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ON A CLASS OF NONHOMOGENOUS QUASILINEAR PROBLEM INVOLVING SOBOLEV SPACES WITH VARIABLE EXPONENT

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Abstract

We study the nonlinear boundary value problem

$$-\operatorname{div}\left(\left(|\nabla u(x)|^{p_1(x)-2}+|\nabla u(x)|^{p_2(x)-2}\right)\nabla u(x)\right)=$$

$$=\lambda|u|^{q(x)-2}u-\mu|u|^{\alpha(x)-2}u$$

in Ω, $u=0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, λ, μ are positive real numbers, p_1, p_2, q and α are a continuous functions on $\overline{\Omega}$ satisfying appropriate conditions. First result we show the existence of infinitely many weak solutions for any $\lambda,\mu>0$. Second we prove that for any $\mu>0$, there exists λ_* sufficiently small, and λ^* large enough such that for any $\lambda\in(0,\lambda_*)\cup(\lambda^*,\infty)$, the above nonhomogeneous quasilinear problem has a non-trivial weak solution. The proof relies on some variational arguments based on a $\mathbb{Z}_2$-symmetric version for even functionals of the mountain pass theorem, the Ekeland's variational principle and some adequate variational methods .

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1 Introduction and preliminary results

In this paper we are concerned with the problem

$$\begin{cases} -\operatorname{div}\left(\left(|\nabla u(x)|^{p_1(x)-2}+|\nabla u(x)|^{p_2(x)-2}\right)\nabla u(x)\right)=\lambda|u|^{q(x)-2}u-\mu|u|^{\alpha(x)-2}u, \\ u\not\equiv 0, \\ u=0, \end{cases} \tag{1}$$

for $x \in \Omega$ where $\Omega \subset \mathbb{R}^N$, $(N \geq 3)$ is a bounded domain with smooth boundary, λ, μ are positive real numbers, p_1, p_2, q and α are continuous functions on $\overline{\Omega}$.

The study of differential equations and variational problems involving operators with variable exponents growth conditions have received more and more interest in the last few years. In fact the interest in studying such problems was stimulated by their application in mathematical physics see [1, 6, 11], more precisely in elastic mechanics or electrorheological fluids.
Next, we recall that the problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u(x)|^{p(x)-2}\nabla u(x)\right) = f(x,u), & \text{for } x\in\Omega \\ u=0, & \text{for } x\in\partial\Omega. \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, has been largely considered in literature.

- Fan, Zhang and Zhao [10] established the existence of infinitely many eigenvalues for problem (2) for $f(x,u)=\lambda|u(x)|^{p(x)-2}u$ on $\overline{\Omega}$. They used an argument based on the Ljusternik-Schnirelmann critical point theory.

- Mihăilescu and Rădulescu [15] used the Ekeland's variational principle for $f(x,u)=\lambda|u(x)|^{q(x)-2}u$, $\min_{\overline{\Omega}} q(x) < \min_{\overline{\Omega}} p(x)$ and $q(x)$ has a subcritical growth to prove the existence of a continuous family of eigenvalues which lies in a neighborhood of the origin.

- Ben Ali and Kefi [4] studied the problem for $f(x,u)=\lambda|u(x)|^{q(x)-2}u - |u(x)|^{\alpha(x)-2}u$ where
$1 < inf_\Omega p(x) \leq sup_\Omega p < N$. In a first part they used the mountain pass theorem to prove that the problem has infinitely many weak solutions if $\max(sup_\Omega p, sup_\Omega \alpha) < inf_\Omega q$ and $q(x) < \frac{Np(x)}{N-p(x)}$. In a second part they used the Ekeland's variational principle to prove that the problem has a non trivial weak solution, if $1 < inf_\Omega q < \min(inf_\Omega p, inf_\Omega \alpha)$ and $max(\alpha(x), q(x)) < \frac{Np(x)}{N-p(x)}$ $\forall x \in \overline{\Omega}$.

- Mihăilescu [17] considered the problem

$$-\operatorname{div}\left(\left(|\nabla u(x)|^{p_1(x)-2}+|\nabla u(x)|^{p_2(x)-2}\right)\nabla u(x)\right)=f(x,u)$$

where $f(x,u) = \pm\big(-\lambda|u|^{m(x)-2}u + |u|^{q(x)-2}u\big)$ and

$$m(x) = max\{p_1(x), p_2(x)\}.$$

Under the assumption $m(x) < q(x) < \frac{Nm(x)}{N-m(x)}$, he established in a first case, using the Mountain pass theorem, the existence of infinitely many weak solutions. In a second case he used a simple variational argument for λ large enough to prove that the problem has a weak solution.

- Allegue and Bezzarga [2] studied the problem

$$-\operatorname{div}\big(a(x,\nabla u)\big) = \lambda u^{\gamma-1} - \mu u^{\beta-1}$$

where λ and μ are positive real numbers, $\operatorname{div}\big(a(x,\nabla u)\big)$ is a $p(x)$-Laplace type operator, with $1 < \beta < \gamma < \inf_{x\in\overline{\Omega}} p(x)$ and

$$p^+ < min\left\{N, Np^-/(N-p^-)\right\}.$$

They proved that if λ is large enough, there exists at least two non-negative non-trivial weak solutions using the Mountain Pass theorem of Ambrosetti and Rabinowitz and some adequate variational methods.

In the sequel, we start with some preliminary basic results on the theory of Lebesgue-Sobolev spaces with variable exponent. We refer to the book of Musielak [18], the papers of Kovacik and Rakosnik [13] and Fan et al. [7, 9]. Set

$$C_+(\overline{\Omega}) = \{h;\ h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u : \text{is a Borel real-valued function on } \Omega : \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}.$$

We define on $L^{p(x)}$, the so-called *Luxemburg norm*, by the formula

$$|u|_{p(x)} := \inf\Big\{\mu > 0;\ \int_\Omega \Big|\frac{u(x)}{\mu}\Big|^{p(x)}\,dx \le 1\Big\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many aspects: they are separable and Banach spaces [13, Theorem 2.5; Corollary

2.7] and the Hölder inequality holds [13, Theorem 2.1]. The inclusions between Lebesgue spaces are also naturally generalized [13, Theorem 2.8]: if $0 < |\Omega| < \infty$ and r_1, r_2 are variable exponents so that $r_1(x) \leq r_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder type inequality

$$|\int_\Omega uv\,dx| \leq (\frac{1}{p^-} + \frac{1}{p'^-})|u|_{p(x)}|v|_{p'(x)}, \tag{3}$$

is held.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

The space $W^{1,p(x)}(\Omega)$ is equiped by the following norm :

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

We recall that if (u_n), u, $\in W^{1,p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold

$$\min(|u|^{p^-}_{p(x)}, |u|^{p^+}_{p(x)}) \leq \rho_{p(x)}(u) \leq \max(|u|^{p^-}_{p(x)}, |u|^{p^+}_{p(x)}), \tag{4}$$

$$\min(|\nabla u|^{p^-}_{p(x)}, |\nabla u|^{p^+}_{p(x)}) \leq \rho_{p(x)}(|\nabla u|) \leq \max(|\nabla u|^{p^-}_{p(x)}, |\nabla u|^{p^+}_{p(x)}), \tag{5}$$

$$\begin{aligned} |u|_{p(x)} \to 0 \;&\Leftrightarrow\; \rho_{p(x)}(u) \to 0, \\ \lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \;&\Leftrightarrow\; \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0, \\ |u_n|_{p(x)} \to \infty \;&\Leftrightarrow\; \rho_{p(x)}(u_n) \to \infty. \end{aligned} \tag{6}$$

We define also $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}.$$

The space $(W_0^{1,p(x)}(\Omega), \|.\|)$ is a separable and reflexive Banach space.

Next, we remember some embedding results regarding variable exponent Lebesgue-Sobolev spaces. We note that if $s(x) \in C_+(\overline{\Omega})$ and $s(x) < p^*(x)$

for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ is compact and continuous, where $p^*(x) = Np(x)/(N - p(x))$ if $p(x) < N$ or $p^*(x) = +\infty$ if $p(x) \geq N$. We refer to [13] for more properties of Lebesgue and Sobolev spaces with variable exponent. We also refer to the recent papers [3, 7, 8, 9, 12, 14, 15, 16] for the treatment of nonlinear boundary value problems in Lebesgue-Sobolev spaces with variable exponent.

Remark 1. *Let $p_1(x), p_2(x) \in C_+(\overline{\Omega})$ and $m(x) = max\{p_1(x), p_2(x)\}$ for any $x \in \overline{\Omega}$, then*
$m(x) \in C_+(\overline{\Omega})$ and $p_1(x), p_2(x) \leq m(x)$ for any $x \in \overline{\Omega}$, consequently by Theorem 2.8, in [13] $W_0^{1,m(x)}(\Omega)$ is continuously embedded in $W_0^{1,p_i(x)}(\Omega)$ for $i \in \{1, 2\}$.

2 Main results

In the following, we consider problem (1). Let p_1, p_2, q and $\alpha \in C_+(\overline{\Omega}), m(x) = max\{p_1(x), p_2(x)\}$ for any $x \in \overline{\Omega}$ and $\lambda, \mu > 0$.

Definition 1. *We say that $u \in W_0^{1,m(x)}(\Omega)$ is a weak solution of (1) if*

$$\int_\Omega \left(\left(|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}\right) \nabla u \nabla v - \lambda |u|^{q(x)-2} uv + \mu |u|^{\alpha(x)-2} uv\right) dx = 0,$$

for any $v \in W_0^{1,m(x)}(\Omega)$.

We prove the following results:

Theorem 1. *For any $\lambda, \mu > 0$ problem (1) has infinitely many weak solutions provided that*
$2 \leq p_i^- \leq p_i^+ < N$ for $i \in \{1, 2\}$, $q^- > max(m^+, \alpha^+)$ and $q^+ < \frac{Nm^-}{N-m^-}$.

Theorem 2. (I) *For any $\mu > 0$ there exists $\lambda_* > 0$ under which problem (1) has a nontrivial weak solution, provided that $2 \leq p_i^- \leq p_i^+ < N$ for $i \in \{1, 2\}$, $q^- < min(p_1^-, p_2^-, \alpha^-)$ and*
$max(\alpha^+, q^+) < \frac{Nm^-}{N-m^-}$.
(II) *If $2 \leq p_i^- \leq p_i^+ < N$ for $i \in \{1, 2\}$, $q^+ < \alpha^-$ and $\alpha^+ < \frac{Nm^-}{N-m^-}$, then for any $\mu > 0$, there exists also a critical value $\lambda^* > 0$ such that for any $\lambda \geq \lambda^*$, problem (1) has a nontrivial weak solution.*

We mention that Theorem 1 and theorem 2 still remain valid for more general classes of differential operators. For example, we can replace the $p(x)$-Laplace type operator div $\left(\left(|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}\right) \nabla u\right)$ by the generalized mean curvature operator div $\left(\left(\left(1 + |\nabla u|^2\right)^{(p(x)-2)/2} \nabla u\right)\right)$.

3 Proof of Theorem 1

The proof of theorem 1 is based on a $\mathbb{Z}_2$-symmetric version for even functionals of the Mountain pass Theorem (see Theorem 9.12 in [19]).
Mountain Pass Theorem. *Let X be an infinite dimensional real Banach space and let $I \in C^1(X, \mathbb{R})$ be even satisfying the Palais-Smale condition and $I(0) = 0$. Suppose that*
(I1) There exist two contants $\rho, a > 0$ such that $I(x) \geq a$ if $\|x\| = \rho$.
(I2) For each finite dimensional subspace $X_1 \subset X$, the set $\{x \in X_1; I(x) \geq 0\}$ is bounded.
Then I has an unbounded sequence of critical values.

Let E denote the generalized Sobolev space $W_0^{1,m(x)}(\Omega)$ and $\|.\|$ denote the norm $\|.\|_{m(x)}$. Let λ and μ be arbitrary but fixed. The energy functional corresponding to the problem (1) is defined as $J_{\lambda,\mu} : E \to \mathbb{R}$,

$$J_{\lambda,\mu}(u) := \int_\Omega \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_\Omega \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx - \lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx +$$
$$+\mu \int_\Omega \frac{1}{\alpha(x)} |u|^{\alpha(x)} dx,$$

Proposition 1. *The functional $J_{\lambda,\mu}$ is well-defined on E and $J_{\lambda,\mu} \in C^1(E, \mathbb{R})$.*

Proof. A simple calculation based on Remark 1, relation (3) and the compact embedding of E into $L^{s(x)}(\Omega)$ for all $s \in C_+(\overline{\Omega})$ with $s(x) < m^*(x)$ on $\overline{\Omega}$ shows that $J_{\lambda,\mu}$ is well-defined on E and $J_{\lambda,\mu} \in C^1(E, \mathbb{R})$ with the derivate given by

$$\langle dJ_{\lambda,\mu}(u), v \rangle = \int_\Omega \big(|\nabla u|^{p_1(x)-2} \nabla u \nabla v + |\nabla u|^{p_2(x)-2} \nabla u \nabla v - \lambda |u|^{q(x)-2} uv +$$
$$+\mu |u|^{\alpha(x)-2} uv \big) dx, \ \forall \ v \in E.$$

□

In order to use the mountain pass theorem, we need the following lemmas.

Lemma 1. *For any $\lambda, \mu > 0$ there exists $r, a > 0$ such that $J_{\lambda,\mu}(u) \geq a > 0$ for any $u \in E$ with $\|u\| = r$.*

Proof. Since $m(x) = max\{p_1(x), p_2(x)\}$ for any $x \in \overline{\Omega}$ then

$$|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)} \geq |\nabla u|^{m(x)} \quad \forall x \in \overline{\Omega}, \tag{7}$$

On the other hand $q(x) < m^*(x)$ for all $x \in \overline{\Omega}$, then E is continuously embedded in $L^{q(x)}(\Omega)$. So there exists a positive constant C such that, for all $u \in E$,

$$|u|_{q(x)} \leq C\|u\|. \tag{8}$$

Suppose that $\|u\| < min(1, \frac{1}{C})$, then for all $u \in E$ with $\|u\| = \rho$ we have

$$|u|_{q(x)} < 1.$$

Furthermore, relations (4) yields for all $u \in E$ with $\|u\| = \rho$ we have

$$\int_\Omega |u|^{q(x)} dx \le |u|_{q(x)}^{q^-}.$$

The above inequality and relation (8) imply that for all $u \in E$ with $\|u\| = \rho$, we have

$$\int_\Omega |u|^{q(x)} dx \le C^{q^-} \|u\|^{q^-}. \tag{9}$$

On the other hand we have

$$\int_\Omega |\nabla u|^{m(x)} dx \ge \|u\|^{m^+}. \tag{10}$$

Then using relations (7), (9) and (10), we deduce that, for any $u \in E$ with $\|u\| = \rho$, the following inequalities hold true

$$\begin{aligned} J_{\lambda,\mu}(u) &\ge \frac{1}{m^+}\int_\Omega |\nabla u|^{m(x)} dx - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)} dx, \\ &\ge \frac{1}{m^+}\|u\|^{m^+} - \frac{\lambda}{q^-} C^{q^-} \|u\|^{q^-}. \end{aligned}$$

Let $h_\lambda(t) = \dfrac{1}{m^+} t^{m^+} - \dfrac{\lambda}{q^-} C^{q^-} t^{q^-}$, $t > 0$. It is easy to see that $h_\lambda(t) > 0$ for all $t \in (0, t_1)$, where $t_1 < \left(\dfrac{q^-}{\lambda m^+ C^{q^-}}\right)^{\frac{1}{q^- - m^+}}$.

So for any $\lambda, \mu > 0$ we can choose $r, a > 0$ such that $J_{\lambda,\mu}(u) \ge a > 0$ for all $u \in E$ with $\|u\| = r$. The proof of Lemma 1 is complete. □

Lemma 2. *If $E_1 \subset E$ is a finite dimensional subspace, the set $S = \{u \in E_1; J_{\lambda,\mu}(u) \ge 0\}$ is bounded in E.*

Proof. We have

$$\int_\Omega \frac{1}{p_i(x)} |\nabla u|^{p_i(x)} dx \le K_i(\|u\|^{p_i^-} + \|u\|^{p_i^+}) \quad \forall u \in E \quad i = \{1, 2\}, \tag{11}$$

where K_i $(i \in \{1,2\})$ are positive constants. Indeed, using relation (4) we have

$$\int_\Omega |\nabla u|^{p_i(x)}dx \leq |\nabla u|^{p_i^-}_{p_i(x)} + |\nabla u|^{p_i^+}_{p_i(x)} = \|u\|^{p_i^-}_{p_i(x)} + \|u\|^{p_i^+}_{p_i(x)} \ \forall u \in E \quad i = \{1,2\} \tag{12}$$

On the other hand, using Remark 1 there exists a positive constant C_i such that

$$\|u\|_{p_i(x)} \leq C_i \|u\| \quad \forall u \in E \quad i \in \{1,2\}. \tag{13}$$

The last two inequality yield

$$\int_\Omega |\nabla u|^{p_i(x)}dx \leq C_i^{p_i^-} \|u\|^{p_i^-} + C_i^{p_i^+} \|u\|^{p_i^+} \quad \forall u \in E \quad i = \{1,2\}, \tag{14}$$

and thus (11) holds true. Also we have

$$\int_\Omega |u|^{\alpha(x)}dx \leq |u|^{\alpha^-}_{\alpha(x)} + |u|^{\alpha^+}_{\alpha(x)} \quad \forall u \in E. \tag{15}$$

The fact that E is continuously embedded in $L^\alpha(\Omega)$ assures the existence of a positive constant C_3 such that

$$|u|_{\alpha(x)} \leq C_3 \|u\| \quad \forall u \in E. \tag{16}$$

The last two inequalities show that there exists a positive constant $K_3(\mu)$ such that

$$\mu \int_\Omega \frac{1}{\alpha(x)} |u|^{\alpha(x)}dx \leq \frac{\mu}{\alpha^-}\left(C_3^{\alpha^-} \|u\|^{\alpha^-} + C_3^{\alpha^+} \|u\|^{\alpha^+}\right) \leq$$

$$\leq K_3(\mu)\left(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}\right) \quad \forall u \in E. \tag{17}$$

By inequality (11) and (17) we get

$$J_{\lambda,\mu}(u) \leq K_1(\|u\|^{p_1^-} + \|u\|^{p_1^+}) + K_2(\|u\|^{p_2^-} + \|u\|^{p_2^+}) +$$

$$+ K_3(\mu)\left(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}\right) - \frac{\lambda}{q^+}\int_\Omega |u|^{q(x)}dx, \tag{18}$$

for all $u \in E$.
Let $u \in E$ be arbitrary but fixed. We define

$$\Omega_< = \{x \in \Omega; |u(x)| < 1\}. \quad \Omega_\geq = \Omega \backslash \Omega_<.$$

Then we have

$$\begin{aligned}
J_{\lambda,\mu}(u) &\leq K_1(\|u\|^{p_1^-} + \|u\|^{p_1^+}) + K_2(\|u\|^{p_2^-} + \|u\|^{p_2^+}) + \\
&+ K_3(\mu)\left(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}\right) - \frac{\lambda}{q^+}\int_\Omega |u|^{q(x)}dx \leq \\
&\leq K_1(\|u\|^{p_1^-} + \|u\|^{p_1^+}) + K_2(\|u\|^{p_2^-} + \|u\|^{p_2^+}) + \\
&+ K_3(\mu)\left(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}\right) - \frac{\lambda}{q^+}\int_{\Omega_\geq} |u|^{q(x)}dx \leq \\
&\leq K_1(\|u\|^{p_1^-} + \|u\|^{p_1^+}) + K_2(\|u\|^{p_2^-} + \|u\|^{p_2^+}) + \\
&+ K_3(\mu)\left(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}\right) - \frac{\lambda}{q^+}\int_{\Omega_\geq} |u|^{q^-}dx \leq \\
&\leq K_1(\|u\|^{p_1^-} + \|u\|^{p_1^+}) + K_2(\|u\|^{p_2^-} + \|u\|^{p_2^+}) + \\
&+ K_3(\mu)\left(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}\right) - \frac{\lambda}{q^+}\int_{\Omega} |u|^{q^-}dx + \frac{\lambda}{q^+}\int_{\Omega_<} |u|^{q^-}dx.
\end{aligned}$$

But for each $\lambda > 0$ there exists positive constant $K_4(\lambda)$ such that

$$\frac{\lambda}{q^+}\int_{\Omega_<} |u|^{q^-}dx \leq K_4(\lambda) \quad \forall u \in E.$$

The functional $|.|_{q^-} : E \to \mathbb{R}$ defined by

$$|u|_{q^-} = \left(\int_\Omega |u|^{q^-}dx\right)^{1/q^-},$$

is a norm in E. In the finite dimensional subspace E_1 the norm $|u|_{q^-}$ and $\|u\|$ are equivalent, so there exists a positive constant $K = K(E_1)$ such that

$$\|u\| \leq K|u|_{q^-} \quad \forall u \in E_1.$$

So that there exists a prositive constant $K_5(\lambda)$ such that

$$J_{\lambda,\mu}(u) \leq K_1(\|u\|^{p_1^-} + \|u\|^{p_1^+}) + K_2(\|u\|^{p_2^-} + \|u\|^{p_2^+}) + K_3(\mu)\left(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}\right) +$$

$$+K_4(\lambda) - K_5(\lambda)\,\|u\|^{q^-}$$

$\forall u \in E_1$.
Hence

$$K_1(\|u\|^{p_1^-} + \|u\|^{p_1^+}) + K_2(\|u\|^{p_2^-} + \|u\|^{p_2^+}) + K_3(\mu)\left(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}\right) +$$

$$+K_4(\lambda) - K_5(\lambda)\,\|u\|^{q^-} \geq 0$$

$\forall u \in S$
. And since $q^- > max(m^+, \alpha^+)$, we conclude that S is bounded in E. □

Lemma 3. *If $\{u_n\} \subset E$ is a sequence which satisfies the properties*

$$|J_{\lambda,\mu}(u_n)| < C_4, \tag{19}$$

$$dJ_{\lambda,\mu}(u_n) \to 0 \quad as \quad n \to \infty, \tag{20}$$

where C_4 is a positive constant, then $\{u_n\}$ possesses a convergent subsequence.

Proof. First we show that $\{u_n\}$ is bounded in E. If not,we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. Thus we may consider that $\|u_n\| > 1$ for any integer n. Using (20) it follows that there exists $N_1 > 0$ such that for any $n > N_1$ we have

$$\|dJ_{\lambda,\mu}(u_n)\| \leq 1.$$

On the other hand, for all $n > N_1$ fixed, the application $E \ni v \to \langle dJ_{\lambda,\mu}(u_n), v\rangle$ is linear and continuous. The above information yield that

$$|\langle dJ_{\lambda,\mu}(u_n), v\rangle| \leq \|dJ_{\lambda,\mu}(u_n)\|\,\|v\| \leq \|v\| \quad v \in E, \quad n > N_1.$$

Setting $v = u_n$ we have

$$-\|u_n\| \leq \int_\Omega |\nabla u_n|^{p_1(x)}dx + \int_\Omega |\nabla u_n|^{p_2(x)}dx - \lambda\int_\Omega |u_n|^{q(x)}dx + \mu\int_\Omega |u_n|^{\alpha(x)}dx \leq \|u_n\|,$$

for all $n > N_1$.
We obtain

$$-\|u_n\| - \int_\Omega |\nabla u_n|^{p_1(x)}dx - \int_\Omega |\nabla u_n|^{p_2(x)}dx - \mu\int_\Omega |u_n|^{\alpha(x)}dx \leq -\lambda\int_\Omega |u_n|^{q(x)}dx, \tag{21}$$

for all $n > N_1$. Provided that $\|u_n\| > 1$ relation (7), (19) and (21) imply

$$\begin{aligned}
C_4 > J_{\lambda,\mu}(u_n) \quad &\geq \quad (\frac{1}{m^+} - \frac{1}{q^-})\left[\int_\Omega |\nabla u_n|^{p_1(x)}dx + \int_\Omega |\nabla u_n|^{p_2(x)}dx\right] + \\
&+ \quad \mu(\frac{1}{\alpha^+} - \frac{1}{q^-})\int_\Omega |u_n|^{\alpha(x)}dx - \frac{1}{q^-}\|u_n\| \geq \\
&\geq \quad (\frac{1}{m^+} - \frac{1}{q^-})\int_\Omega |\nabla u_n|^{m(x)}dx - \frac{1}{q^-}\|u_n\| \geq \\
&\geq \quad (\frac{1}{m^+} - \frac{1}{q^-})\|u_n\|^{m^-} - \frac{1}{q^-}\|u_n\|.
\end{aligned}$$

Letting $n \to \infty$ we obtain a contradiction. It follows that $\{u_n\}$ is bounded in E. And we deduce that there exists a subsequence, again denoted by $\{u_n\}$, and $u \in E$ such that $\{u_n\}$ converges weakly to u in E. Since E is compactly embedded in $L^{q(x)}(\Omega)$ and $L^{\alpha(x)}(\Omega)$, then $\{u_n\}$ converges strongly to u in $L^{q(x)}(\Omega)$ and $L^{\alpha(x)}(\Omega)$ respectively. The above information and relation (20) imply

$$|\langle dJ_{\lambda,\mu}(u_n) - dJ_{\lambda,\mu}(u), u_n - u\rangle| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

On the other hand we have

$$\int_\Omega (|\nabla u_n|^{p_1(x)-2}\nabla u_n + |\nabla u_n|^{p_2(x)-2}\nabla u_n - |\nabla u|^{p_1(x)-2}\nabla u-$$

$$-|\nabla u|^{p_2(x)-2}\nabla u)(\nabla u_n - \nabla u)dx = \langle dJ_{\lambda,\mu}(u_n)-$$

$$-dJ_{\lambda,\mu}(u), u_n - u\rangle + \lambda \int_\Omega \left(|u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u\right)(u_n - u)\, dx-$$

$$-\mu \int_\Omega \left(|u_n|^{\alpha(x)-2}u_n - |u|^{\alpha(x)-2}u\right)(u_n - u)\, dx. \tag{22}$$

Now we need the following proposition:

Proposition 2. *Let $r \in C^+(\overline{\Omega})$ such that $r(x) < m^*(x)$ $\forall x \in \overline{\Omega}$ then*

$$\lim_{n\to\infty} \int_\Omega |u_n|^{r(x)-2}u_n(u_n - u)dx = 0.$$

Proof. Using (3) we have $\int_\Omega |u_n|^{r(x)-2}u_n(u_n-u)dx \le ||u_n|^{r(x)-2}u_n|_{\frac{r(x)}{r(x)-1}}|u_n-u|_{r(x)}$. Then if $||u_n|^{r(x)-2}u_n|_{\frac{r(x)}{r(x)-1}} > 1$, by (4), there exists $C > 0$ such that $||u_n|^{r(x)-2}u_n|_{\frac{r(x)}{r(x)-1}} \le |u_n|^C_{r(x)}$ and this ends the proof. □

Combining proposition 2, and the relation (22) we deduce that

$$\lim_{n\to\infty} \int_\Omega (|\nabla u_n|^{p_1(x)-2}\nabla u_n + |\nabla u_n|^{p_2(x)-2}\nabla u_n - |\nabla u|^{p_1(x)-2}\nabla u - |\nabla u|^{p_2(x)-2}\nabla u)(\nabla u_n - \nabla u)\, dx = 0. \tag{23}$$

It is known that

$$\left(|\xi|^{r-2}\xi - |\psi|^{r-2}\psi\right)(\xi - \psi) \ge \left(\frac{1}{2}\right)^r |\xi - \psi|^r, \forall r \ge 2, \xi, \psi \in \mathbb{R}^N. \tag{24}$$

From (23) and (24) it follows that

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n - \nabla u|^{p_1(x)}dx + \int_\Omega |\nabla u_n - \nabla u|^{p_2(x)}dx = 0.$$

Using relation (7) we get

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n - \nabla u|^{m(x)}dx = 0.$$

That fact and the relation (6) imply $\|u_n - u\| \to 0$ as $n \to \infty$. The proof of Lemma 3 is complete. □

PROOF OF THEOREM 1. It is clear that the functional $J_{\lambda,\mu}$ is even and verifies $J_{\lambda,\mu}(0) = 0$. Lemma 1 and lemma 2 show that conditions *(I1)* and *(I2)* are satisfied. Lemma 3 implies that $J_{\lambda,\mu}$ satisfies the Palais- Smale condition. Thus the Mountain Pass Theorem can be applied to the functional $J_{\lambda,\mu}$. We conclude that problem (1) has infinitely many weak solutions in E. The proof of theorem 1 is complete.

4 Proof of Theorem 2

First, we prove the assertion (I) in Theorem 2. We show that for any $\mu > 0$ there exists $\lambda_* > 0$ such that for every $\lambda \in (0, \lambda_*)$ the problem (1) has a nontrivial weak solution. The key argument in the proof is related to Ekeland's variational principle.
In order to apply it we need the following lemmas:

Lemma 4. *For all $\mu > 0$ and all $\rho \in (0, 1)$ there exist $\lambda_* > 0$ and $b > 0$ such that, for all $u \in E$ with $\|u\| = \rho$, $J_{\lambda,\mu}(u) \geq b > 0$ for any $\lambda \in (0, \lambda_*)$.*

Proof. Since $q^+ < \frac{Nm^-}{N-m^-}$ for all $x \in \overline{\Omega}$, we have the continuous embedding $E \hookrightarrow L^{q(x)}(\Omega)$. This implies that there exists a positive constant M such that

$$|u|_{q(x)} \leq M\|u\| \quad \forall u \in E. \tag{25}$$

We fix $\rho \in (0, 1)$ such that $\rho < \min(1, 1/M)$. Then for all $u \in E$ with $\|u\| = \rho$ we deduce that

$$|u|_{q(x)} < 1.$$

Furthermore, relations (4) yield for all $u \in E$ with $\|u\| = \rho$, we have

$$\int_\Omega |u|^{q(x)}dx \leq |u|_{q(x)}^{q^-}.$$

The above inequalitiy and relations (25) imply, for all $u \in E$ with $\|u\| = \rho$, that

$$\int_\Omega |u|^{q(x)}dx \leq M^{q^-}\|u\|^{q^-}. \tag{26}$$

Using relations (7) and (26) we deduce that, for any $u \in E$ with $\|u\| = \rho$, the following inequalities hold true.

$$\begin{aligned}
J_{\lambda,\mu}(u) &\geq \frac{1}{p_1^+}\int_\Omega |\nabla u|^{p_1(x)}dx + \frac{1}{p_2^+}\int_\Omega |\nabla u|^{p_2(x)}dx - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}dx + \\
&+ \frac{\mu}{\alpha^+}\int_\Omega |u|^{\alpha(x)}dx \geq \frac{1}{max(p_1^+,p_2^+)}\left[\int_\Omega |\nabla u|^{p_1(x)}dx + \int_\Omega |\nabla u|^{p_2(x)}dx\right] - \\
&- \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}dx \geq \frac{1}{m^+}\int_\Omega |\nabla u|^{m(x)}dx - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}dx, \\
&\geq \frac{1}{m^+}\|u\|^{m^+} - \frac{\lambda}{q^-}M^{q^-}\|u\|^{q^-}, \\
&\geq \frac{1}{m^+}\rho^{m^+} - \frac{\lambda}{q^-}M^{q^-}\rho^{q^-} = \rho^{q^-}\left(\frac{1}{m^+}\rho^{m^+-q^-} - \frac{\lambda}{q^-}M^{q^-}\right).
\end{aligned}$$

By the above inequality we remark that for

$$\lambda_* = \frac{q^-}{2m^+M^{q^-}}\rho^{m^+-q^-}, \tag{27}$$

and for any $\lambda \in (0, \lambda_*)$, there exists $b = \dfrac{\rho^{m^+}}{2m^+} > 0$ such that

$$J_{\lambda,\mu}(u) \geq b > 0, \forall \mu > 0; \quad \forall u \in E \quad \text{with} \quad \|u\| = \rho.$$

The proof of Lemma 4 is complete. □

Lemma 5. *There exists $\phi \in E$ such that $\phi \geq 0$, $\phi \neq 0$ and $J_{\lambda,\mu}(t\phi) < 0$, for $t > 0$ small enough.*

Proof. Let $l = min\{p_1^-, p_2^-, \alpha^-\}$. Since $q^- < l$, then let $\epsilon_0 > 0$ be such that $q^- + \epsilon_0 < l$. On the other hand, since $q \in C(\overline{\Omega})$ it follows that there exists an open set $\Omega_0 \subset\subset \Omega$ such that $|q(x) - q^-| < \epsilon_0$ for all $x \in \Omega_0$. Thus, we conclude that $q(x) \leq q^- + \epsilon_0 < l$ for all $x \in \overline{\Omega}_0$.

Let $\phi \in C_0^\infty(\Omega)$ be such that $\text{supp}(\phi) \supset \overline{\Omega}_0$, $\phi(x) = 1$ for all $x \in \overline{\Omega}_0$ and $0 \le \phi \le 1$ in Ω. Then using the above information for any $t \in (0,1)$ we have

$$\begin{aligned}
J_{\lambda,\mu}(t\phi) &= \int_\Omega \frac{t^{p_1(x)}}{p_1(x)}|\nabla\phi|^{p_1(x)}dx + \int_\Omega \frac{t^{p_2(x)}}{p_2(x)}|\nabla\phi|^{p_2(x)}dx - \\
&- \lambda\int_\Omega \frac{t^{q(x)}}{q(x)}|\phi|^{q(x)}dx + \mu\int_\Omega \frac{t^{\alpha(x)}}{\alpha(x)}|\phi|^{\alpha(x)}dx \le \\
&\le \frac{t^{p_1^-}}{p_1^-}\int_\Omega |\nabla\phi|^{p_1(x)}dx + \frac{t^{p_2^-}}{p_2^-}\int_\Omega |\nabla\phi|^{p_2(x)}dx - \\
&- \frac{\lambda}{q^+}\int_\Omega t^{q(x)}|\phi|^{q(x)} + \mu\frac{t^{\alpha^-}}{\alpha^-}\int_\Omega |\phi|^{\alpha(x)}dx \le \\
&\le \frac{t^l}{l}\left[\int_\Omega \left(|\nabla\phi|^{p_1(x)} + |\nabla\phi|^{p_2(x)}\right)dx + \mu\int_\Omega |\phi|^{\alpha(x)}dx\right] - \\
&- \frac{\lambda t^{q^-+\epsilon_0}}{q^+}\int_{\Omega_0} |\phi|^{q(x)}dx = \\
&= \frac{t^l}{l}\left[\int_\Omega \left(|\nabla\phi|^{p_1(x)} + |\nabla\phi|^{p_2(x)}\right)dx + \mu\int_\Omega |\phi|^{\alpha(x)}dx\right] - \\
&- \frac{\lambda t^{q^-+\epsilon_0}}{q^+}|\Omega_0|.
\end{aligned}$$

Therefore

$$J_{\lambda,\mu}(t\phi) < 0,$$

for $t < \delta^{1/(l-q^- -\epsilon_0)}$ with

$$0 < \delta < \min\left\{1, \frac{l\mu|\Omega_0|}{q^+\left[\int_\Omega \left(|\nabla\phi|^{p_1(x)} + |\nabla\phi|^{p_2(x)}\right)dx + \mu\int_\Omega |\phi|^{\alpha(x)}dx\right]}\right\}.$$

Finally, we point out that $\int_\Omega \left(|\nabla\phi|^{p_1(x)} + |\nabla\phi|^{p_2(x)}\right)dx + \mu\int_\Omega |\phi|^{\alpha(x)}dx > 0$. In fact if $\int_\Omega \left(|\nabla\phi|^{p_1(x)} + |\nabla\phi|^{p_2(x)}\right)dx + \mu\int_\Omega |\phi|^{\alpha(x)}dx = 0$, then $\int_\Omega |\phi|^{\alpha(x)}dx = 0$. Using relation (4), we deduce that $|\phi|_{\alpha(x)} = 0$ and consequently $\phi = 0$ in Ω which is a contradiction. The proof of lemma is complete. $\square$

PROOF OF (I)

Let $\mu > 0$, λ_* be defined as in (27) and $\lambda \in (0, \lambda_*)$. By Lemma 4 it follows that on the boundary of the ball centered at the origin and of radius ρ in E, denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)} J_{\lambda,\mu} > 0. \tag{28}$$

On the other hand, by Lemma 5, there exists $\phi \in E$ such that $J_{\lambda,\mu}(t\phi) < 0$, for all $t > 0$ small enough. Moreover, relations (4), (7) and (25) imply, that for any $u \in B_\rho(0)$, we have

$$J_{\lambda,\mu}(u) \geq \frac{1}{m^+}\|u\|^{m^+} - \frac{\lambda}{q^-}M^{q^-}\|u\|^{q^-}.$$

It follows that

$$-\infty < \underline{c} := \inf_{B_\rho(0)} J_{\lambda,\mu} < 0.$$

We let now $0 < \epsilon < \inf_{\partial B_\rho(0)} J_{\lambda,\mu} - \inf_{B_\rho(0)} J_{\lambda,\mu}$. Using the above information, the functional $J_{\lambda,\mu} : \overline{B_\rho(0)} \longrightarrow \mathbb{R}$, is lower bounded on $\overline{B_\rho(0)}$ and $J_{\lambda,\mu} \in C^1(\overline{B_\rho(0)}, \mathbb{R})$. Then by Ekeland's variational principle there exists $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$\begin{cases} \underline{c} \leq J_{\lambda,\mu}(u_\epsilon) \leq \underline{c} + \epsilon, \\ 0 < J_{\lambda,\mu}(u) - J_{\lambda,\mu}(u_\epsilon) + \epsilon \cdot \parallel u - u_\epsilon \parallel, \quad u \neq u_\epsilon. \end{cases}$$

Since

$$J_{\lambda,\mu}(u_\epsilon) \leq \inf_{\overline{B_\rho(0)}} J_{\lambda,\mu} + \epsilon \leq \inf_{B_\rho(0)} J_{\lambda,\mu} + \epsilon < \inf_{\partial B_\rho(0)} J_{\lambda,\mu},$$

we deduce that $u_\epsilon \in B_\rho(0)$.
Now, we define $I_{\lambda,\mu} : \overline{B_\rho(0)} \longrightarrow \mathbb{R}$ by $I_{\lambda,\mu}(u) = J_{\lambda,\mu}(u) + \epsilon \cdot \parallel u - u_\epsilon \parallel$. It is clair that u_ϵ is a minimum point of $I_{\lambda,\mu}$ and thus

$$\frac{I_{\lambda,\mu}(u_\epsilon + t \cdot v) - I_{\lambda,\mu}(u_\epsilon)}{t} \geq 0,$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{J_{\lambda,\mu}(u_\epsilon + t \cdot v) - J_{\lambda,\mu}(u_\epsilon)}{t} + \epsilon \cdot \parallel v \parallel \geq 0.$$

Letting $t \to 0$ it follows that $< dJ_{\lambda,\mu}(u_\epsilon), v > +\epsilon \cdot \parallel v \parallel \geq 0$ and we infer that $\parallel dJ_{\lambda,\mu}(u_\epsilon) \parallel \leq \epsilon$.
We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$J_{\lambda,\mu}(w_n) \longrightarrow \underline{c} \quad \text{and} \quad dJ_{\lambda,\mu}(w_n) \longrightarrow 0_{E^*}. \tag{29}$$

It is clair that $\{w_n\}$ is bounded in E. Thus, there exists a subsequence again denoted by $\{w_n\}$, and w in E such that, $\{w_n\}$ converges weakly to w in E. Since E is compactly embedded in $L^{q(x)}(\Omega)$ and in $L^{\alpha(x)}(\Omega)$, then $\{w_n\}$ converges strongly in $L^{q(x)}(\Omega)$ and $L^{\alpha(x)}(\Omega)$. Using similar arguments than those used in proof of lemma 3 we deduce that $\{w_n\}$ converges strongly to w in E. Since $J_{\lambda,\mu} \in C^1(E,\mathbb{R})$, we conclude

$$dJ_{\lambda,\mu}(w_n) \to dJ_{\lambda,\mu}(w), \quad \text{as} \quad n \to \infty. \tag{30}$$

Relations (28) and (29) show that $dJ_{\lambda,\mu}(w) = 0$ and thus w is a weak solution for problem (1). Moreover, by relation (29) it follows that $J_{\lambda,\mu}(w) < 0$ and thus, w is a nontrivial weak solution for (1).
The proof of (I) in theorem 2 is complete.
Now we need to prove (II) in theorem 2. For this purpose, we will show that $J_{\lambda,\mu}$ possesses a nontrivial global minimum point in E. With that end of view we start by proving two auxiliary results.

Lemma 6. *The functional $J_{\lambda,\mu}$ is coercive on E.*

Proof. For any a, $b > 0$ and $0 < k < l$ the following inequality holds (see lemma 4 in [17])

$$a.t^k - b.t^l \le a.\left(\frac{a}{b}\right)^{k/l-k}, \quad \forall t \ge 0.$$

Using the above inequality we deduce that for any $x \in \Omega$ and $u \in E$ we have

$$\begin{aligned} \frac{\lambda}{q^-}|u|^{q(x)} - \frac{\mu}{\alpha^+}|u|^{\alpha(x)} &\le \frac{\lambda}{q^-}\left(\frac{\lambda\alpha^+}{\mu q^-}\right)^{q(x)/\alpha(x)-q(x)} \\ &\le \frac{\lambda}{q^-}\left[\left(\frac{\lambda\alpha^+}{\mu q^-}\right)^{q^+/\alpha^- -q^+} + \left(\frac{\lambda\alpha^+}{\mu q^-}\right)^{q^-/\alpha^+ -q^-}\right] = C, \end{aligned}$$

where C is a positive constant independent of u and x. Integrating the above inequality over Ω we obtain

$$\frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}dx - \frac{\mu}{\alpha^+}\int_\Omega |u|^{\alpha(x)}dx \le D. \tag{31}$$

Where D is a positive constant independent of u.
Using inequalities (5), (7) and (31) we obtain that, for any $u \in E$ with $\|u\| > 1$, we have

$$\begin{aligned} J_{\lambda,\mu}(u) &\ge \frac{1}{m^+}\int_\Omega |\nabla u|^{m(x)}dx - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}dx + \frac{\mu}{\alpha^+}\int_\Omega |u|^{\alpha(x)}dx, \\ &\ge \frac{1}{m^+}\|u\|^{m^-} - D. \end{aligned}$$

Then $J_{\lambda,\mu}$ is coercive and the proof of lemma is complete. $\square$

Lemma 7. *The functional $J_{\lambda,\mu}$ is weakly lower semicontinuous.*

Proof. Since the functionals $\Lambda_i : E \to \mathbb{R}$,

$$\Lambda_i = \int_\Omega \frac{1}{p_i(x)} |\nabla u|^{p_i(x)} dx, \quad \forall i \in \{1, 2\}$$

is convex (see lemma 5 in [17]), it follows that $\Lambda_1 + \Lambda_2$ is convex. Thus to show that the functional $\Lambda_1 + \Lambda_2$ is weakly lower semicontinuous on E, it is enough to show that $\Lambda_1 + \Lambda_2$ is strongly lower semicontinuous on E (see corollary III.8 in [5]).
We fix $u \in E$ and $\epsilon > 0$ and let $v \in E$ be arbitrary.
Since $\Lambda_1 + \Lambda_2$ is convex and inequality (3) holds true, we have

$$\begin{aligned}
\Lambda_1(v) + \Lambda_2(v), \quad &\geq \Lambda_1(u) + \Lambda_2(u) + \left\langle \Lambda_1'(u) + \Lambda_2'(u), v - u \right\rangle, \\
&\geq \Lambda_1(u) + \Lambda_2(u) - \int_\Omega |\nabla u|^{p_1(x)-1} |\nabla(v-u)| \, dx - \\
&- \int_\Omega |\nabla u|^{p_2(x)-1} |\nabla(v-u)| \, dx \geq \\
&\geq \Lambda_1(u) + \Lambda_2(u) - D_1 \left| |\nabla u|^{p_1(x)-1} \right|_{\frac{p_1(x)}{p_1(x)-1}} |\nabla(v-u)|_{p_1(x)} - \\
&- D_2 \left| |\nabla u|^{p_2(x)-1} \right|_{\frac{p_2(x)}{p_2(x)-1}} |\nabla(v-u)|_{p_2(x)} \geq \\
&\geq \Lambda_1(u) + \Lambda_2(u) - D_3 \|u - v\|_{m(x)} \geq \\
&\geq \Lambda_1(u) + \Lambda_2(u) - \epsilon,
\end{aligned}$$

for all $v \in E$ with $\|u - v\| < \epsilon / \left[\left| |\nabla u|^{p_1(x)-1} \right|_{\frac{p_1(x)}{p_1(x)-1}} + \left| |\nabla u|^{p_2(x)-1} \right|_{\frac{p_2(x)}{p_2(x)-1}} \right]$.
We denote by D_1, D_2 and D_3 three positive constants. It follows that $\Lambda_1 + \Lambda_2$ is strongly lower semicontinuous and since it is convex we obtain that $\Lambda_1 + \Lambda_2$ is weakly lower semicontinuous.
Finally, if $\{w_n\} \subset E$ is a sequence which converges weakly to w in E then $\{w_n\}$ converges strongly to w in $L^{q(x)}(\Omega)$ and $L^{\alpha(x)}(\Omega)$ thus, $J_{\lambda,\mu}$ is weakly lower semicontinuous. The proof of lemma is complete. □

PROOF OF (II)

Proof. By lemmas 6 and 7 we deduce that $J_{\lambda,\mu}$ is coercive and weakly lower semicontinuous on E. Then Theorem 1.2 in [20] implies that there exists $u_{\lambda,\mu} \in E$ a global minimizer of $J_{\lambda,\mu}$ and thus a weak solution of problem.
We show that $u_{\lambda,\mu}$ is not trivial for λ large enough. Indeed, letting $t_0 > 1$

be a fixed real and Ω_1 be an open subset of Ω with $|\Omega_1| > 0$ we deduce that there exists $u_0 \in C_0^\infty(\Omega) \subset E$ such that $u_0(x) = t_0$ for any $x \in \overline{\Omega}_1$ and $0 \leq u_0(x) \leq t_0$ in $\Omega \backslash \Omega_1$. We have

$$\begin{aligned} J_{\lambda,\mu}(u_0) &= \int_\Omega \frac{1}{p_1(x)} |\nabla u_0|^{p_1(x)} dx + \int_\Omega \frac{1}{p_2(x)} |\nabla u_0|^{p_2(x)} dx - \\ &- \lambda \int_\Omega \frac{1}{q(x)} |u_0|^{q(x)} dx + \mu \int_\Omega \frac{1}{\alpha(x)} |u_0|^{\alpha(x)} dx \leq \\ &\leq L(\mu) - \frac{\lambda}{q^+} t_0^{q^-} |\Omega_1| . \end{aligned}$$

where $L(\mu)$ is a positive constant.
Thus there exists $\lambda^* > 0$ such that $J_{\lambda,\mu}(u_0) < 0$ for any $\lambda \in [\lambda^*, \infty)$. It follows that $J_{\lambda,\mu}(u_0) < 0$ for any $\lambda \geq \lambda^*$ and thus $u_{\lambda,\mu}$ is a nontrivial weak solution of problem (1) for λ large enough. The proof of the assertion (II) is complete. □

References

[1] E. Acerbi and G. Mingione, Regulary results for a class of functionals with nonstandard growth, *Arch. Rtional Mech. Anal.* **156** (2001), 121-140.Zbl 0984.49020

[2] O. Allegue and M.Bezzarga, A multiplicity of solutions for a nonlinear degenerate problem involving a p(x)-Laplace type operator, *Complex Variables and Elliptic Equations*, to appear.

[3] K.Ben Ali and Bezzarga , On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, Potential Theory and Stochastics in Albac Aurel Cornea Memorial Volume, Conference Proceedings Albac, September 4-8, 2007, Theta 2008, pp. 21-34.

[4] K.Ben Ali and K. Kefi , Mountain pass and Ekeland's principle for eigenvalue problem with variable exponent , *Complex variables and elliptic equations* ,**54**, (2009) NO. 8, 795-809.

[5] H. Brezis, *Analyse fonctionnelle theorie et applications*, Masson, Paris, 1992.

[6] L. Diening, *Theorical and numerical results for electrorheological fluids,* Ph.D. thesis, University of frieburg, Germany, 2002. Zbl 1022.76001

[7] X. Fan, J. Shen and D. Zhao, Sobolev Embedding Theorems for Spaces $W^{k,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **262** (2001), 749-760.

[8] X. L. Fan and Q. H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, *Nonlinear Anal.* **52** (2003), 1843-1852.

[9] X. L. Fan and D. Zhao, On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263** (2001), 424-446.

[10] X. Fan, Q.Zhang and D. Zhao, Eigenvalues of p(x)-Laplacian Dirichlet problem, *J. Math. Anal. Appl.*, **302** (2005), 306-317.

[11] T. C. Halsey, Electrorheological fluids, *Science* **258** (1992), 761-766.

[12] K. Kefi, Nonhomogeneous boundary value problems in Sobolev spaces with variable exponent, *International Journal of Applied Mathematical Sciences.* ISSN 0973-0176 Vol.3 No.2 (2006), pp. 103-115.

[13] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.* **41** (1991), 592-618.

[14] M. Mihăilescu and V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proc. Roy. Soc. London Ser. A* **462** (2006), 2625-2641.

[15] M. Mihăilescu and V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proceedings of the American Mathematical Society*, **135** (2007).

[16] M. Mihăilescu and V. Rădulescu, Existence and multiplicity of solutions for quasilinear nonhomogeneous problems: an Orlicz-Sobolev space setting, *J. Math. Anal. Appl.*, **330** (2007), 416-432.

[17] M. Mihăilescu , On a class of nonlinear problem involving a p(x)-Laplace type operator, *Czechoslovak Mathematical Journal* **133** vol.58 (2008), 155-172.

[18] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, Vol. 1034, Springer, Berlin, 1983.

[19] P. Rabinowitz, Minimaux methods in critical point theory with applications to differential equation, Expository Lectures from the CBMS Regional Conference held at the University of Miami, *American Mathematical Society*, Providence, RI, (1984).

[20] M. Struwe, Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer, Heidelberg, 1996. Zbl 0864.49001

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