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Mahler's measure : proof of two conjectured formulae

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Abstract

In this note we prove the two formulae conjectured by D. W. Boyd [Experiment. Math. 7 (1998), 37-82],

$$m(y^2(x+1)^2 + y(x^2+6x+1) + (x+1)^2) = \frac{8}{3}L^{'}(\chi_{-4}, -1),$$

$$m(y^2(x+1)^2 + y(x^2-10x+1) + (x+1)^2) = \frac{20}{3}L^{'}(\chi_{-3}, -1),$$

where m denotes the logarithmic Mahler measure for two-variable polynomials.

1 Introduction

The logarithmic Mahler measure of a non-zero Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$ is defined as

$$m(P) := \frac{1}{(2\pi i)^n} \int_{\mathbf{T}^n} \log |P(x_1, ..., x_n)| \frac{dx_1}{x_1} ... \frac{dx_n}{x_n}.$$

Here,

$$\mathbf{T}^n = \{(x_1, ..., x_n) \in \mathbb{C}^n / \ |x_1| = ... |x_n| = 1\}$$

is the n-torus. This integral is not singular and $m(P)$ always exists. Moreover, if P has integral coefficients, this number is nonnegative.

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In [3] Boyd computed the measure of polynomials of the form

$$P_k(x,y) = k + Q(x,y)$$

where Q is a Laurent polynomial and k an integer parameter. He found families of (conjectural) formulas of the type $m(P_k) \stackrel{?}{=} r_k\ b_{E_k}$, where $P_k = 0$ defines a curve E_k of genus 1, r_k is a rational number and

$$b_{E_k} := \frac{N_k}{4\pi^2} L(E_k, 2)$$

Here N_k is the conductor of the elliptic curve E_k and $L(E_k, 2)$ its L-series. In particular many experimental relations between the Mahler measure of different polynomials are founded.

Some of these experimental relations are proved Rodriguez-Villegas [5], Bertin [1, 2], Touafek and Kerada [6]. The main idea is to view $m(P)$ as an elliptic regulator, so, expressed in terms of the elliptic dilogarithm.

Also in [3], formulas of the type $m(P_k) \stackrel{?}{=} r_k d_f$ are given, where r_k is a rational number,

$$d_f := L^{'}(\chi_{-f}, -1) = \frac{f^{\frac{3}{2}}}{4\pi} L(\chi_{-f}, 2)$$

and $P_k = 0$ defines a curve of genus 0. Here $L(\chi_{-f}, 2)$ is the Dirichlet L-function associated to the odd primitive caracter χ_{-f}.
Bloch's formula gives $L^{'}(\chi_{-f}, -1)$ for odd primitive character χ_{-f} as a combination of Bloch-Wigner dilogarithms,

$$L^{'}(\chi_{-f}, -1) = \frac{f}{4\pi} \sum_{m=1}^{f} \chi_{-f}(m) D(\xi_f^m) \tag{1}$$

where ξ_f denotes a primitive of roots of unity. So, we may get $m(P)$ as a combination of Bloch-Wigner dilogarithm.
The notation $A \stackrel{?}{=} B$, means " A is conjectured to be equal to B ", that is A and B are numerically equal to at least 25 decimal places.

After some preliminaries, we prove in section 3 the two following identities guessed by Boyd [3]

$$m(y^2(x+1)^2 + y(x^2+6x+1) + (x+1)^2) = \frac{8}{3} L^{'}(\chi_{-4}, -1),$$

$$m(y^2(x+1)^2 + y(x^2-10x+1) + (x+1)^2) = \frac{20}{3} L^{'}(\chi_{-3}, -1).$$

2 Preliminaries

2.1 Polylogarithms

For a positive integer k, the kth polylogarithm function is defined for $|x| < 1$ by

$$Li_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k},\ x \in \mathbb{C}.$$

This function can be continued analytically to $\mathbb{C}\backslash[1, \infty)$.

In order to avoid discontinuities, and to extend this function to the whole complex plane, Zagier [7] propose the following version

$$\widehat{\mathbb{L}}_k(x) := \widehat{\Re}_k\Big(\sum_{j=0}^{k-1} \frac{2^j B_j}{j!} (\log|x|)^j Li_{k-j}(x)\Big)$$

where B_j is the jth Bernoulli number and $\widehat{\Re}_k$ denotes $\Re$ or $i\Im$ depending on whether k is odd or even.

This function is one-valued, real analytic in $\mathbb{P}^1(\mathbb{C})\backslash\{0, 1, \infty\}$ and continuous in $\mathbb{P}^1(\mathbb{C})$. Moreover, $\widehat{\mathbb{L}}_k$ satisfy some functional equations, for example

$$\widehat{\mathbb{L}}_k(\frac{1}{x}) = (-1)^{k-1}\widehat{\mathbb{L}}_k(x).$$

For $k = 2$,

$$\widehat{\mathbb{L}}_2(x) := \Im Li_2(x) + \log|x| arg(1 - x)$$

is well-known as $D(x)$, the Bloch-Wigner dilogarithm.
The Bloch-Wigner dilogarithm satisfies the following properties

$D(x) = -D(\overline{x})$, $D(x) = D(\frac{x-1}{x})$, $D(x) = D(\frac{1}{1-x})$, $D(x) = -D(\frac{1}{x})$, $D(x) = -D(1 - x)$, $D(x) = -D(\frac{x}{x-1})$, $D(x^n) = n\sum_{k=0}^{n-1} D(e^{\frac{2\pi ik}{n}}x)$ (distribution formula).

2.2 Mahler measure of two-variable polynomials

Let $P \in \mathbb{C}[x, y]$ be a polynomial in two variables, we may think of it as a polynomial in x with coefficients which are polynomials in y and write

$$P(x, y) = a_0(y)\prod_{j=1}^{k}(x - x_j(y))$$

where $x_j(y)$ are algebraic functions of y. Integrating the x variable using Jensen's formula we obtain

$$m(P) = m(a_0) + \sum_{j=1}^{k} \frac{1}{2\pi i} \int_{|y|=1} \log^+ |x_j(y)| \frac{dy}{y}. \tag{2}$$

where $\log^+ |z| = \log |z|$ if $|z| \geq 1$ and 0 otherwise.

Let

$$\eta_2(2)(x, y) = i \log |x| d(argy) - i \log |y| d(argx)$$

it's a differential form on the variety

$$\gamma = \{P(x, y) = 0\} \cap \{|x| = 1, |y| \geq 1\}.$$

The differential form $\eta_2(2)$ satisfy the following properties

- $\eta_2(2)(x, y) = -\eta_2(2)(y, x)$
- $\eta_2(2)(x_1 x_2, y) = \eta_2(2)(x_1, y) + \eta_2(2)(x_2, y)$
- $\eta_2(2)(x, 1 - x) = d\widehat{D}(x)$, where $\widehat{D}(x) = iD(x)$
- if $\alpha \neq \beta$

$$\begin{aligned} \eta_2(2)(t-\alpha, t-\beta) &= \eta_2(2)(\frac{t-\alpha}{\beta-\alpha}, 1 - \frac{t-\alpha}{\beta-\alpha}) + \eta_2(2)(t-\alpha, \alpha-\beta) \\ &+ \eta_2(2)(\beta-\alpha, t-\beta).(Tate's\ formula) \end{aligned}$$

Hence $\eta_2(2)(t - 1, t) = -\eta_2(2)(t, 1 - t)$ etc...

For each j, the set

$$\gamma_j = \{(x_j(y), y) : |y| = 1 \text{ and } |x_j(y)| \geq 1\}$$

is a direct path (or a union of such) inside of $C = \{P(x, y) = 0\}$. The set $\cup\gamma_j$ precisely coincides with

$$\gamma = \{(x, y) \in C : |y| = 1, |x| \geq 1\}$$

Therefore (2) may be rewritten as

$$m(P) = m(a_0) + \frac{1}{2\pi i} \int_{\gamma} \log |x| \frac{dy}{y} = m(a_0) + \frac{1}{2\pi i} \int_{\gamma} \eta_2(2)(x, y).$$

Or equivalently

$$m(P) = m(a_0) + \frac{1}{2\pi i} \int_{\gamma} \log |x| \frac{dy}{y} = m(a_0) + \frac{1}{2\pi i} \sum_j \int_{\gamma_j} \eta_2(2)(x_j(y), y).$$

When the differential form $\eta_2(2)$ is exact and $\partial\gamma \neq 0$ where

$$\partial\gamma = \{P(x,y)=0\} \cap \{|x|=1, |y|=1\},$$

in this case we can integrate using Stokes formula.

We can guarantee that $\eta_2(2)$ is exact by having $\{x,y\}$ is trivial in $K_2(\mathbb{C}(C))\otimes\mathbb{Q}$ where $C = \{P(x,y)=0\}$. When the polynomial P is tempered, this condition is satisfied (see Villegas [4]).

2.3 Tempered polynomial

A polynomial in two variables is tempered if the polynomials of the faces of its Newton polygon has only roots of unity.

When drawing the convex hull of points $(i,j) \in \mathbb{Z}^2$ corresponding to the monomials $a_{i,j}x^iy^j$, $a_{i,j} \neq 0$, you also draw points located on the faces. The polynomial of the face is a polynomial in one variable t which is a combination of the monomials 1, t, t^2,.... The coefficients of the combination are given when going along the face, that is $a_{i,j}$ if the lattice point of the face belongs to the convex hull and 0 otherwise. The two polynomials

$$P(x,y) = y^2(x+1)^2 + y(x^2+6x+1) + (x+1)^2$$

and

$$Q(x,y) = y^2(x+1)^2 + y(x^2-10x+1) + (x+1)^2$$

are tempered.

3 Results

For the polynomials

$$P(x,y) = y^2(x+1)^2 + y(x^2+6x+1) + (x+1)^2$$

and

$$Q(x,y) = y^2(x+1)^2 + y(x^2-10x+1) + (x+1)^2$$

Boyd [3] guessed

$$m(P) \stackrel{?}{=} \frac{8}{3}L^{'}(\chi_{-4},-1),\ m(Q) \stackrel{?}{=} \frac{20}{3}L^{'}(\chi_{-3},-1) \tag{3}$$

Therefore, by Bloch's formula (1), (3) may be rewritten as

$$m(P) \stackrel{?}{=} \frac{16}{3\pi}D(i),\ m(Q) \stackrel{?}{=} \frac{10}{\pi}D(j)$$

where $j = e^{\frac{2\pi i}{3}}$.

In the following we prove these two formulae.

Theorem 1 *We have the following identity*

$$m(y^2(x+1)^2 + y(x^2+6x+1) + (x+1)^2) = \frac{8}{3}L^{'}(\chi_{-4}, -1).$$

Proof. Let P be the polynomial defined by

$$P(x,y) = y^2(x+1)^2 + y(x^2+6x+1) + (x+1)^2.$$

It defines a rational curve with the double point $(1,-1)$.

Let $x = X+1$ et $y = Y-1$, so $P(x,y) = 0$ becomes,

$$((X+2)Y)^2 = X^2(Y-1).$$

i.e.

$$Y - 1 = (\frac{(X+2)Y}{X})^2.$$

Let $t = \frac{(X+2)Y}{X}$, so $y = t^2$ and the equation $P(x,y) = 0$ becomes

$$(x+1)^2t^4 + (x^2+6x+1)t^2 + (x+1)^2 = 0.$$

Hence we get the following parametrization

$$\begin{aligned} y(t) &= t^2 \\ x_1(t) &= -\frac{(t^3-1)(t+1)}{(t-1)(t^3+1)} \\ x_2(t) &= -\frac{(t^3+1)(t-1)}{(t+1)(t^3-1)}. \end{aligned}$$

We have

$$m(P) = \frac{1}{2\pi i}\int_{|y|=1} \log|x|\frac{dy}{y} = \frac{1}{2\pi i}\int_{\gamma} \eta_2(2)(x,y)$$

where

$$\gamma = \left\{(x+1)^2y^2 + (x^2+6x+1)y + (x+1)^2 = 0\right\} \cap \{|y| = 1, |x| \geq 1\}$$

and

$$\eta_2(2)(x,y) = i\log|x|d(argy) - i\log|y|d(argx).$$

Using the parametrization, we get

$$m(P) = \frac{1}{2\pi i}\int_{\gamma_1} \eta_2(2)(x_1(t),y(t)) + \frac{1}{2\pi i}\int_{\gamma_2} \eta_2(2)(x_2(t),y(t)) \qquad (4)$$

where

$$\gamma_1 = \{t : |x_1(t)| \geq 1, |y(t)| = 1\} = \left\{t : t = e^{i\frac{\theta}{2}}, \theta \in [0, \pi]\right\}$$

and

$$\gamma_2 = \{t : |x_2(t)| \geq 1, |y(t)| = 1\} = \left\{t : t = e^{i\frac{\theta}{2}}, \theta \in [\pi, 2\pi]\right\}$$

Using Tate's formula, we get

$$\eta_2(2)(t-1, t) = -\eta_2(2)(t, 1-t),$$

$$\eta_2(2)(t+1, t) = -\eta_2(2)(-t, 1+t),$$

$$\eta_2(2)(1+t^3, t^3) = -\eta_2(2)(-t^3, 1+t^3).$$

So

$$\begin{aligned}
\eta_2(2)(x_1(t), y(t)) &= \frac{2}{3}\eta_2(2)(1-t^3, t^3) - \frac{2}{3}\eta_2(2)(1+t^3, t^3) \\
&+ 2\eta_2(2)(t+1, t) - 2\eta_2(2)(t-1, t) \\
&= -\frac{2}{3}\eta_2(2)(t^3, 1-t^3) + \frac{2}{3}\eta_2(2)(-t^3, 1+t^3) \\
&- 2\eta_2(2)(-t, 1+t) + 2\eta_2(2)(t, 1-t).
\end{aligned}$$

Hence

$$\frac{1}{2\pi i}\int_{\gamma_1} \eta_2(2)(x_1(t), y(t)) = \frac{i}{2\pi i}\left[-\frac{2}{3}D(t^3) + \frac{2}{3}D(-t^3) - 2D(-t) + 2D(t)\right]_1^i$$

which gives

$$\frac{1}{2\pi i}\int_{\gamma_1} \eta_2(2)(x_1(t), y(t)) = \frac{8}{3\pi}D(i). \tag{5}$$

By the same arguments we get

$$\frac{1}{2\pi i}\int_{\gamma_2} \eta_2(2)(x_2(t), y(t)) = \frac{8}{3\pi}D(i). \tag{6}$$

Using (5) and (6), (4) becomes

$$m(P) = \frac{16}{3\pi}D(i) = \frac{8}{3}L^{'}(\chi_{-4}, -1).$$

■

Theorem 2 *We have the following identity*

$$m(y^2(x+1)^2 + y(x^2-10x+1) + (x+1)^2) = \frac{20}{3}L^{'}(\chi_{-3}, -1).$$

Proof. Let Q be the polynomial defined by

$$Q(x,y) = y^2(x+1)^2 + y(x^2-10x+1) + (x+1)^2.$$

It defines a rational curve with the double point $(1,1)$.

Let $x = X+1$ et $y = Y+1$, so $Q(x,y) = 0$ becomes,

$$((X+2)Y)^2 = -3X^2(Y+1).$$

i.e.

$$Y + 1 = -(\frac{(X+2)Y}{\sqrt{3}X})^2.$$

Let $t = \frac{(X+2)Y}{\sqrt{3}X}$, so $y = -t^2$ and the equation $Q(x,y) = 0$ becomes

$$(x+1)^2t^4 - (x^2-10x+1)t^2 + (x+1)^2 = 0.$$

Hence we get the following parametrization

$$y(t) = -t^2, x_1(t) = -\frac{t^2+\sqrt{3}t+1}{t^2-\sqrt{3}t+1}, x_2(t) = -\frac{t^2-\sqrt{3}t+1}{t^2+\sqrt{3}t+1}.$$

We have

$$m(Q) = \frac{1}{2\pi i}\int_{|y|=1} \log|x|\frac{dy}{y} = \frac{1}{2\pi i}\int_{\gamma} \eta_2(2)(x,y),$$

where

$$\gamma = \left\{(x+1)^2y^2 + (x^2-10x+1)y + (x+1)^2 = 0\right\} \cap \{|y| = 1, |x| \geq 1\}.$$

Using the parametrization , we get

$$m(Q) = \frac{1}{2\pi i}\int_{\gamma_1} \eta_2(2)(x_1(t), y(t)) = \frac{1}{2\pi i}\int_{\gamma_2} \eta_2(2)(x_2(t), y(t)),$$

where

$$\gamma_1 = \{t : |x_1(t)| \geq 1, |y(t)| = 1\} = \left\{t : t = ie^{i\frac{\theta}{2}}, \theta \in [0, 2\pi]\right\}$$

and

$$\gamma_2 = \{t : |x_2(t)| \geq 1, |y(t)| = 1\} = \left\{t : t = -ie^{i\frac{\theta}{2}}, \theta \in [0, 2\pi]\right\}.$$

Using Tate's formula, we get

$$\begin{aligned} \eta_2(2)(x_1(t), y(t)) &= 2\eta_2(2)(\frac{t-\xi_6}{-\xi_6}, 1 - \frac{t-\xi_6}{-\xi_6}) + 2\eta_2(2)(\frac{t-\overline{\xi_6}}{-\overline{\xi_6}}, 1 - \frac{t-\overline{\xi_6}}{-\overline{\xi_6}}) \\ &+ 2\eta_2(2)(\frac{t}{-\xi_6}, 1 - \frac{t}{-\xi_6}) + 2\eta_2(2)(\frac{t}{-\overline{\xi_6}}, 1 - \frac{t}{-\overline{\xi_6}}). \end{aligned}$$

Hence

$$\begin{aligned} m(Q) &= \frac{2}{2\pi i}\int_{\gamma_1} (d\widehat{D}(\frac{t-\xi_6}{-\xi_6}) + d\widehat{D}(\frac{t-\overline{\xi_6}}{-\overline{\xi_6}}) + d\widehat{D}(\frac{t}{-\xi_6}) + d\widehat{D}(\frac{t}{-\overline{\xi_6}})) \\ &= \frac{2i}{2\pi i}\left[(D(\frac{t-\xi_6}{-\xi_6}) + D(\frac{t-\overline{\xi_6}}{-\overline{\xi_6}}) + D(\frac{t}{-\xi_6}) + D(\frac{t}{-\overline{\xi_6}}))\right]_i^{-i} \end{aligned}$$

where $\xi_6 = e^{i\frac{\pi}{6}}$. So

$$m(Q) = \frac{4}{\pi}\left[D(i\xi_6) - D(-i\xi_6)\right]. \tag{7}$$

Using the fact that $i\xi_6 = j$ and $D(-j) = -\frac{3}{2}D(j)$, (7) becomes

$$m(Q) = \frac{10}{\pi}D(j) = \frac{20}{3}L^{'}(\chi_{-3}, -1).$$

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