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A RADON-NIKODYM TYPE THEOREM

Christiane Godet-Thobie and Anca Croitoru

Abstract

In [6] and [7], the authors introduced and studied an integral for multifunctions with respect to a multimeasure which contains different multivalued integrals as particular cases. If $\mathcal{P}_k(X)$ is the family of nonempty compact subsets of a locally convex algebra X, both the multifunction and the multimeasure take values in a subset $\widetilde{X}$ of $\mathcal{P}_k(X)$ which satisfies certain conditions. In this paper, we continue this work and establish a Radon-Nikodym theorem, using a method of Maynard [13] which bases on the notion of exhaustion.

Introduction

The study of multifunctions was intensified and diversified in the last period thanks to their multiple applications in mathematical economics, theory of games, optimization and optimal control.

In [6] and [7], we constructed an integration theory for multifunctions with respect to multimeasures. If $\mathcal{P}_k(X)$ is the family of nonempty compact subsets of a locally convex algebra X, then the multifunctions and the multimeasures take values in a subset $\widetilde{X}$ of $\mathcal{P}_k(X)$ which satisfies certain conditions. For different choices of the space X, of the multifunctions and of the multimeasures, this set-valued integral contains, like particular cases, the classical integrals of Dunford [10], Brooks [3] and the integrals introduced in Sambucini [14], Croitoru [4].

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One of the most interesting problems in the theory of integration is the existence of a Radon-Nikodym derivative. In this paper, we obtain a Radon-Nikodym type theorem in the context of the integration theory constructed in [6]. According to this result, we can express a multimeasure Γ like a set-valued integral of a multifunction with respect to a multimeasure φ, under a condition of absolute continuity: $\Gamma \ll \varphi$. In this case, the construction of the Radon-Nikodym derivative follows the method of Maynard [13], using the notion of exhaustion.

1 Terminology and notations

The terminology and different notations are those of [6] and [7]. Let S be a nonempty set, $\mathcal{A}$ an algebra of subsets of S. Let X be a Hausdorff locally convex commutative algebra and let Q be a filtering family of seminorms which defines the topology of X and satisfies the following property for every $x, y \in X$ and every $p \in Q$:

(*) $\quad p(xy) \leq p(x)p(y)$.

1.1. Examples

(a) $X = \{f \mid f : T \to \mathbb{R}\}$ where T is a nonempty set.
Let $Q = \{p_t | t \in T\}$ where $p_t(f) = |f(t)|$, for every $f \in X$

(b) $X = \{f \mid f : T \to \mathbb{R}$ is bounded$\}$ where T is a topological space.
Let $\mathcal{K} = \{K \subset T | K$ is compact$\}$ and $Q = \{p_K | K \in \mathcal{K}\}$ where, for every $f \in X$, $p_K(f) = \sup\limits_{t \in K} |f(t)|$.

(c) $X = \{f \mid f : T \to \mathbb{R}$ is continuous $\} = \mathcal{C}(T)$ where T is a topological space. Let $\mathcal{K} = \{K \subset T | K$ is compact$\}$ and $Q = \{p_K | K \in \mathcal{K}\}$ where, for every $f \in X$, $p_K(f) = \sup\limits_{t \in K} |f(t)|$.

(d) As a particular case, we may consider $X = L^1(\mathbb{R})$ which is a Banach algebra with the sum and the convolution as operations.

We denote by $\mathcal{P}_k(X) = \mathcal{P}_k$ the family of all nonempty compact subsets of X. For every $p \in Q$ and every A, $B \in \mathcal{P}_k$, let $h_p(A, B)$ be the Hausdorff semimetric defined by p on $\mathcal{P}_k$ analogously to the definition of Hausdorff metric [12]. We define $\|A\|_p = h_p(A, O) = \sup\limits_{x \in A} p(x)$ where $O = \{0\}$. It is known that $\{h_p\}_{p \in Q}$ is a filtering family of semimetrics on $\mathcal{P}_k$ which defines a Hausdorff topology on $\mathcal{P}_k$.

For greater convenience of the reader, we now recall some definitions and properties which be used in the following.

1.2. Definition.
$M : \mathcal{A} \to \mathcal{P}_k$ is said to be an **additive multimeasure** *if:*
(i) $M(\emptyset) = O$,
(ii) $M(A \cup B) = M(A) + M(B)$, for every $A, B \in \mathcal{A}$ such that $A \cap B = \emptyset$.

1.3. Examples and applications.

I. If ν_1, ν_2 are two finite measures defined on $\mathcal{A}$, so that $\nu_1 \leq \nu_2$ and ν_2 is a probability measure, then one obtains a particular multimeasure $M : \mathcal{A} \to \mathcal{P}_0([0,1])$, $M(A) = [\nu_1(A), \nu_2(A)]$, $\forall A \in \mathcal{A}$, which is the simplest example of a probability multimeasure. We recall that a multimeasure $M : \mathcal{A} \to \mathcal{P}_0([0,1])$ is said to be *a probability multimeasure* if $1 \in M(S)$. These probability multimeasures are used in control, robotics and decision theory (in Bayesian estimation).

II. We now give an example of such a multimeasure used by Wasserman [16] in robust Bayesian inference. In this paper, Wasserman generalizes previous works of Shafer [15] and Dempster [8] who defines the upper and lower probabilities generated by a multifunction. Following [16] p.454-455, let Θ be a Polish space with Borel σ-algebra $\mathcal{B}(\Theta)$ and let X be a convex compact metrizable subset of a locally convex topological vector space with Borel σ-algebra $\mathcal{B}(X)$. Let μ be a probability measure on $(X, \mathcal{B}(X))$ and let Γ be a multifunction defined on X with values in $\mathcal{P}_f(\Theta)$ the family of nonempty closed subsets of Θ. For each $A \subset \Theta$, we denote

$$A_* = \{x \in X; \Gamma(x) \subset A\} \text{ and } A^* = \{x \in X; \Gamma(x) \cap A \neq \emptyset\}.$$

Now, if, as in [16], we define on $(\Theta, \mathcal{B}(\Theta))$ *the belief function Bel* and *the plausibility function Pl* by, for any $A \in \mathcal{B}(\Theta)$,

$$Bel(A) = \mu(A_*) \text{ and } Pl(A) = \mu(A^*),$$

then we can consider the set Π of all probability measures P satisfying, for any $A \in \mathcal{B}(\Theta))$, $Bel(A) \leq P(A) \leq Pl(A)$. It can be shown that Π is non empty and that, for every $A \in \mathcal{B}(\Theta)$,

$$Bel(A) = \inf_{P \in \Pi} P(A) \text{ and } Pl(A) = \sup_{P \in \Pi} P(A).$$

So, Bel and Pl may be thought as the lower and upper bounds of the family of the selections measures of the multimeasure M such that, for every $A \in \mathcal{B}(\Theta)$, $M(A) = [Bel(A), Pl(A)]$.

The next definition is a natural extension of the concept of the variation of a vector measure [10].

1.4. Definition.

Let $M : \mathcal{A} \to \mathcal{P}_k$. For every $p \in Q$, the p**-variation** *of M is the non-negative (possibly infinite) set function $v_p(M, \cdot)$ defined on $\mathcal{A}$ as follows:*

$$v_p(M, A) = \sup \left\{ \sum_{i=1}^{n} \|M(E_i)\|_p \;\middle|\; (E_i)_{i=1}^{n} \subset \mathcal{A}, E_i \cap E_j = \emptyset \text{ for } i \neq j, \bigcup_{i=1}^{n} E_i = A, n \in \mathbb{N}^* \right\}, \forall A \in \mathcal{A}.$$

If M is an additive multimeasure, then $v_p(M, \cdot)$ is finitely additive for every $p \in Q$. We say that M is **with bounded** $p-$**variation** *iff $v_p(M, \cdot)$ is bounded for every $p \in Q$.*

In the sequel, multimeasures and multifunctions take their values in a subset $\widetilde{X}$ of $\mathcal{P}_k$ satisfying the conditions:

- $\widetilde{X}$ is complete with respect to $\{h_p\}_{p \in Q}$,
- $O \in \widetilde{X}$,
- $A + B, A \cdot B \in \widetilde{X}$, for all $A, B \in \widetilde{X}$,
- $A \cdot (B + C) = A \cdot B + A \cdot C$, for all $A, B, C \in \widetilde{X}$.

1.5. Examples

(a) $\widetilde{X} = \{\{f\} | f \in X\}$ for X like in Example 1.1, (a) and (b).

(b) $\widetilde{X} = \{A \mid A \subset [0, +\infty[, A$ is nonempty compact convex$\}$ for $X = \mathbb{R}$.

(c) $\widetilde{X} = \{[f, g] \mid f, g \in X, 0 \le f \le g\}$ for X like in 1.1-(a), where $[f, g] = \{u \in X \mid f \le u \le g\} = \{u \in X \mid f(t) \le u(t) \le g(t),$ for every $t \in T\}$ and $[f, f] = \{f\}$.

(c) $\widetilde{X}$ is the family of nonempty compact subsets of X like in Example 1.1 (c).

In the sequel, we also suppose that $\varphi : \mathcal{A} \to \widetilde{X}$ is an additive multimeasure such that its $p-$variation $v_p(\varphi, \cdot)$, denoted by ν_p, is bounded and there exists at least one $p \in Q$ such that $(S, \mathcal{A}, \nu_p)$ is complete (cf. [10]-III).

1.6. Definition

A multimeasure $\Gamma : \mathcal{A} \to \widetilde{X}$ *is said to be* **absolutely continuous with respect to the multimeasure** φ *if, for every* $p \in Q$ *and* $\varepsilon > 0$, *there exists* $\delta(p,\varepsilon) = \delta > 0$ *such that for every* $E \in \mathcal{A}$,

$$\nu_p(E) < \delta \Rightarrow v_p(\Gamma, E) < \varepsilon$$

that is denoted: $\Gamma \ll \varphi$.

Now we recall some integral notions already used in [6] and [7].

If $F : S \to \widetilde{X}$ is the simple multifunction $F = \sum_{i=1}^{n} B_i \cdot \mathcal{X}_{A_i}$, where $B_i \in \widetilde{X}$, $A_i \in \mathcal{A}, i \in \{1, 2, ..., n\}$, $A_i \cap A_j = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^{n} A_i = S$ and $\mathcal{X}_{A_i}$ is the characteristic function of A_i, the integral of F over $E \in \mathcal{A}$ with respect to φ is:

$$\int_E F d\varphi = \sum_{i=1}^{n} B_i \cdot \varphi(A_i \cap E) \in \widetilde{X}.$$

1.7. Definition (Definition 2.2 of [6])

A multifunction $F : S \to \widetilde{X}$ *is called* φ**-totally measurable in seminorm** *if for every* $p \in Q$, *there is a sequence* $(F_n^p)_n$ *of simple multifunctions* $F_n^p : S \to \widetilde{X}$ *such that* $h_p(F_n^p, F) \xrightarrow{\nu_p} 0$.

1.8. Definition (Definition 2.3 of [6])

A multifunction $F : S \to \widetilde{X}$ *is called* $\varphi-$**integrable in seminorm** *if, for every* $p \in Q$, *there exists a sequence* $(F_n^p)_n$ *of simple multifunctions,* $F_n^p : S \to \widetilde{X}$, *satisfying the following conditions:*

(i) $h_p(F_n^p, F) \xrightarrow{\nu_p} 0$ *(that is: F is φ-totally measurable in seminorm),*

(ii) $h_p(F_n^p, F)$ *is* ν_p-*integrable, for every* $n \in \mathbb{N}$,

(iii) $\lim_{n\to\infty} \int_E h_p(F_n^p, F) d\nu_p = 0$, *for every* $E \in \mathcal{A}$,

(iv) For every $E \in \mathcal{A}$, *there exists* $I_E \in \widetilde{X}$ *such that, for every* $p \in Q$, $\lim_{n\to\infty} h_p\left(\int_E F_n^p d\varphi, I_E\right) = 0$.

We denote $I_E = \int_E F d\varphi$ and call it the **integral of F on E with respect to** φ. The sequence $(F_n^p)_n$ is said to be a $p-$**defining sequence** for F.

1.9. Remark (**Connections with previous integrals**)

I. In the above definition of φ-integrability in seminorm, the p-defining sequence depends on the seminorm. This setting is weaker and differs from that of [5], where the defining sequence is independent of the seminorm.

II. In [6] and [7], we have shown that this integral contains different classical integrals or multivalued integrals ([1], [2], [3], [4], [10] and [14]) and has some of the classical properties of an integral.
For examples:

(a) If X is a real Banach algebra, then we obtain the integral defined in [4].

(b) If $X = \mathbb{R}$, $\widetilde{X} = \{A | A \subset [0, +\infty), A$ is a nonvoid compact convex set$\}$ and $\varphi = \{\mu\}$ (where μ is finitely additive), then we obtain the integral (defined in [14]) of the multifunction F with respect to μ.

The next convergence theorem of Vitali type (Theorem 3.1 of [7]) will be used in the next section.

1.10. Theorem(Vitali)
Let $F : S \to \widetilde{X}$ be a multifunction and, for every $p \in Q$, $(F_n^p)_{n\in\mathbb{N}^*}$ be a sequence of $\varphi-$integrable in seminorm multifunctions $F_n^p : S \to \widetilde{X}$. We denote, for every $E \in \mathcal{A}$, every $n \in \mathbb{N}^*$ and every $p \in Q$, $\Gamma_n^p(E) = \int_E \|F_n^p\|_p d\nu_p$ and, for every $p \in Q$, we suppose the following conditions:

(i) $h_p(F_n^p, F) \xrightarrow{\nu_p} 0$,

(ii) $\Gamma_n^p \ll \nu_p$, *uniformly in $n \in \mathbb{N}^*$ (i.e. for every $p \in Q$ and $\varepsilon > 0$, there is $\delta(p, \varepsilon) = \delta > 0$ such that $\Gamma_n^p(E) < \varepsilon$ for all $E \in \mathcal{A}$ with $\nu_p(E) < \delta$ and for every $n \in \mathbb{N}^*$).*

Then the multifunction F is $\varphi-$integrable in seminorm and, for every $E \in \mathcal{A}$, $\lim_{n\to\infty} \int_E F_n^p d\varphi = \int_E F d\varphi$.

2 Radon-Nikodym type theorem

In this section, the approach to be used in obtaining a Radon-Nikodym theorem will be analogous to that of Maynard in [13]. We begin by recalling some definitions.
In the sequel, we denote $\mathcal{A}^+ = \{E \in \mathcal{A} \mid \nu_p(E) > 0, \forall p \in Q\}$.

2.1. Definition

(i) A finite or countable family of pairwise disjoint sets $(E_i)_i \subset \mathcal{A}^+$ *will be called an* **uniform exhaustion** *of* S *if for every* $\varepsilon > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $\nu_p\left(S \setminus \bigcup_{i=1}^{n_0} E_i\right) < \varepsilon$ *for every* $p \in Q$.

(ii) A set property P *is said to be* **uniformly exhaustive** *on* $E \in \mathcal{A}$ *if there exists an uniform exhaustion* $(E_i)_i$ *of* E *such that every* E_i *has* P.

2.2. Definition

A set property P *is called* **"uniform null difference"** *if whenever* $A, B \in \mathcal{A}^+$, *from* $\nu_p(A \triangle B) = 0$ *for every* $p \in Q$, *it follows that either* A *and* B *both have* P *or neither does.*

2.3. Theorem

Let P *be an uniform null difference property such that* P *is uniformly exhaustive on* S. *Then there exists* $I \subset \mathbb{N}^*$ *and* $(E_i)_{i \in I}$ *an uniform exhaustion of* S, *such that every* E_i *has* P *and* $S = \bigcup_{i \in I} E_i$.

Proof

Since P is uniformly exhaustive on S, there exists $I \subset \mathbb{N}^*$ and $(A_i)_{i \in I}$ an uniform exhaustion of S, such that every A_i has P. Thus, we have:

$$(1) \qquad \forall \varepsilon > 0, \exists n_0(\varepsilon) = n_0 \in \mathbb{N}^* \text{ such that } \nu_p(S \setminus \bigcup_{i=1}^{n_0} A_i) < \varepsilon, \ \ \forall p \in Q.$$

Let $A_0 = S \setminus \bigcup_{i \in I} A_i$. By the inclusion $A_0 \subset S \setminus \bigcup_{i=1}^{n_0} A_i$ and from (1), it results that $\nu_p^*(A_0) < \varepsilon, \forall \varepsilon > 0$ (where $\nu_p^*(A_0) = \inf\{\nu_p(C) | A_0 \subset C, C \in \mathcal{A}\}$, cf. Dunford and Schwartz [10]-III). So $\nu_p^*(A_0) = 0$. If there is $q \in Q$ such that $(S, \mathcal{A}, \nu_q)$ is complete, from $\nu_q^*(A_0) = 0$, it follows that $A_0 \in \mathcal{A}$ and, for every $p \in Q$, $\nu_p(A_0) = 0$.

Let $(E_i)_{i \in I}$ be the family of sets defined by: $E_1 = A_0 \cup A_1 \in \mathcal{A}$ and for $i \geq 2$, $E_i = A_i \in \mathcal{A}$. We have, for every $p \in Q$, for $i = 1$, $\nu_p(E_1) \geq \nu_p(A_1) > 0$ and for every $i \geq 2$, $\nu_p(E_i) = \nu_p(A_i) > 0$. Evidently, $S = \bigcup_{i \in I} E_i$.

Let $\varepsilon > 0$. For n_0 of (1) we have $\bigcup_{i=1}^{n_0} E_i = A_0 \cup \bigcup_{i=1}^{n_0} A_i$.

By the inclusion $S \setminus \bigcup_{i=1}^{n_0} E_i \subset S \setminus \bigcup_{i=1}^{n_0} A_i$ and from (1), it follows:

$$\nu_p\left(S \setminus \bigcup_{i=1}^{n_0} E_i\right) \leq \nu_p\left(S \setminus \bigcup_{i=1}^{n_0} A_i\right) < \varepsilon, \ \ \forall p \in Q$$

which assures that $(E_i)_{i\in I}$ is an uniformly exhaustion of S. Now, for every $i \geq 2, E_i = A_i$ has P. So, we have only to prove that E_1 has P. By the relations:

$$E_1 \triangle A_1 = (A_0 \cup A_1)\triangle A_1 = A_0 \backslash A_1 \subset A_0$$

it follows

$$\nu_p(E_1 \triangle A_1) \leq \nu_p(A_0) = 0, \quad \forall p \in Q$$

and $\nu_p(E_1 \triangle A_1) = 0$, for every $p \in Q$. Since P is uniform null difference and A_1 has P, we obtain that E_1 has P. □

Now, we give two properties of a set-valued integral Γ, properties which will be of use in the next Radon-Nikodym theorem.

2.4. Theorem

Let $F : S \to \widetilde{X}$ be a φ-integrable in seminorm bounded multifunction and $\Gamma(E) = \int_E F d\varphi$, $E \in \mathcal{A}$ (Γ is a multimeasure according to Theorem 2.8-(a) of [6]). Then we have:
(i) $\Gamma \ll \varphi$;
(ii) for every $p \in Q$, there exists $r_p > 0$ such that, for every $E \in \mathcal{A}$ with $\nu_p(E) > 0$, $\|\Gamma(E)\|_p \leq r_p\, \nu_p(E)$.

Proof
(i) It follows from Theorem 2.8-(c) of [6].
(ii) Since the boundedness of F, for every $p \in Q$, there exists $r_p > 0$ such that:

$$\|F(s)\|_p \leq r_p, \quad \forall s \in S. \tag{2}$$

From (2) and Theorem 2.7-(b) of [6], for each $E \in \mathcal{A}$ with $\nu_p(E) > 0$, we have:

$$\|\Gamma(E)\|_p = \left\| \int_E F d\varphi \right\|_p \leq \int_E \|F\|_p d\nu_p \leq r_p\, \nu_p(E).$$

□

The next sentence follows from classical properties of the variation of a vector measure ([9], [10]).

2.5. Proposition

If Γ is a multimeasure which satisfy condition (ii) of Theorem 2.4, then Γ is a multimeasure with bounded $p-$variation.

We now give a definition of approximate average ranges which is adapted from that of [13] for set-valued case.

2.6. Definition
For a multifunction $\Gamma : \mathcal{A} \to \widetilde{X}$, $p \in Q, \varepsilon > 0$ *and* $E \in \mathcal{A}$, *let:*

$$D_p(\Gamma, E, \varepsilon) = \{C \in \widetilde{X} \mid h_p(\Gamma(B), \nu_p(B) \cdot C) \leq \varepsilon \nu_p(B), \forall B \in \mathcal{A}, B \subset E\},$$
$$\widetilde{D}_p(\Gamma, E, \varepsilon) = \{C \in \widetilde{X} \mid h_p(\Gamma(B), \varphi(B) \cdot C) \leq \varepsilon \nu_p(B), \forall B \in \mathcal{A}, B \subset E\}.$$

The next proposition give examples of uniform null difference properties which will used in this paper. Its demonstration is adapted from that of Theorem 3.6 of [5].

2.7. Proposition
Let Γ *be an additive multimeasure with bounded* $p-$*variation which is absolutely continuous with respect to* φ. *Then,*
(i) $D_p(\Gamma, E, \gamma) \neq \emptyset$,
(ii) $\widetilde{D}_p(\Gamma, E, \gamma) \neq \emptyset$ *and*
(iii) $D_p(\Gamma, E, \gamma) \cap \widetilde{D}_p(\Gamma, E, \gamma) \neq \emptyset$
are uniform null difference properties.

Proof
It is clear it is sufficient to prove *(i)* and *(ii)*.
(i) Since $\Gamma \ll \varphi$, for every $p \in Q$ and every $\varepsilon > 0$, there exists $\delta(p, \varepsilon) = \delta > 0$ such that, for every $E \in \mathcal{A}$ such that $\nu_p(E) < \delta$, $\|\Gamma(E)\|_p \leq v_p(\Gamma, E) < \varepsilon$. Now, we have to show that, if A and $B \in \mathcal{A}^+$ such that $\nu_p(A \triangle B) = 0$ for every $p \in Q$, $D_p(\Gamma, A, \gamma) = D_p(\Gamma, B, \gamma)$ for each $p \in Q$. Now we fix one $p \in Q$ and consider $C \in D_p(\Gamma, A, \gamma)$.
$C \in \widetilde{X}$ and, for every $B \in \mathcal{A}, B \subset A$, $h_p(\Gamma(B), \nu_p(B)C) \leq \gamma \nu_p(B)$.
Let $H \in \mathcal{A}$, $H \subset B$. Since $\nu_p(A \triangle B) = 0$, $\nu_p(H \backslash A) \leq \nu_p(B \backslash A) = 0$ and $\nu_p(H) = \nu_p(H \cap A) + \nu_p(H \backslash A) \leq \nu_p(H \cap A) + \nu_p(B \backslash A) = \nu_p(H \cap A) \leq \nu_p(H)$. So, $\nu_p(H) = \nu_p(H \cap A)$ for every $H \in \mathcal{A}, H \subset B$.
From $H \cap A \subset A$, it follows $h_p(\Gamma(A \cap H), \nu_p(A \cap H) \cdot C) \leq \gamma \nu_p(A \cap H)$. Since $\nu_p(H \backslash A) = 0 < \delta$, $\|\Gamma(H \backslash A)\|_p < \varepsilon$ and this for all ε. So, $\|\Gamma(H \backslash A)\|_p = 0$. Now, we can write:
$h_p(\Gamma(H), \nu_p(H) \cdot C) = h_p(\Gamma(H \cap A) + \Gamma(H \backslash A), (\nu_p(H \cap A) + \nu_p(H \backslash A)) \cdot C) \leq$ $\leq h_p(\Gamma(H \cap A), \nu_p(H \cap A) \cdot C) + \|\Gamma(H \backslash A)\|_p \leq \gamma \nu_p(H \cap A) = \gamma \nu_p(H)$. So for every $H \in \mathcal{A}, H \subset B$, we have $h_p(\Gamma(H), \nu_p(H) \cdot C) \leq \gamma \nu_p(H)$. That is: $C \in D_p(\Gamma, B, \gamma)$. The inclusion $D_p(\Gamma, B, \gamma) \subset D_p(\Gamma, A, \gamma)$ is similarly proved by exchange of A and B. So, the first part " $D_p(\Gamma, E, \gamma) \neq \emptyset$ is an uniform null difference property" is proved.
(ii) For the second property, with same notations as in the first part, if $H \in \mathcal{A}, H \subset B$, since, for every $p \in Q$, $\nu_p(A \triangle B) = 0$, it follows from $\|\varphi(H \backslash A)\|_p \leq$

$\nu_p(H\backslash A) = 0$, as in the first part, that $\|\Gamma(H\backslash A)\|_p = 0$ and $h_p(\ \Gamma(H),\ \varphi(H)\cdot C\ ) = h_p(\ \Gamma(\ H\cap A\ ),\ \varphi(H\cap A)\cdot C\ )\ +\ \|\Gamma(\ H\backslash A\ )\|_p \leq$
$\leq \gamma\nu_p(H\cap A) = \gamma\nu_p(H)$. The proof is finished as in the first part. □

2.8. Theorem

Let $F : S \to \widetilde{X}$ be a φ-integrable in seminorm bounded multifunction and $\Gamma(E) = \int_E F d\varphi$, $E \in \mathcal{A}$. Then,
$\forall p \in Q$, $\forall\varepsilon > 0$ and $\forall E \in \mathcal{A}$ such that $\nu_p(E) > 0$, there exists $B \in \mathcal{A}$, $B \subset E$ $\nu_p(B) > 0$ such that $\widetilde{D}_p(\Gamma, B, \varepsilon) \neq \emptyset$.

Proof

Since F is $\varphi-$integrable in seminorm, for every $p \in Q$, there exists $(F_n^p)_n$ a $p-$defining sequence of simple multifunctions $F_n^p : S \to \widetilde{X}$. So,

(i) $h_p(F_n^p, F) \xrightarrow{\nu_p} 0$,

(ii) $h_p(F_n^p, F)$ is ν_p-integrable, for every $n \in \mathbb{N}$,

(iii) $\lim\limits_{n\to\infty} \int_E h_p(F_n^p, F)d\nu_p = 0$, for every $E \in \mathcal{A}$.

Thanks to (i), there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}^*} \subset \mathbb{N}$ such that $\nu_p(\{s \in S \mid h_p(F_{n_k}^p(s), F(s)) > \frac{1}{2^k}\}) \leq \frac{1}{2^k}$. Let $A_k^p = \{s \in S \mid h_p(F_{n_k}^p(s), F(s)) > \frac{1}{2^k}\}$. If we denote $G_k^p = F_{n_k}^p \mathcal{X}_{\complement A_k^p}$, G_k^p is a simple function and for every $k \in \mathbb{N}^*$, $h_p(G_k^p, F)$ is ν_p -measurable. It is easy to see that for every $\varepsilon > 0$ and every $k \in \mathbb{N}^*$, $\nu_p(\{s \in S | h_p(G_k^p(s), F(s)) > \varepsilon\}) \leq \nu_p(A_k^p) + \nu_p(\{s \in S | h_p(F_{n_k}^p(s), F(s)) > \varepsilon\})$. So $h_p(G_k^p, F) \xrightarrow{\nu_p} 0$. And, with notations of Theorem 2.4,

$$\int_S h_p(G_k^p, F)d\nu_p = \int_{A_k^p} \|F\|_p d\nu_p + \int_{\complement A_k^p} h_p(G_k^p, F)d\nu_p \leq r_p\nu_p(A_k^p) + \frac{1}{2^k}\nu_p(S).$$

Then, $\lim\limits_{k\to\infty} \int_S h_p(G_k^p, F)d\nu_p = 0$ and $\lim\limits_{k\to\infty} \int_E h_p(G_k^p, F)d\nu_p = 0$, $\forall E \in \mathcal{A}$. Since $\nu_p(E) > 0$ and $\lim\limits_{k\to\infty} \nu_p(A_k^p) = 0$, $\lim\limits_{k\to\infty} \nu_p(E \cap \complement A_k^p) = \nu_p(E)$. So, there exists k_0 such that, for every $k \geq k_0$, $\nu_p(E \cap \complement A_k^p) > 0$.

If $G_k^p = \sum\limits_{i=1}^{l} C_i \mathcal{X}_{A_i}$, $\nu_p(E \cap \complement A_k^p) = \sum\limits_{i=1}^{l} \nu_p(E \cap \complement A_k^p \cap A_i)$ and there exists at least one $i = i_0(k) = i_0$ such that $\nu_p(E \cap \complement A_k^p \cap A_{i_0}) > 0$.

Denoting $B = E \cap \complement A_k^p \cap A_{i_0}$, for every $H \in \mathcal{A}, H \subset B$, we have
$h_p(\Gamma(H), \varphi(H)\cdot C_{i_0}) = h_p(\int_H F d\varphi,\ \int_H G_k^p d\varphi) \leq \int_H h_p(G_k^p, F)d\nu_p \leq \frac{1}{2^k}\nu_p(H)$
That is: $C_{i_0} \in \widetilde{D}_p(\Gamma, B, \frac{1}{2^k})$. And, since, for every $\varepsilon > 0$, there exists $k \geq k_0$ such that $\frac{1}{2^k} \leq \varepsilon$, we can conclude that $C_{i_0} \in \widetilde{D}_p(\Gamma, B, \varepsilon)$. □

2.9. Remark

Let $\Gamma : \mathcal{A} \to \widetilde{X}$ be a bounded multimeasure such that:

$$\text{for every } \varepsilon > 0 \text{ and every } p \in Q,\ D_p(\Gamma, E, \varepsilon) \cap \widetilde{D}_p(\Gamma, E, \varepsilon) \neq \emptyset \tag{3}$$

is an uniformly exhaustive property on every $E \in \mathcal{A}^+$.

Since (3) and Theorem 2.3, for every $p \in Q$ and $\varepsilon > 0$, there is $(E_i^{p,\varepsilon})_i$ an uniform exhaustion of each $E \in \mathcal{A}^+$, such that $E = \bigcup\limits_i E_i^{p,\varepsilon}$ and:

$$D_p(\Gamma, E_i^{p,\varepsilon}, \varepsilon) \cap \widetilde{D}_p(\Gamma, E_i^{p,\varepsilon}, \varepsilon) \neq \emptyset, \ \ \forall i.$$

By induction, following the same way as in Hagood [11], we can obtain a sequence $E_\alpha^{p,n} (n \in \mathbb{N}, \alpha \in \mathbb{N}^n)$ of uniform exhaustions of S such that:

$$D_p(\Gamma, E_\alpha^{p,n}, 2^{-n}) \cap \widetilde{D}_p(\Gamma, E_\alpha^{p,n}, 2^{-n}) \neq \emptyset, \ \ \forall \alpha \in \mathbb{N}^n, n \in \mathbb{N}, \tag{4}$$

$$E_\alpha^{p,n} = \bigcup_{i \in \mathbb{N}} E_{\alpha,i}^{p,n}, \text{ where } \left(E_{\alpha,i}^{p,n+1}\right)_i \text{ is an uniform exhaustion of } E_\alpha^{p,n}, \ \forall \alpha \in \mathbb{N}^n, n \in \mathbb{N}, \tag{5}$$

$$S = \bigcup_\alpha E_\alpha^{p,n} \text{ and } (E_\alpha^{p,n})_\alpha \text{ is an uniform exhaustion of } S, \forall n \in \mathbb{N}. \tag{6}$$

From (6), for $\varepsilon = \frac{1}{n}$, there exists $k(n) = k \in \mathbb{N}$ such that:

$$\nu_p\left(S \setminus \bigcup_{i=1}^{k} E_i^{p,n}\right) < \frac{1}{n}. \tag{7}$$

If we consider $G_n^p = \sum\limits_{i=1}^{k} C_i^{p,n} \cdot \mathcal{X}_{E_i^{p,n}} + O \cdot \mathcal{X}_{S \setminus \bigcup\limits_{i=1}^{k} E_i^{p,n}}$, where $C_i^{p,n} \in D_p(\Gamma, E_i^{p,n}, 2^{-n}) \cap \widetilde{D}_p(\Gamma, E_i^{p,n}, 2^{-n})$, G_n^p is simple and therefore φ-integrable in seminorm. The sequence $(G_n^p)_n$ is called *associate* to Γ.

2.10. Theorem(Radon-Nikodym)

Let $\Gamma : \mathcal{A} \to \widetilde{X}$ be a multimeasure satisfying the three following conditions:
(i) $\Gamma \ll \varphi$,
(ii) for every $p \in Q$, there exists $r_p > 0$ such that $\|\Gamma(E)\|_p \leq r_p \nu_p(E)$, for every $E \in \mathcal{A}$ with $\nu_p(E) > 0$,

(iii) for every $p \in Q$ and $\varepsilon > 0$, the property "$D_p(\Gamma, E, \varepsilon) \cap \widetilde{D}_p(\Gamma, E, \varepsilon) \neq \emptyset$" is an uniform exhaustive property on every $E \in \mathcal{A}^+$, and the sequence $(G_n^p)_n$, associate to Γ (see Remark 2.9), is convergent in ν_p-measure.

Then there exists a φ-integrable in seminorm multifunction $F : S \to \widetilde{X}$, such that $\Gamma(E) = \int_E F d\varphi$, for every $E \in \mathcal{A}$.

Proof

Thanks to Propositions 2.7 and 2.5, Γ has a bounded p-variation and, for every $p \in Q$ and $\varepsilon > 0$, the property "$D_p(\Gamma, E, \varepsilon) \cap \widetilde{D}_p(\Gamma, E, \varepsilon) \neq \emptyset$" is an uniform null difference property on every $E \in \mathcal{A}^+$. So, we can use Theorem2.3.

According to Remark 2.9, the sequence $(G_n^p)_n$, associate to Γ, is given by:

$$G_n^p = \sum_{i=1}^{k} C_i^{p,n} \cdot \mathcal{X}_{E_i^{p,n}} + O \cdot \mathcal{X}_{S \setminus \bigcup_{i=1}^{k} E_i^{p,n}},$$

where

$$C_i^{p,n} \in D_p(\Gamma, E_i^{p,n}, 2^{-n}) \cap \widetilde{D}_p(\Gamma, E_i^{p,n}, 2^{-n}). \tag{8}$$

Since *(iii)*, there exists a multifunction $F : S \to \widetilde{X}$ such that:

$$h_p(G_n^p, F) \xrightarrow{\nu_p} 0. \tag{9}$$

From (8) and *(ii)*, it results:

$$\|C_i^{p,n}\|_p = h_p(C_i^{p,n}, O) = \frac{1}{\nu_p(E_i^{p,n})} h_p(\nu_p(E_i^{p,n}) C_i^{p,n}, O) \leq \tag{10}$$

$$\leq \frac{1}{\nu_p(E_i^{p,n})} h_p(\nu_p(E_i^{p,n}) C_i^{p,n}, \Gamma(E_i^{p,n})) + \frac{1}{\nu_p(E_i^{p,n})} \|\Gamma(E_i^{p,n})\|_p \leq 2^{-n} + r_p.$$

Since (10), we have for every $E \in \mathcal{A}$:

$$\int_E \|G_n^p\|_p d\nu_p = \int_E \sum_{i=1}^{k} \|C_i^{p,n}\|_p \mathcal{X}_{E_i^{p,n}} d\nu_p = \sum_{i=1}^{k} \int_E \|C_i^{p,n}\|_p \mathcal{X}_{E_i^{p,n}} d\nu_p \leq$$

$$\leq \sum_{i=1}^{k} (2^{-n} + r_p) \int_E \mathcal{X}_{E_i^{p,n}} d\nu_p = (2^{-n} + r_p) \sum_{i=1}^{k} \nu_p(E \cap E_i^{p,n}) = \tag{11}$$

$$= (2^{-n} + r_p) \nu_p(E).$$

If we consider now $\delta(p,\varepsilon)=\delta=\frac{\varepsilon}{2^{-n}+r_p}>0$, then for every $E\in\mathcal{A}$ with $\nu_p(E)<\delta$, using (11), we obtain $\int_E \|G_n^p\|_p d\nu_p<\varepsilon$. So, we have:

(12) for every $p\in Q$ and $\varepsilon>0$, there exists $\delta(p,\varepsilon)=\delta>0$ such that for every $E\in\mathcal{A}$ with $\nu_p(E)<\delta$, it follows $\int_E \|G_n^p\|_p d\nu_p<\varepsilon$.

From (9) and (12), using Theorem 1.10 (Vitali), it results that F is φ-integrable in seminorm and:

$$\lim_{n\to\infty}\int_E G_n^p d\varphi=\int_E F d\varphi,\ \ \forall E\in\mathcal{A}. \tag{13}$$

Now we prove that $\Gamma(E)=\int_E F d\varphi, \forall E\in\mathcal{A}$. Let us fix $\varepsilon>0$, $p\in Q$ and let $\delta(p,\frac{\varepsilon}{3})=\delta>0$ given from (i). According to (13), let $n\in\mathbb{N}^*$ such that $\frac{1}{n}<\delta$ and:

$$h_p(\int_E G_n^p d\varphi,\int_E F d\varphi)<\frac{\varepsilon}{3}. \tag{14}$$

Then we have:

$$\begin{aligned}
&h_p(\Gamma(E),\int_E F d\varphi)\le h_p\left(\Gamma(E),\Gamma(\bigcup_{i=1}^k(E\cap E_i^{p,n}))\right)+\\
&+h_p\left(\Gamma(\bigcup_{i=1}^k(E\cap E_i^{p,n})),\sum_{i=1}^k C_i^{p,n}\cdot\varphi(E\cap E_i^{p,n})\right)+\\
&+\underbrace{h_p\left(\int_E G_n^p d\varphi,\int_E G d\varphi\right)}_{<\frac{\varepsilon}{3},\ cf.(14)}<\|\Gamma(E\setminus\bigcup_{i=1}^k(E\cap E_i^{p,n}))\|_p+\\
&+\sum_{i=1}^k h_p(\Gamma(E\cap E_i^{p,n}),C_i^{p,n}\cdot\varphi(E\cap E_i^{p,n}))+\frac{\varepsilon}{3}\le\\
&\le\underbrace{v(\Gamma,E\setminus\bigcup_{i=1}^k(E\cap E_i^{p,n}))}_{<\frac{\varepsilon}{3},\ cf.\ (i)\text{ and }(7)}+2^{-n}\sum_{i=1}^k\nu_p(E\cap E_i^{p,n})+\frac{\varepsilon}{3}<\\
&<\frac{2\varepsilon}{3}+2^{-n}\nu_p(E)\le\frac{2\varepsilon}{3}+2^{-n}\nu_p(S)<\varepsilon,
\end{aligned}$$

which shows that $\Gamma(E)=\int_E F d\varphi,\forall E\in\mathcal{A}$. □

2.11. Remark

In the previous Theorem of Radon-Nikodym type (in [5]) the first three conditions fulfilled by Γ are the following:

(a) Γ is uniformly bounded;

(b) $\Gamma \ll \nu_p$, uniformly in $p \in \mathbb{Q}$;

(c) there exists $r > 0$ such that $\|\Gamma(E)\| \le r\nu_p(E)$, for every $E \in \mathcal{A}$ with $\nu_p(E) > 0$ and for every $p \in Q$.

As we may observe, Theorem 2.10 applies to a wider class of multimeasures Γ, that are not necessarily uniformly bounded.

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References

[1] Blondia, C., *Integration in locally convex spaces*, Simon Stevin J., **55**(3) (1981), 81–102.

[2] Bochner, S., *Integration von Functionen deren werte die Elemente eines Vectorraumes sind*, Fund.Math. **20** (1933), 262–276.

[3] Brooks, J.K., *An integration theory for set-valued measures I, II*, Bull. Soc. Roy. Sciences de Liège **37** (1968), 312–319, 375–380.

[4] Croitoru, A., *A set-valued integral*, An. Ştiint. Univ. Al.I. Cuza Iaşi **44** (1998), 101–112.

[5] Croitoru, A., *Multivalued version of Radon-Nikodym theorem*, Carpathian J. Math, **21** (2005), 27–38.

[6] Croitoru, A., Godet-Thobie, C., *Set-valued integration in seminorm. I*, Annals of University of Craiova, Math. Comp. Sci. Ser., **33** (2006), 16–25.

[7] Croitoru A., Godet-Thobie C., *Set-valued integration in seminorm II*, An. Ştiint. Univ. Ovidius Constanţa **13** (2005), 55–66.

[8] Dempster A. P., *Upper and lower probabilities induced by a multivalued mapping,* Ann. Math. Statist., **38** (1967), 325–339.

[9] Diestel, J., Uhl, J.J., *Vector measures*, Mat. Surveys 15, Amer. Math. Soc., Providence, 1977.

[10] Dunford, N., Schwartz, J., *Linear Operators I. General Theory*, Interscience, New York, 1958.

[11] Hagood, J.W., *A Radon - Nikodym theorem and L_p completeness for finitely additive vector measure*, J. Math. Anal. Appl. **113** (1986), 266–279.

[12] Hu, S., Papageorgiou, N.S., *Handbook of Multivalued Analysis*, Vol. I, Kluwer Acad. Publ., 1997.

[13] Maynard, H.B., *A Radon - Nikodym theorem for finitely additive bounded measures*, Pacific J. Math. **83** (1979), 401–413.

[14] Sambucini, A.R., *Integrazione per seminorme in spazi localmente convessi*, Riv. Mat. Univ. Parma **3** (1994), 371–381.

[15] Shafer, G., *Allocations of Probability*, Annals of Proba., **7** (1979), 827–839.

[16] Wasserman L. A., *Prior envelopes based on belief functions*, Annals Stat, **18** (1990), 454–464.

Christiane GODET-THOBIE
Université de Bretagne Occidentale,
Laboratoire de Mathématiques-UMR 6205,
6, avenue Victor Le Gorgeu - CS 93837
29238 - BREST CEDEX 3, FRANCE
christiane.godet-thobie@univ-brest.fr

Anca CROITORU
Faculty of Mathematics, University “Al.I.Cuza”,
Carol I Street, No. 11, Iaşi 700506, ROMANIA
croitoru@uaic.ro