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Common fixed points for a pair of commuting mappings in complete cone metric spaces

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Abstract

This paper is devoted to prove the S. L. Singh's common fixed point Theorem for commuting mappings in cone metric spaces. In this framework, we introduce the notions of generalized Kannan contraction, generalized Zamfirescu contraction and generalized Weak contraction for a pair of mappings, proving afterward fixed point results.

1 Introduction and Preliminaries.

In 1977, S.L. Singh [15] proved the following result.

Theorem 1.1. *Let (M, d) be a complete metric space. Let S and T be mappings from M into itself such that,*

(a) T is continuous.

(b) $S(M) \subset T(M)$.

(c) S and T commute.

(d) The following inequality holds

$$\begin{aligned} d(Sx, Sy) \leq & ad(Tx, Ty) + b[d(Sx, Ty) + d(Sx, Sy)] \\ & + c[d(Sx, Ty) + d(Sy, Tx)] \end{aligned}$$

for all $x, y \in M$, where a, b, c are nonnegative real numbers such that $0 < a + 2b + 2c < 1$.

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Then, S and T have a unique common fixed point in M.

As a consequence of this theorem, the results of Jungck ([9]) can be obtained by considering the particular case $a = b = 0$. On the other hand, when in Theorem 1.1 is considered $T = Id$, the identity mapping, then are obtained the results given by the Hardy & Rogers in [7] .

The main goal of this paper is to present Singh's fixed point Theorem 1.1 in the setting of complete cone metric spaces. Furthermore, we will introduce some contractive conditions for a pair of mappings on these spaces which generalize some well-known notions given in complete metric spaces.

First, we must recall that the cone metric spaces were introduced in 2007 by Huang and Zhang in [8]. They also obtained several fixed point theorems for contractive single valued mappings in such spaces. Since then, a lot of works in this subject were already published, including various coincidence and common fixed point theorems for a pair of weakly compatible mapping ([1]) as well as generalized contraction and Zamfirescu pair ([2]).

Definition 1.1 ([8])**.** Let $(E, \|\cdot\|)$ be a real Banach space. A subset $P \subset E$ is called a cone if and only if:

(P1) P is closed, nonempty and $P \neq \{0\}$.

(P2) $a, b \in \mathbb{R}, \quad a, b \geq 0, \quad x, y \in P$ implies $ax + by \in P$.

(P3) $x \in P$ and $-x \in P \Rightarrow x = 0$, that is, $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by

$$x \leq y \quad \text{if and only if} \quad y - x \in P.$$

We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{Int } P$, where $\text{Int } P$ denote the interior of P.

Definition 1.2 ([8])**.** Let $(E, \|\cdot\|)$ be a Banach space and $P \subset E$ a cone. The cone P is a called normal if there is a number $K > 0$ such that for all $x, y \in M$

$$0 \leq x \leq y \quad \text{implies} \quad \|x\| \leq K\|y\|.$$

The least positive number satisfying the above is called the normal constant of P.

In 2008, Sh. Rezapour and R. Hamlbarani [14] proved that there are no normal cones with normal constant $K < 1$ and that for each $h > 1$ there are cones with normal constant $K > h$.

In the following, we always suppose that $(E, \|\cdot\|)$ is a real Banach space, P is a cone in E with $\text{Int } P \neq \emptyset$ and $\leq$ is partial ordering with respect to P.

Definition 1.3 ([8]). Let M be a nonempty set. Suppose that the mapping $d : M \times M \longrightarrow E$ satisfies:

(CM1) $0 \leq d(x, y)$ for all $x, y \in M$ and $d(x, y) = 0$ if and only if $x = y$.

(CM2) $d(x, y) = d(y, x)$ for all $x, y \in M$.

(CM3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in M$.

Then, d is called a cone metric on M, and the pair (M, d) is called a cone metric space. It will be denoted by CMS.

Note that the notion of a cone metric space is more general that the concept of a metric space.

Definition 1.4 ([8]). Let (M, d) be a CMS. Let (x_n) be a sequence in M and $x \in M$.

(i) (x_n) is said convergent to x whenever for every $c \in E$, with $0 \ll c$, there is a positive integer n_0 such that $d(x_n, x) \ll c$ for all $n \geq n_0$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) (x_n) is said to be a Cauchy sequence in M whenever for every $c \in E$ with $0 \ll c$ there is a positive integer n_0 such that $d(x_n, x_m) \ll c$ for all $n, m \geq n_0$.

(iii) (M, d) is called a complete CMS if every Cauchy sequence is convergent in M.

(iv) A set $A \subseteq M$ is said to be closed if for any sequence $(x_n) \subset A$ convergent to x, we have that $x \in A$.

(v) A set $A \subseteq M$ is called sequentially compact if for any sequence $(x_n) \subset A$, there exists a subsequence (x_{n_k}) of (x_n) which is convergent to an element of A.

Lemma 1.2 ([8]). *Let (M, d) be a CMS, $P \subset E$ a normal cone with normal constant K. Let (x_n) be a sequence in M and $x, y \in M$.*

(i) (x_n) converges to x if and only if $\lim\limits_{n\to\infty} d(x_n, x) = 0$.

(ii) If (x_n) converges to x, and (x_n) converges to y, then $x = y$.

(iii) If (x_n) converges to x, then (x_n) is a Cauchy sequence.

(iv) (x_n) is a Cauchy sequence if and only if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.

Lemma 1.3 ([11]). *Let (M, d) be a CMS and let $P \subset E$ be a normal cone with normal constant K. If there exists a sequence (x_n) in M and a real number $a \in (0, 1)$ such that for every $n \in \mathbb{N}$,*

$$d(x_{n+1}, x_n) \leq ad(x_n, x_{n-1}),$$

then (x_n) is a Cauchy sequence.

Definition 1.5 ([16]). Let (M, d) be a CMS and $A \subset M$.

(i) A point $b \in A$ is called an interior point of A whenever there exists a point c, $0 \ll c$, such that

$$B(b, c) \subseteq A$$

where $B(b, c) = \{y \in M \,:\, d(y, b) \ll c\}$.

(ii) A subset $A \subset M$ is called open if each element of A is an interior point of A.

The family $\mathcal{B} = \{B(b, c) \,/\, b \in M,\ 0 \ll c\}$ is a sub-basis for a topology on M. We denote this cone topology by τ_c.

The topology τ_c is Hausdorff and first countable, ([16]). Hence, we conclude that any CMS (M, d) is Hausdorff and the limits are unique.

Definition 1.6 ([16]). Let (M, d) be a CMS. A mapping $T : M \longrightarrow M$ is called continuous at $x \in M$, if for each $V \in \tau_c$ containing Tx, there exists $U \in \tau_c$ containing x such that

$$T(U) \subset V.$$

If T is continuous at each $x \in M$, then it is called continuous.

Definition 1.7 ([16]). Let (M, d) be a CMS. A mapping $T : M \longrightarrow M$ is called sequentially continuous if $(x_n) \subset M$, $x_n \to x$ implies $Tx_n \longrightarrow Tx$.

Proposition 1.4 ([16]). *Let (M, d) be a CMS and $T : M \longrightarrow M$ be any mapping. Then, T is continuous if and only if T is sequentially continuous.*

2 On the Singh's common fixed point Theorem for commuting mappings in cone metric spaces

In this section we will prove Singh's common fixed point Theorem for commuting mappings in the framework of cone metric spaces. Afterwards, we are going to give some consequences of this result.

Definition 2.1 ([17]). Let (M, d) be CMS, and $P \subset E$ a normal cone with normal constant K. Let $S, T : M \longrightarrow M$ be mappings such that $S(M) \subset T(M)$ and for every $x_0 \in M$ we define the sequence (x_n) by $T(x_n) = S(x_{n-1})$, $n = 1, 2, \ldots$. We say that $S(x_n)$ is an $(S, T)-$sequence with initial point x_0. Notice that, in general, an (S, T)-sequence is not uniquely defined.

We would like to recall that a pair of mappings (T, S) is called a *commuting pair* if the two mappings T and S satisfy $TS = ST$.

Theorem 2.1. *Let (M, d) be a complete CMS, and $P \subset E$ a normal cone with normal constant K. Let S and T be self-mappings of M such that,*

(a) T is continuous.

(b) $S(M) \subset T(M)$.

(c) (S, T) is a commuting pair.

(d) The following inequality holds

$$\begin{aligned} d(Sx, Sy) \leq & ad(Tx, Ty) + b[d(Sx, Tx) + d(Sy, Ty)] \\ & + c[d(Sx, Ty) + d(Sy, Ty)] \end{aligned} \tag{S}$$

for all $x, y \in M$, where where a, b, c are nonnegative real numbers such that $0 < a + 2b + 2c < \frac{1}{K}$.

Then, S and T have a unique common fixed point.

Proof: Suppose that $x_0 \in M$ is an arbitrary point. We will prove that the $(S, T)-$sequence $(S(x_n))$ with initial point x_0 is a Cauchy sequence in M. In fact, notice that

$$\begin{aligned} d(Sx_{n+1}, Sx_n) \leq & ad(Tx_{n+1}, Tx_n) + b[d(Sx_{n+1}, Tx_{n+1}) + d(Sx_n, Tx_n)] \\ & + c[d(Sx_{n+1}, Tx_n) + d(Sx_n, Tx_n)]. \end{aligned}$$

Thus,

$$d(Sx_{n+1}, Sx_n) \leq \frac{a + b + c}{1 - b - c} d(Sx_n, Sx_{n-1})$$

i.e.

$$d(Sx_{n+1}, Sx_n) \leq \alpha\, d(Sx_n, Sx_{n-1}) \tag{2.1}$$

with

$$\alpha = \frac{a + b + c}{1 - b - c} < 1.$$

In this way, from inequality (2.1) and Lemma 1.3, we have that $(S(x_n))$ is a Cauchy sequence in M.

On the other hand, repeating the procedure above we can conclude that for all $n \in \mathbb{N}$

$$d(Sx_{n+1}, Sx_n) \leq \alpha^n \, d(Sx_1, Sx_0).$$

Since $P \subset E$ is a normal cone with normal constant K, we have

$$\|d(Sx_{n+1}, Sx_n)\| \leq \alpha^n \, K \|d(Sx_1, Sx_0)\|.$$

Taking limits in the inequality above we conclude that

$$\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = 0.$$

Since M is a complete CMS, there exists $z_0 \in M$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_{n+1} = z_0. \tag{2.2}$$

Since T is continuous (and therefore sequentially continuous by Proposition 1.4), and due to the fact that S and T commute, we have

$$Tz_0 = T\left(\lim_{n\to\infty} Tx_n\right) = \lim_{n\to\infty} T^2 x_n$$

as well as

$$Tz_0 = T\left(\lim_{n\to\infty} Sx_n\right) = \lim_{n\to\infty} TSx_n = \lim_{n\to\infty} STx_n.$$

Now,

$$\begin{aligned} d(STx_n, Sz_0) \leq{}& ad(T^2x_n, Tz_0) + b[d(STx_n, T^2x_n) + d(Sz_0, Tz_0)] \\ &+ c[d(STx_n, Tz_0) + d(Sz_0, Tz_0)]. \end{aligned}$$

Again, since P is a normal cone with normal constant K we have

$$\begin{aligned} \|d(STx_n, Sz_0)\| \leq{}& K[a\|d(T^2x_n, Tz_0)\| + b\|d(STx_n, T^2x_n)\| \\ &+ c\|d(STx_n, Tz_0)\| + (b+c)\|d(Sz_0, Tz_0)\|] \end{aligned}$$

taking the limit as $n \to \infty$ we obtain

$$\begin{aligned} \|d(Tz_0, Sz_0)\| \leq{}& K[a\|d(Tz_0, Tz_0)\| + b\|d(Tz_0, Tz_0)\| + c\|d(Tz_0, Tz_0)\| \\ &+ (b+c)\|d(Sz_0, Tz_0)\|] \end{aligned}$$

or, rewriting the inequality above,

$$\|d(Tz_0, Sz_0)\| \leq K(b+c)\|d(Sz_0, Tz_0)\|.$$

Hence, since $0 \leq b + c < \frac{1}{K}$, we have then $d(Tz_0, Sz_0) = 0$, that is $Tz_0 = Sz_0$.

Now,

$$\begin{aligned} d(Sx_n, Sz_0) \leq & ad(Tx_n, Tz_0) + b[d(Sx_n, Tx_n) + d(Sz_0, Tz_0)] \\ & + c[d(Sx_n, Tz_0) + d(Sz_0, Tz_0)] \\ = & ad(Tx_n, Tz_0) + bd(Sx_n, Tx_n) + cd(Sx_n, Tz_0) \\ & + (b + c)d(Sz_0, Tz_0), \end{aligned}$$

since $P \subset E$ is a normal cone with normal constant K, then we have

$$\begin{aligned} \|d(Sx_n, Sz_0)\| \leq K[& a\|d(Tx_n, Tz_0)\| + b\|d(Sx_n, Tx_n)\| + c\|d(Sx_n, Tz_0)\| \\ & + (b + c)\|d(Sz_0, Tz_0)\|]. \end{aligned}$$

Again, taking the limit as $n \to \infty$ we obtain,

$$\begin{aligned} \|d(z_0, Sz_0)\| \leq & K[a\|d(z_0, Tz_0)\| + b\|d(z_0, z_0)\| + c\|d(z_0, Tz_0)\| \\ & + (b + c)\|d(Sz_0, Tz_0)\|] \\ = & K(a + c)\|d(z_0, Tz_0)\|. \end{aligned}$$

As above, we conclude that $d(z_0, Sz_0) = 0$, which implies that $z_0 = Sz_0$ and thus we have proved that

$$Sz_0 = Tz_0 = z_0.$$

The uniqueness of the common fixed point z_0 follows from inequality (S). In fact, let us suppose that $y_0 = Sy_0 = Ty_0$. Then,

$$\begin{aligned} d(y_0, z_0) = & d(Sy_0, Sz_0) \leq ad(Ty_0, Tz_0) + b[d(Sy_0, Ty_0) + d(Sz_0, Tz_0)] \\ & + c[d(Sy_0, Tz_0) + d(Sz_0, Tz_0)] \\ = & (a + c)d(Sy_0, Sz_0) = (a + c)d(y_0, z_0). \end{aligned}$$

As before, the conclusion follows from the fact that $0 \leq a + c < 1$. Thus the theorem is proved. □

2.1 Some consequences of Theorem 2.1

In this section we are going to mention some results, which now can be obtained as a consequence of Theorem 2.1. First, notice that if in the Theorem 2.1 we take $E = \mathbb{R}_+$ and $P = [0, +\infty)$ (in which case $K = 1$) we obtain Theorem 1.1 for a pair (S, T) of mappings. Now if we take $b = c = 0$ in inequality (S), we obtain the following.

Corollary 2.2. *Let (M, d) be a complete CMS, and $P \subset E$ a normal cone with normal constant K. Let S and T be self-mappings of M such that,* *(a) T is continuous.*

(b) $S(M) \subset T(M)$.

(c) (S, T) is a commuting pair.

(d) The inequality

$$d(Sx, Sy) \leq ad(Tx, Ty) \qquad (J)$$

holds for all $x, y \in M$, where $0 \leq a < \frac{1}{K}$.

Then, S and T have a unique common fixed point.

In this case, if we take in the Corollary 2.2, $E = \mathbb{R}_+$ and $P = [0, +\infty)$, then we obtain the result given in 1976 by G. Jungck in [9]. On the other hand, if we consider $a = c = 0$ in (S), then we obtain the next result.

Corollary 2.3. *Let (M, d) be a complete CMS and $P \subset E$ a normal cone with normal constant K. Let S and T be self-mappings of M such that,*

(a) T is continuous.

(b) $S(M) \subset T(M)$.

(c) (S, T) is a commuting pair.

(d) The inequality

$$d(Sx, Sy) \leq b[d(Sx, Tx) + d(Sy, Ty)] \qquad (GKC)$$

is satisfies for all $x, y \in M$ and $0 \leq b < \frac{1}{2K}$.

Then, S and T have a unique common fixed point.

The mappings satisfying inequality (GKC) are called *generalized Kannan contractions*. If in the Corollary 2.3 we take $E = \mathbb{R}_+$, $P = [0, +\infty)$ and $T = Id$ (the identity mapping) we obtain the Kannan's result [10]. Finally, if we consider $a = b = 0$ in inequality (S), then we obtain the following result.

Corollary 2.4. *Let (M, d) be a complete CMS, and $P \subset E$ a normal cone with normal constant K. Let S and T be self-mappings of M such that,*

(a) T is continuous.

(b) $S(M) \subset T(M)$.

(c) (S, T) is a commuting pair.

(d) *The inequality*

$$d(Sx, Sy) \leq c[d(Sx, Ty) + d(Sy, Tx)] \qquad (GCC)$$

holds for all $x, y \in M$ *and* $0 \leq c < \frac{1}{2K}$.

Then, S and T have a unique common fixed point.

Notice that if in the Corollary 2.4, we take $E = \mathbb{R}_+$, $P = [0, +\infty)$ and $T = Id$, then we obtain the Chatterjea's results [6]. The mappings satisfying condition (GCC) are called *generalized Chatterjea contractions*.

3 Common fixed points for generalized Zamfrescu and weak contraction mappings on CMS

Using the ideas of T. Zamfrescu [18] (see also, [13]) we introduce the notion of Generalized Zamfirescu mappings (GZ0) in the framework of complete cone metric spaces.

Definition 3.1. Let (M, d) be a CMS, $P \subset E$ a normal cone with normal constant K and let $S, T : M \longrightarrow M$ be two mappings. The pair (S, T) is called generalized Zamfirescu mappings, (GZ0), if there are $0 \leq a < 1/K$, $0 \leq b < 1/2K$ and $0 \leq c < 1/2K$ such that for all $x, y \in M$, at least one of the next conditions is true:

(GZ01) $d(Sx, Sy) \leq ad(Tx, Ty)$.

(GZ02) $d(Sx, Sy) \leq b[d(Sx, Tx) + d(Sy, Ty)]$.

(GZ03) $d(Sx, Sy) \leq c[d(Sx, Ty) + d(Sy, Tx)]$.

Remark 3.2.

1. If in Definition 3.1 we take M a Banach space, $E = \mathbb{R}_+$ and $P = [0, +\infty)$, then we obtain the definition given by M. O. Olantinwo and C. O. Imoru [13].

2. If in Definition 3.1 we take $E = \mathbb{R}_+$, $P = [0, +\infty)$ and $T = Id$, then we get the Zamfirescu's definition [18].

From Definition 3.1 is immediate the following.

Proposition 3.1. *Let* (M, d) *be a CMS,* $P \subset E$ *a normal cone with normal constant* K *and* $S, T : M \longrightarrow M$ *a pair of* (GZ0)*, then we have*

(a) $d(Sx, Sy) \leq \delta d(Tx, Ty) + 2\delta d(Sx, Tx)$.

(b) $d(Sx, Sy) \leq \delta d(Tx, Ty) + 2\delta d(Sy, Tx)$

for all $x, y \in M$ *and where*

$$\delta = \max\left\{a,\ \frac{b}{1-b},\ \frac{c}{1-c}\right\}, \quad 0 \leq \delta < 1.$$

The following result generalizes the well–known theorem given by T. Zamfirescu in [18].

Theorem 3.2. *Let* (M, d) *be a complete CMS, and* $P \subset E$ *a normal cone with normal constant* K. *Suppose that* $S, T : M \longrightarrow M$ *are* (GZ0) *such that,*

(a) T *is continuous.*

(b) $S(M) \subset T(M)$.

(c) (S, T) *is a commuting pair.*

Then, S *and* T *have a unique common fixed point.*

Proof: Since the pair (S, T) is a (GZ0), then by Proposition 3.1 (b) we have that

$$d(Sx, Sy) \leq \delta d(Tx, Ty) + 2\delta d(Sy, Tx), \qquad \forall x, y \in M,$$

where

$$1 > \delta = \max\left\{a,\ \frac{b}{1-b},\ \frac{c}{1-c}\right\}.$$

Suppose that $x_0 \in M$ is an arbitrary point. We are going to prove that the (S, T)-sequence $S(x_n)$ with initial point x_0 is a Cauchy sequence in M.

Notice that

$$\begin{aligned} d(Sx_{n+1}, Sx_n) \leq& \delta d(Tx_{n+1}, T_n x) + 2\delta d(Sx_n, Tx_{n+1}) \\ =& \delta d(Sx_n, Sx_{n-1}) + 2\delta d(Sx_n, Sx_n) \\ =& \delta d(Sx_n, Sx_{n-1}). \end{aligned}$$

Therefore, from Lemma 1.3 we conclude that $S(x_n)$ is a Cauchy sequence. Repeating the procedure above, we get

$$d(Sx_{n+1}, Sx_n) \leq \delta^n d(Sx_1, Sx_0).$$

Thus, taking norm we obtain

$$\|d(Sx_{n+1}, Sx_n)\| \leq \delta^n K \|d(Sx_1, Sx_0)\|,$$

therefore, as in the proof of Theorem 2.1, taking the limit as $n \to \infty$ and using the fact that M is a complete CMS, we guarantee the existence of a $z_0 \in M$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_{n+1} = z_0. \tag{3.3}$$

By the continuity of T and the commuting property of T and S, we have

$$Tz_0 = \lim_{n\to\infty} T^2 x_n \tag{3.4}$$

$$Tz_0 = \lim_{n\to\infty} STx_n. \tag{3.5}$$

On the other hand, using Proposition 3.1 (a), we get

$$d(STx_n, Sz_0) \leq \delta d(T^2x_n, Tz_0) + 2\delta d(STx_n, T^2x_n).$$

Taking the norm and the limit as $n \to \infty$ to the above inequality we have

$$\lim_{n\to\infty} \|d(STx_n, Sz_0)\| \leq K \lim_{n\to\infty} [\delta\|d(T^2x_n, Tz_0)\| + 2\delta\|d(STx_n, T^2x_n)\|].$$

From (3.4) and (3.5) we conclude that,

$$d(Tz_0, Sz_0) = 0,$$

that is, $Tz_0 = Sz_0$.

Now, using Proposition 3.1 (a), the following inequality holds,

$$d(Sx_n, Sz_0) \leq \delta d(Tx_n, Tz_0) + 2\delta d(Sx_n, Tx_n).$$

Repeating the argument above we obtain

$$\lim_{n\to\infty} \|d(Sx_n, Sz_0)\| \leq K \lim_{n\to\infty} [\delta\|d(Tx_n, Tz_0)\| + 2\delta\|d(Sx_n, Tx_n)\|],$$

using (3.3) and the fact that $Tz_0 = Sz_0$ we conclude that

$$\|d(z_0, Sz_0)\| \leq K\delta\|d(z_0, Sz_0)\|.$$

Since $K\delta < 1$, then $z_0 = Sz_0$.

Finally, we are going to prove the uniqueness of the fixed point. Let us suppose that $y_0 \in M$ is such that $y_0 = Sy_0 = Ty_0$. Then

$$\begin{aligned} d(z_0, y_0) = d(Sz_0, Sy_0) \leq& \delta d(Tz_0, Ty_0) + 2\delta d(Sz_0, Tz_0) \\ =& \delta d(z_0, y_0). \end{aligned}$$

From the fact that $\delta < 1$, we conclude that $z_0 = y_0$. Thus the theorem is proved. $\square$

Now we introduce an equivalent notion of (GZ0) in CMS as follows.

Definition 3.3. Let (M,d) be a CMS, $P \subset E$ a normal cone with normal constant K and $S,T: M \longrightarrow M$ be mappings for which there exists $0 \leq h < 1/K$ such that for all $x,y \in M$

$$d(Sx,Sy) \leq h \max\left\{d(Tx,Ty), \frac{d(Sx,Tx)+d(Sy,Ty)}{2}, \frac{d(Sx,Ty)+d(Sy,Tx)}{2}\right\}. \quad (3.6)$$

It is not difficult to see that Definitions 3.1 and 3.3 are equivalent. Therefore, our results remain valid for mappings satisfying (3.6) as well.

In 2003, V. Berinde, ([4] and [5]), introduced a new class of contraction mappings on metric spaces, which are called *weak contraction*. Now we will extend this kind of contractive condition to a pair of mappings in the setting of CMS.

Definition 3.4. Let (M,d) be a CMS, $P \subset E$ a normal cone with normal constant K and $S,T: M \longrightarrow M$ two mappings. The pair (S,T) is called a generalized weak contraction, (GWC) if there exist constants $0 < \delta < 1$ and $L \geq 0$ such that

$$d(Sx,Sy) \leq \delta d(Tx,Ty) + Ld(Sx,Tx)$$

for all $x,y \in M$.

The next proposition gives examples of (GWC). This can be proved in a similar form as Proposition 3.3 of [12].

Proposition 3.3. *Let (M,d) be a CMS, $P \subset E$ a normal cone with normal constant K and $S,T: M \longrightarrow M$ two mappings. Then,*

(a) If the pair (S,T) satisfies the condition (J), then (S,T) is a (GWC).

(b) If the pair (S,T) is a (GKC)*, then (S,T) is a* (GWC).

(c) If the pair (S,T) is a (GCC)*, then (S,T) is a* (GWC).

(d) If the pair (S,T) is a (GZ0)*, then (S,T) is a* (GWC).

The next result can be proved by following the procedure used in the proof of Theorem 3.2.

Theorem 3.4. *Let (M,d) be a complete CMS, $P \subset E$ a normal cone with normal constant K and $S,T: M \longrightarrow M$ two mappings. If the pair (S,T) is a* (GWC) *such that,*

(a) T is continuous.

(b) $S(M) \subset T(M)$.

(c) (S, T) *is a commuting pair.*

Then, S *and* T *have a unique common fixed point.*

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