceil & \text{if} \quad G \text{ is non-bipartite.} \end{cases} \tag{18}$$
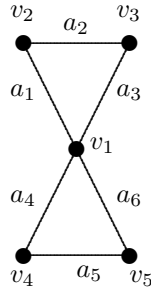
Figure 1: A connected non-bipartite graph with two cycles of length 3.

Proof. The claim follows directly because of (14), (16), and the fact that the regularity index corresponding to the torus $\mathbb{T}_{n-1}$ is exactly $(q-2)(n-1)$. $\quad\square$

In the first example of the following section we will realize that this lower bound is attained in some cases.

4   Examples

In this section we will give three different examples. In the first example we will consider a particular connected non-bipartite graph and we will compute the main characteristics of the corresponding projective parameterized codes arising from the incidence matrix of that graph. In the second example we will define clutters as particular cases of hypergraphs and a specific example of projective parameterized codes arising from uniform clutters will be given. Finally in the third example we will compute the main parameters of the projective parameterized codes associated to a matrix that does not represent a clutter and then it does not represent a graph. In these examples we will use the notation appeared in the previous sections and we will use Macaulay2 [10] for the main computations. Also we will use δ'_d to represent the lower bound showed in (14) and b_d will represent the Singleton bound, i.e.,

$$\delta'_d = \left\lceil \frac{|X| \cdot \delta_{\mathbb{T}_{n-1}}(\alpha d)}{(q-1)^{n-1}} \right\rceil \ \text{ and } \ b_d = |X| - H_X(d) + 1.$$

4.1   Example 1

Let G be the graph given in Fig. 1 where $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ is its vertex set and its edge set is given by $E(G) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. The

incidence matrix of G is the 5×6 matrix given by

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \tag{19}$$

Let $K = \mathbb{F}_7$ be a finite field with 7 elements. The toric set arising from the matrix (19) (or associated to the graph G showed in Fig. 1) is given by

$$X = \{[(t_1 t_2, t_2 t_3, t_1 t_3, t_1 t_4, t_4 t_5, t_1 t_5)] \in \mathbb{P}^5 : t_i \in K^*\}.$$

In this case we have five subsets Y_i with $|Y_i| = 6$ for all $i = 1, \ldots, 5$. The corresponding subset M is

$$M = \{(i, i, i, i, i) : i = 1, \ldots, 6\},$$

and therefore, by using Theorem 3.1,

$$|X| = \tfrac{1}{|M|} \prod_{i=1}^{5} |Y_i| = 1296.$$

We notice that $\delta'_d = \delta_{\mathbb{T}_4}(2d)$ because of Corollary 3.11. By using Macaulay2 we compute the following values.

d	1	2	3	4	5
$H_X(d)$	6	21	55	120	231
$H_{\mathbb{T}_5}(d)$	6	21	56	126	252
$\overline{H}(d)$	0	0	1	6	21
δ'_d	864	432	180	108	36
b_d	1291	1276	1242	1177	1066

d	6	7	8	9	10
$H_X(d)$	401	627	885	1130	1296
$H_{\mathbb{T}_5}(d)$	457	762	1182	1722	2373
$\overline{H}(d)$	56	135	297	592	1077
δ'_d	24	12	5	3	1
b_d	896	670	412	167	1

Moreover in this case the regularity index is $r_X = 10 = \frac{(q-2)(n-1)}{2}$ and it shows that the lower bound given in (18) works very well. This lower bound is attained in this particular case.

4.2   Example 2

In this example we continue using the notation used in the introduction.

A *clutter* $\mathcal{C}$ is a family E of subsets of a finite ground set $Z = \{Z_1, \ldots, Z_n\}$ such that if $h_1, h_2 \in E$, then $h_1 \not\subset h_2$. The ground set Z is called the vertex set of $\mathcal{C}$ and E is called the edge set of $\mathcal{C}$ and they are denoted by $V_{\mathcal{C}}$ and $E_{\mathcal{C}}$ respectively.

Clutters are special hypergraphs and are sometimes called Sperner families in the literature. One example of a Clutter is a graph with the vertices and edges defined in the usual way for graphs.

Let $\mathcal{C}$ be a clutter with vertex set $V_{\mathcal{C}} = \{Z_1, \ldots, Z_n\}$ and let h be an edge of $\mathcal{C}$. The characteristic vector of h is the vector $a = \sum_{Z_i \in h} e_i$ where e_i is the *ith* unit vector in $\mathbb{R}^n$. Throughout this example we assume that $a_1, \ldots, a_m$ is the set of all characteristic vectors of the edges of $\mathcal{C}$. In this case the matrix (2) is known as the incidence matrix of the clutter $\mathcal{C}$ and the set X defined in (1) is the toric set associated to the clutter $\mathcal{C}$. The clutter $\mathcal{C}$ is called uniform if the sum of the elements of the columns of its incidence matrix is a constant.

We realize that in any clutter, like in graphs, $|X| = \frac{(q-1)^n}{|M|}$ because $|Y_i| = q - 1$ for all $i = 1, \ldots, n$.

Let $K = \mathbb{F}_9$ be a finite field with 9 elements and X be the toric set associated to the uniform clutter $(\alpha = 3)$ whose incidence matrix is the 6×6 matrix given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \tag{20}$$

The toric set X associated to (20) becomes

$$X = \{[(t_1 t_2 t_3, t_2 t_3 t_4, t_3 t_4 t_5, t_4 t_5 t_6, t_1 t_5 t_6, t_1 t_2 t_6)] \in \mathbb{P}^5 : t_i \in K^*\}.$$

In this case we have six subsets Y_i with $|Y_i| = 8$ for $i = 1, \ldots, 6$. The corresponding set M used in Theorem 3.1 has 512 elements and therefore, by (5),

$$|X| = \tfrac{1}{|M|} \textstyle\prod_{i=1}^{6} |Y_i| = 512.$$

In the same way that in the last example we obtain, by using Macaulay2, the following values.

d	1	2	3	4	5	6
$H_X(d)$	6	19	44	85	146	231
$H_{\mathbb{T}_5}(d)$	6	21	56	126	252	462
$\overline{\overline{H}}(d)$	0	2	12	41	106	231
δ'_d	320	128	48	24	7	4
b_d	507	494	469	428	367	282

d	7	8	9	10	11	12
$H_X(d)$	344	442	492	510	512	512
$H_{\mathbb{T}_5}(d)$	792	1282	1972	2898	4088	5558
$\overline{\overline{H}}(d)$	448	840	1480	2388	3576	5046
δ'_d	1	1	1	1	1	1
b_d	169	71	21	3	1	1

It is immediate from the last table that $r_X = 11$.

4.3 Example 3

In this example we will give the main characteristics of the projective parameterized codes arising from a matrix that does not represent a clutter.

Let $K = \mathbb{F}_{11}$ be a finite field with 11 elements and X be the toric set associated to the 3×4 matrix given by

$$A = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 0 & 4 & 2 & 2 \\ 3 & 1 & 4 & 3 \end{pmatrix} \tag{21}$$

In this case $\alpha = 6$ and the set X becomes

$$X = \{[(t_1^3 t_3^3, t_1 t_2^4 t_3, t_2^2 t_3^4, t_1 t_2^2 t_3^3)] \in \mathbb{P}^3 : t_i \in K^*\}.$$

We have three subsets Y_i with $|Y_1| = |Y_3| = 10$ and $|Y_2| = 5$. The corresponding subset M has 10 elements and then, by Theorem 3.1,

$$|X| = \frac{1}{|M|} \prod_{i=1}^{3} |Y_i| = 50.$$

By using Macaulay2 we obtain the following values.

d	1	2	3	4	5	6
$H_X(d)$	4	10	20	32	44	50
$H_{\mathbb{T}_3}(d)$	4	10	20	35	56	84
$\overline{\overline{H}}(d)$	0	0	0	3	12	34
δ'_d	20	3	1	1	1	1
b_d	47	41	31	19	7	1

We conclude that $r_X = 6$.

References

[1] W. W. Adams and P. Loustaunau, *An Introduction to Gröbner Bases* (GSM **3**, American Mathematical Society, 1994).

[2] P. Delsarte, J. M. Goethals and F. J. MacWilliams, *On generalized Reed-Muller codes and their relatives,* Information and Control **16** (1970) 403–442.

[3] I. M. Duursma, C. Rentería and H. Tapia-Recillas, *Reed-Muller codes on complete intersections*, Appl. Algebra Engrg. Comm. Comput. **11** (6) (2001) 455–462.

[4] D. Eisenbud, *The geometry of syzygies: A second course in commutative algebra and algebraic geometry* (Graduate Texts in Mathematics, **229**, Springer-Verlag, New York, 2005).

[5] L. Gold, J. Little and H. Schenck, *Cayley-Bacharach and evaluation codes on complete intersections*, J. Pure Appl. Algebra **196** (1) (2005) 91–99.

[6] M. González-Sarabia and C. Rentería, *Evaluation codes associated to complete bipartite graphs*, Int. J. Algebra **2** (2008) 163–170.

[7] M. González-Sarabia and C. Rentería, *Evaluation Codes Associated to some Matrices*, Int. J. Contemp. Math. Sci. **2** (13) (2007) 615–625.

[8] M. González-Sarabia, C. Rentería and M. Hernández de la Torre, *Minimum distance and second generalized Hamming weight of two particular linear codes*, Congr. Numer. **161** (2003) 105–116.

[9] M. González-Sarabia, C. Rentería and H. Tapia-Recillas, *Reed-Muller-type codes over the Segre variety*, Finite Fields Appl. **8** (4) (2002) 511–518.

[10] D. Grayson and M. Stillman, *Macaulay*2, Available via anonymous ftp from math.uiuc.edu, 1996.

[11] J. Hansen, *Toric surfaces and error-correcting codes*, Coding theory, Criptography and related areas (2000) 132–142.

[12] J. Harris, *Algebraic Geometry. A first course* (Graduate Texts in Mathematics **133**, Springer-Verlag, New York, 1992).

[13] D. Joyner, *Toric codes over finite fields*, Appl. Algebra Engrg. Comm. Comput. **15** (2004) 63–79.

[14] F.J. MacWilliams and N.J.A. Sloane, *The Theory of Error-correcting Codes* (North-Holland, 1977).

[15] J. Neves, M. Vaz Pinto, R.H. Villarreal, *Vanishing ideals over graphs and even cycles*, To appear in Comm. Algebra. (2011) arXiv:1111.6278[pdf, ps, other].

[16] C. Rentería, A. Simis and R. H. Villarreal, *Algebraic methods for parameterized codes and invariants of vanishing ideals over finite fields*, Finite Fields Appl.**17** (2011) 81–104.

[17] E. Reyes, R. H. Villarreal and L. Zárate, *A note on affine toric varieties*, Linear Algebra Appl. **318** (2000) 173–179.

[18] E. Sarmiento, M. Vaz Pinto and R. H. Villarreal, *The minimum distance of parameterized codes on projective tori*, Appl. Algebra Engrg. Comm. Comput. **22** (4) (2011) 249–264.

[19] I. Soprunov, J. Soprunova, *Bringing toric codes to the next dimension*, SIAM J. Discrete Math. **24** (1) (2010) 655–665.

[20] A. Sørensen, *Projective Reed-Muller codes*, IEEE Trans. Inform. Theory **37** (6) (1991) 1567–1576.

[21] R. Stanley, *Hilbert functions of graded algebras,* Adv. Math. **28** (1978) 57–83.

[22] H. Stichtenoth, *Algebraic function fields and codes* (Universitext, Springer-Verlag, Berlin, 1993).

[23] S. Tohăneanu, *Lower bounds on minimal distance of evaluation codes,* Appl. Algebra Engrg. Comm. Comput. **20** (5–6) (2009) 351–360.

[24] M. Tsfasman, S. Vladut and D. Nogin, *Algebraic geometric codes*: *basic notions* (Mathematical Surveys and Monographs **139**, American Mathematical Society, Providence, RI, 2007).

[25] R. H. Villarreal, *Monomial Algebras* (Monographs and Textbooks in Pure and Applied Mathematics **238**, Marcel Dekker, New York, 2001).

Manuel GONZÁLEZ SARABIA,
Departamento de Ciencias Básicas,
Unidad Profesional Interdisciplinaria en
Ingeniería y Tecnologías Avanzadas,
Instituto Politécnico Nacional,
07340, México, D.F.
Partially supported by COFAA-IPN and SNI.
Email: mgonzalezsa@ipn.mx

Carlos RENTERÍA MÁRQUEZ,
Departamento de Matemáticas,
Escuela Superior de Física y Matemáticas,
Instituto Politécnico Nacional,
07300, México, D.F.
Partially supported by COFAA-IPN and SNI.
Email: renteri@esfm.ipn.mx

Eliseo SARMIENTO ROSALES,
Departamento de Matemáticas,
Escuela Superior de Física y Matemáticas,
Instituto Politécnico Nacional,
07300, México, D.F.
Partially supported by SNI.
Email: esarmiento@ipn.mx