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On pointwise measurability of multifunctions

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Abstract

We present some results concerning pointwise measurability of multifunctions and some relationships with global measurability of multifunctions.

1 Introduction

Measurability of multifunctions and measurable selections have important applications in economic mathematics (e.g. Aumann [1]), optimization and optimal control, for example, in studying the measurability of the demand relation in a perfectly competitive (pure exchange) economy (Klein and Thompson [5]), in the existence of subgame-perfect equilibria with public randomization (Mariotti [6]), in probabilistic control models (Estigneev [3]).

Motivated by their research work in nonstandard analysis, H. Render and L. Rogge [8] have defined a concept of pointwise measurability for functions. In a previous work (Croitoru and Văideanu [2]), we extended this notion and introduced the concept of pointwise measurable multifunction, establishing some relations with continous multifunctions and different kinds of global measurable multifunctions.

In this paper, we introduce a new concept of pointwise measurability of a multifunction, that is *strong pointwise measurability* and present some sufficient conditions so that strong pointwise measurability implies measurability or strong measurability.

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In the sequel, $(X,\tau_X),(Y,\tau_Y),(Z,\tau_Z)$ are topological spaces, $\mathcal{A}$ is a σ-algebra of subsets of X, $\mathcal{P}(Y)$ is the family of all subsets of Y and $\mathcal{P}_0(Y)$ is the family of nonvoid subsets of Y.

Definition 1.1 *The topological space (X,τ) is said to be:*

(a) Lindelöf if every open covering of X has a countable subcovering.

(b) Lindelöf hereditary if every subspace of X is Lindelöf.

(c) cosmic if X is regular and there exists $(B_n)_{n\in\mathbb{N}}\subset\mathcal{P}(X)$ such that for every $x\in X$ and $V\in\tau_X$ with $x\in V$, there exists $n\in\mathbb{N}$, such that $x\in B_n\subset V$.

Remark 1.2 *(Michael [7])*

I. Every cosmic space is Lindelöf hereditary.

II. A space (X,τ) is cosmic if and only if it is a continuous image of separable metric space.

Definition 1.3 *A function $f:X\to Y$ is said to be $\mathcal{A}$-measurable (shortly measurable) if $f^{-1}(D)\in\mathcal{A}$, for every $D\in\tau_Y$.*

Definition 1.4 *If $F:X\to\mathcal{P}(Y)$ is a multifunction and $B\in\mathcal{P}(Y)$, then:*

(a) *the weak inverse image* of B under F is

$$F^{-1}(B)=\{x\in X|\ F(x)\cap B\neq\emptyset\},$$

(b) *the strong inverse image* of B under F is

$$F^{+1}(B)=\{x\in X|\ F(x)\subset B\}.$$

Definition 1.5 *(Hu and Papageorgiou [4]-p.141) A multifunction $F:X\to\mathcal{P}(Y)$ is said to be:*

(a) $\mathcal{A}$-measurable (shortly measurable), if $F^{-1}(D)\in\mathcal{A}$, for every $D\in\tau_Y$.

(In literature, this notion of measurability is sometimes called weak measurability.)

(b) strong $\mathcal{A}$-measurable (shortly strong measurable) if $F^{-1}(E)\in\mathcal{A}$, for every closed set $E\in\mathcal{P}(Y)$.

Remark 1.6 *If Y is a metric space, then every strong measurable multifunction is measurable.*

2 Pointwise measurability of multifunctions

Definition 2.1 *(Croitoru and Văideanu [2]) A multifunction $F:X\to\mathcal{P}(Y)$ is called $\mathcal{A}$-measurable (shortly measurable) at $x\in X$ if for all $y\in Y$, for* *every $V \in \tau_Y$ with $y \in V$, there exist $U \in \tau_Y$ with $y \in U$, $D \in \tau_X$ with $x \in D$ and $A \in \mathcal{A}$ such that*

$$F^{-1}(U) \cap D \subset A \subset F^{-1}(V).$$

Taking into account that we defined a pointwise measurability for multifunctions in terms of "weak inverse image", it is natural to introduce a pointwise measurability in terms of "strong inverse image".

Definition 2.2 *A multifunction $F : X \to \mathcal{P}(Y)$ is said to be strong $\mathcal{A}$-measurable (shortly strong measurable) at $x \in X$ if for every $y \in Y$ and every $V \in \tau_Y$, with $y \in V$, there exist $U \in \tau_Y$ with $y \in U, D \in \tau_X$ with $x \in D$ and $A \in \mathcal{A}$ such that*

$$F^{-1}(U) \cap D \subset A \subset F^{+1}(V).$$

Example 2.3 *I. Let $f : X \to Y$ be a measurable function and $F : X \to \mathcal{P}(Y)$ the multifunction defined by $F(x) = \{f(x)\}$, for every $x \in X$. Then F is strong measurable at every $x \in X$.*

II. Let $X = Y = [0,1]$ endowed with the usual topology, $\mathcal{A} = \mathcal{B}(X)$ and the multifunction $F : X \to \mathcal{P}(Y)$, $F(x) = [0,x]$, for every $x \in [0,1]$. We prove that F is strong measurable at 0.

Let $y \in [0,1]$ and $V \in \tau_Y$, such that $y \in V$.

(i) If $y = 0$, there exists $\varepsilon > 0$, such that $0 < y + \varepsilon < 1$ and $[0, y+\varepsilon) \subset V$. We consider $U = D = A = [0, y+\varepsilon)$ and we have:

$$\begin{aligned} F^{-1}(U) \cap D &= [0,1] \cap [0, y+\varepsilon) = [0, y+\varepsilon) \\ &= A = F^{+1}([0, y+\varepsilon)) \subset F^{+1}(V). \end{aligned}$$

(ii) If $y \in (0,1)$, there exists $\varepsilon > 0$, such that $0 < y - \varepsilon < y < y + \varepsilon < 1$. We take $U = (y-\varepsilon, y+\varepsilon)$, $D = [0, y-\varepsilon)$, $A = \emptyset$ and we obtain:

$$F^{-1}(U) \cap D = (y-\varepsilon, 1] \cap [0, y-\varepsilon) = \emptyset = A \subset F^{+1}(V).$$

(iii) If $y = 1$, we consider $U = \left(\frac{1}{2}, 1\right]$, $D = \left[0, \frac{1}{2}\right)$, $A = \emptyset$ and we have:

$$F^{-1}(U) \cap D = \left(\frac{1}{2}, 1\right] \cap \left[0, \frac{1}{2}\right) = \emptyset = A \subset F^{+1}(V).$$

Thus F is strong measurable at 0.

Theorem 2.4 *A multifunction $F : X \to \mathcal{P}(Y)$ is strong measurable at $x \in X$ if and only if for every compact set $K \subset Y$ and every $V \in \tau_Y$ with $K \subset V$, there exist $U \in \tau_Y$ with $K \subset U, D \in \tau_X$ with $x \in D$ and $A \in \mathcal{A}$ such that*

$$F^{-1}(U) \cap D \subset A \subset F^{+1}(V). \tag{1}$$

Proof. Suppose F is strong measurable at x. Let $K \subset Y$ be a compact set and $V \in \tau_Y$ with $K \subset V$. Let $y \in K \subset V$. Since F is strong measurable at x, there are $U_y \in \tau_Y$ with $y \in U_y, D_y \in \tau_X$ with $x \in D_y$ and $A_y \in \mathcal{A}$ such that

$$F^{-1}(U_y) \cap D_y \subset A_y \subset F^{+1}(V). \tag{2}$$

So, $K \subset \bigcup_{y \in K} U_y$ and since K is compact, there exist $y_1, y_2, \ldots, y_n \in K$ such that $K \subset \bigcup_{i=1}^{n} U_{y_i}$ and (2) is satisfied for all $\{y_i\}_{i=1}^{n}$. Denoting $U = \bigcup_{i=1}^{n} U_{y_i}$, $D = \bigcap_{i=1}^{n} D_{y_i}$ and $A = \bigcup_{i=1}^{n} A_{y_i} \in \mathcal{A}$, we now obtain (1).

The reverse implication is obvious.

Remark 2.5 *Evidently, if F is strong measurable at $x \in X$, then F is measurable at x since $F^{+1}(B) \subset F^{-1}(B)$, for every $B \in \mathcal{P}(Y)$.*

The converse is not true. Indeed, let $X = Y = \mathbb{R}$ endowed with the usual topology, $\mathcal{A} = \mathcal{B}(\mathbb{R})$ and $F : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ defined by

$$F(x) = \begin{cases} \{0\}, & x \neq 0 \\ [-1, 1], & x = 0. \end{cases}$$

Because F is measurable (even upper semicontinous), it follows that F is measurable at 0.

We now prove that F is not strong measurable at 0. Let $y > 1$ and $V = (y - \varepsilon, y + \varepsilon)$, such that $\varepsilon > 0$ and $y - \varepsilon > 1$. Then $F^{+1}(V) = \emptyset$.

If $A \in \mathcal{A}$, $A \neq \emptyset$, then $A \not\subset F^{+1}(V)$.

If $A = \emptyset$, let $\alpha > 0$ such that $0 < y - \alpha < 1$ and $U = (y - \alpha, y + \alpha)$. Then $F^{-1}(U) = \{0\}$ and for every $D \in \tau_X$ with $0 \in D$, we have $F^{-1}(U) \cap D = \{0\} \not\subset A$. Thus, F is not strong measurable at 0.

We now establish some sufficient criteria, so that pointwise strong measurability implies measurability or strong measurability.

Theorem 2.6 *Suppose X is Lindelöf and $F : X \to \mathcal{P}_0(Y)$ is strong measurable at every $x \in X$. Then:*

(i) For every compact $K \subset Y$ and $V \in \tau_Y$ with $K \subset V$, there is $A \in \mathcal{A}$ such that

$$F^{-1}(K) \subset A \subset F^{+1}(V).$$

(ii) For every G_δ compact $K \subset Y$, we have $F^{+1}(K) = F^{-1}(K) \in \mathcal{A}$.

(iii) Suppose Y is a $\sigma-$compact metric space. Then F is measurable.

Proof. (i) Let K be a compact subset of Y and let $V \in \tau_Y$ with $K \subset V$. By Theorem 2.4, for every $x \in X$, there exist $U_x \in \tau_Y$ with $K \subset U_x$, $D_x \in \tau_X$ with $x \in D_x$ and $A_x \in \mathcal{A}$ such that

$$F^{-1}(U_x) \cap D_x \subset A_x \subset F^{+1}(V). \tag{3}$$

So, $X = \bigcup\limits_{x \in X} D_x$ and since X is Lindelöf, there is a countable subcover, $X = \bigcup\limits_{n=1}^{\infty} D_{x_n}$. Denoting $A = \bigcup\limits_{n=1}^{\infty} A_{x_n}$ and using (3) it follows:

$$F^{-1}(K) = \bigcup_{n=1}^{\infty} [F^{-1}(K) \cap D_{x_n}] \subset \bigcup_{n=1}^{\infty} A_{x_n} = A \subset F^{+1}(V).$$

(ii) Since K is G_δ, we can write $K = \bigcap\limits_{n=1}^{\infty} V_n$, where $V_n \in \tau_Y$, for every $n \in \mathbb{N}$. According to (i), there is $A_n \in \mathcal{A}$ such that

$$F^{+1}(K) \subset F^{-1}(K) \subset A_n \subset F^{+1}(V_n), \;\; \forall n \in \mathbb{N},$$

which implies

$$F^{+1}(K) \subset F^{-1}(K) \subset \bigcap_{n=1}^{\infty} A_n \subset \bigcap_{n=1}^{\infty} F^{+1}(V_n) = F^{+1}(K).$$

Thus,

$$F^{+1}(K) = F^{-1}(K) = \bigcap_{n=1}^{\infty} A_n \in \mathcal{A}.$$

(iii) Let $D \in \tau_Y$. Since Y is $\sigma-$compact, there exists a sequence $(K_n)_{n \in \mathbb{N}^*}$ of compact (and consequently G_δ) subsets of Y, such that $D = \bigcup\limits_{n=1}^{\infty} K_n$. According to (ii), $F^{-1}(D) = \bigcup\limits_{n=1}^{\infty} F^{-1}(K_n) \in \mathcal{A}$, thus F is measurable.

Theorem 2.7 *Suppose X is Lindelöf hereditary and Y is cosmic. If $F : X \to \mathcal{P}_0(Y)$ is strong measurable at every $x \in X$, then F is strong measurable. Moreover, if Y is a metric space, then F is measurable.*

Proof. Since Y is cosmic, there exists $(B_n)_n \subset \mathcal{P}(Y)$ such that for every $y \in Y$ and $V \in \tau_Y$ with $y \in V$, there is $n \in \mathbb{N}$ such that $y \in B_n \subset V$. Let $V \in \tau_Y$ and we set:

$$E_n = \{x \in X \mid \exists D \in \tau_X, \exists A \in \mathcal{A} : x \in F^{-1}(B_n) \cap D \subset A \subset F^{+1}(V)\}.$$

Since F is strong measurable at every $x \in E_n$, there is $D_x^n \in \tau_X$ and $A_x^n \in \mathcal{A}$ such that $x \in F^{-1}(B_n) \cap D_x^n \subset A_x^n \subset F^{+1}(V)$. Then $E_n \subset \bigcup_{x \in E_n} D_x^n$. Since E_n is Lindelöf, there is $(D_{x_i}^n)_{i \in \mathbb{N}}$ such that $E_n \subset \bigcup_{i=1}^{\infty} D_{x_i}^n$. Since $E_n \subset F^{-1}(B_n)$, it results:

$$\bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{n} A_{x_i}^n \subset F^{+1}(V). \tag{4}$$

Let $x \in F^{+1}(V)$ and $y \in F(x)$. Since F is strong measurable at x, there are $U \in \tau_Y$ with $y \in U$, $D \in \tau_X$ with $x \in D$ and $A \in \mathcal{A}$ such that $F^{-1}(U) \cap D \subset A \subset F^{+1}(V)$. Now, there is $n \in \mathbb{N}$ so that $y \in B_n \subset U$. It follows $x \in F^{-1}(B_n) \subset F^{-1}(U)$. Consequently, $x \in F^{-1}(B_n) \cap D \subset A \subset F^{+1}(V)$, which shows that $x \in E_n$. Thus, $F^{+1}(V) \subset \bigcup_{n=1}^{\infty} E_n$ which, together with (4), proves that $F^{+1}(V) = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} A_{x_i}^n \in \mathcal{A}$. So, $F^{+1}(V) \in \mathcal{A}$, for every $V \in \tau_Y$. Now let E be a closed subset of Y. Then $F^{-1}(E) = cF^{+1}(cE) \in \mathcal{A}$, which implies that F is strong measurable.

If Y is a metric space, then Remark 1.6 ensures that F is measurable.

Using the definitions, we obtain the following properties of strong pointwise measurability.

Proposition 2.8 *Let $F, G : X \to \mathcal{P}_0(Y)$ be strong measurable at $x_0 \in X$ and $g : Y \to Z$ a homeomorphism. Then $F \times G$ and $g \circ F$ are strong measurable at x_0. Moreover, if $(F \cap G)(x) \neq \emptyset$ for every $x \in X$, then $F \cap G$ is also strong measurable at x_0.*

Open problems.

In this paper we obtained some properties of strong pointwise measurable multifunctions. According to the Kuratowski-Ryll Nardzewski selection theorem, if Y is a Polish space, then every measurable multifunction $F : X \to \mathcal{P}_f(Y)$ admits a measurable selection (where $\mathcal{P}_f(Y)$ is the family of all closed nonempty subsets of Y). It would be interesting to see if a pointwise strong measurable multifunction has a pointwise measurable selection.

Also we might see if the set of strong measurable points of a multifunction is a G_δ-set or if the pointwise continuity of a multifunction F implies the pointwise strong measurability of F.

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