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On non-commutative Minkowski spheres

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Abstract

The purpose of the following is to try to make sense of the stereographic projection in a non-commutative setup. To this end, we consider the open unit ball of a ternary ring of operators, which naturally comes equipped with a non-commutative version of a hyperbolic metric and ask for a manifold onto which the open unit ball can be mapped so that one might think of this situation as providing a noncommutative analog to mapping the open disk of complex numbers onto the hyperboloid in three space, equipped with the restriction of the Minkowskian metric. We also obtain a related result on the Jordan algebra of self-adjoint operators.

1 Introduction

By definition, the classical Minkowski sphere is the set

$$\mathbf{M} = \mathbf{M}(\mathbb{R}^4) := \big\{(t,x,y,z) \in \mathbb{R}^4 :\ t^2 - (x^2+y^2+z^2) = 1,\ t>0\big\}.$$

It is straightforward to verify that the Hilbert ball

$$\mathbf{B} = \mathbf{B}(\mathbb{R}^3) := \big\{(a_1,a_2,a_3) \in \mathbb{R}^3 :\ a_1^2+a_2^2+a_3^2 = 1\big\}$$

is mapped injectively onto $\mathbf{M}$ by the transformation

$$\Phi(\mathbf{a}) = \Phi(a_1,a_2,a_3) := \frac{1}{1-(a_1^2+a_2^2+a_3^2)}\Big(1+a_1^2+a_2^2+a_3^2, 2a_1, 2a_2, 2a_3\Big).$$

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Namely, we have

$$\Phi : \mathbf{B} \leftrightarrow \mathbf{M}, \qquad \Phi^{-1}(t, \mathbf{x}) = (1+t)^{-1}\mathbf{x} \quad \text{at} \quad (t, \mathbf{x}) \in \mathbf{M}.$$

Notice that, by identifying $\mathbb{R}^3$ with $\mathbf{E} := \mathrm{Mat}(1,3,\mathbb{R})$ the set of all row 3-vectors and $\mathbb{R}^4$ with $\mathbb{R} \times \mathbf{E} \equiv \mathrm{Mat}(1,1,\mathbb{R}) \times \mathrm{Mat}(1,3,\mathbb{R})$, respectively, in matrix terms we can write $\Phi(\mathbf{a}) = \big(\Phi_0(\mathbf{a}), \Phi_1(\mathbf{a})\big)$ where

$$\Phi_0(\mathbf{a}) = (1-\mathbf{a}\mathbf{a}^*)^{-1}(1+\mathbf{a}\mathbf{a}^*), \quad \Phi_1(\mathbf{a}) = 2(1-\mathbf{a}\mathbf{a}^*)^{-1}\mathbf{a}. \tag{1.1}$$

It is a more interesting fact that Φ lifts the natural hyperbolic geometry of $\mathbf{B}$ to $\mathbf{M}$ in a manner such that vector fields corresponding to hyperbolic translation flows of $\mathbf{B}$ will be mapped to restrictions of $\mathbb{R}^4$-vector fields to $\mathbf{M}$ depending linearly on the coordinates $t, \mathbf{x}$ and the 3×3-matrix

$$\widetilde{t} := (1+\mathbf{a}^*\mathbf{a})(1+\mathbf{a}^*\mathbf{a})^{-1} \quad \text{at} \quad (t, \mathbf{x}) = \Phi(\mathbf{a})$$

of a non-commutative time. That is, for the vector fields

$$v_{\mathbf{u}}(\mathbf{a}) := \mathbf{u} - \mathbf{a}\mathbf{u}^*\mathbf{a} \qquad \big(\mathbf{a} \in \mathbf{B},\ \mathbf{u} \in \mathbf{E}\big) \tag{1.2}$$

we get

$$\begin{aligned} \big[\Phi^{\#} v_{\mathbf{u}}\big](t, \mathbf{x}) \quad &:= \quad \frac{d}{d\tau}\Big|_{\tau=0} \Phi\Big(\Phi^{-1}(t,\mathbf{x}) + \tau v_{\mathbf{u}}\big(\Phi^{-1}(t,\mathbf{x})\big)\Big) = \\ &= \quad \Big(2\mathbf{u}\mathbf{x}^*, t\mathbf{u} + \mathbf{u}\widetilde{t}\Big) \quad \text{at} \quad (t, \mathbf{x}) = \Phi(\mathbf{a}). \end{aligned}$$

The appearance of the non-commutative time term suggests that we should regard an embedding of $\mathbf{B}$ instead of $\mathrm{Mat}(1,1) \times \mathrm{Mat}(1,3)$ into $\widehat{\mathbf{E}} := \mathrm{Mat}(1,1) \times \mathrm{Mat}(3,3) \times \mathrm{Mat}(1,3) \times \mathrm{Mat}(3,1)$ by the mapping

$$\begin{aligned} &\widehat{\Phi}(\mathbf{a}) := \Big(\Phi_0(\mathbf{a}), \widetilde{\Phi}_0(\mathbf{a}), \Phi_1(\mathbf{a}), \widetilde{\Phi}_1(\mathbf{a})\Big); \\ &\widetilde{\Phi}_0(\mathbf{a}) := \widetilde{t}(\mathbf{a}) = (1+\mathbf{a}^*\mathbf{a})(1+\mathbf{a}^*\mathbf{a})^{-1}, \\ &\widetilde{\Phi}_1(\mathbf{a}) := \Phi_1(\mathbf{a})^* = 2\mathbf{a}^*(1-\mathbf{a}\mathbf{a}^*)^{-1} = 2(1-\mathbf{a}^*\mathbf{a})^{-1}\mathbf{a}^*\ . \end{aligned} \tag{1.3}$$

In this way, the lifted fields $\widehat{\Phi}^{\#} v_{\mathbf{u}}$ automatically become the restriction of a real linear vector on $\widehat{\mathbf{M}} := \mathrm{ran}(\widehat{\Phi})$ to a real-linear vector field of $\widehat{\mathbf{E}}$, since

$$\begin{aligned} &\big[\widehat{\Phi}^{\#} v_{\mathbf{u}}\big](t, \widetilde{t}, \mathbf{x}, \widetilde{\mathbf{x}}) = \Big(\mathbf{u}\mathbf{x}^* + \mathbf{x}\mathbf{u}^*, \mathbf{u}^*\mathbf{x} + \mathbf{x}^*\mathbf{u}, t\mathbf{u} + \mathbf{u}\widetilde{t}, \mathbf{u}^* t + \widetilde{t}\mathbf{u}^*\Big) \\ &\text{if} \quad (t, \widetilde{t}, \mathbf{x}, \widetilde{\mathbf{x}}) = \widehat{\Phi}(\mathbf{a}), \quad \mathbf{a} \in \mathbf{B}. \end{aligned} \tag{1.4}$$

Our purpose in this note is to generalize the above considerations to the setting of *ternary rings of operators* (TRO in the sequel). As a by-product of our main theorem, we obtain a result of possible independent interest concerning the Jordan algebra of self-adjoint operators.

2 Results

Henceforth $\mathbf{H},\mathbf{K}$ will stand for two arbitrarily fixed real or complex Hilbert spaces and $\mathbf{E}$ denotes a TRO in $\mathcal{L}(\mathbf{H},\mathbf{K})\big(=\{\text{bounded linear operators } \mathbf{H}\to\mathbf{K}\}\big)$. That is $\mathbf{E}\subset\mathcal{L}(\mathbf{H},\mathbf{K})$ is a closed linear subspace such that $\big[\mathbf{abc}\big]:=\mathbf{ab}^*\mathbf{c}\in\mathbf{E}$ whenever $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbf{E}$. We write

$$\mathcal{A}(\mathbf{E}):=\big\{t\in\mathcal{L}(\mathbf{K}):\ t=t^*,\ t\mathbf{E}\subset\mathbf{E}\big\},\quad \widetilde{\mathcal{A}}(\mathbf{E}):=\big\{\widetilde{t}\in\mathcal{L}(\mathbf{H}):\ \widetilde{t}=\widetilde{t}^*,\ \mathbf{E}\widetilde{t}\subset\mathbf{E}\big\}$$

and, by setting also $\widetilde{\mathbf{E}}:=\mathbf{E}^*=\{\mathbf{z}^*:\ \mathbf{z}\in\mathbf{E}\}\subset\mathcal{L}(\mathbf{K},\mathbf{H})$, we define the operator $\widehat{\Phi}:\mathbf{B}:=\{\mathbf{a}\in\mathbf{E}:\ \|\mathbf{a}\|<1\}\to\widehat{\mathbf{E}}:=\mathcal{A}(\mathbf{E})\times\widetilde{\mathcal{A}}(\mathbf{E})\times\mathbf{E}\times\widetilde{\mathbf{E}}$ ranging in the linking algebra [2] by (1.1) and (1.3). Indeed $\mathbf{xx}^*\in\mathcal{A}(\mathbf{E})$ and $\mathbf{x}^*\mathbf{x}\in\widetilde{\mathcal{A}}(\mathbf{E})$ for any $\mathbf{x}\in\mathbf{E}$ whence, with norm-convergence, also

$$\begin{aligned}\Phi_0(\mathbf{a}) &= 1_{\mathbf{K}}+2\sum_{n=1}^{\infty}(\mathbf{aa}^*)^n\in\mathcal{A}(\mathbf{E}),\quad \widetilde{\Phi}_0(\mathbf{a})=1_{\mathbf{H}}+2\sum_{n=1}^{\infty}(\mathbf{a}^*\mathbf{a})^n\in\widetilde{\mathcal{A}}(\mathbf{E}),\\ \Phi_1(\mathbf{a}) &= 2\sum_{n=0}^{\infty}(\mathbf{aa}^*)^n\mathbf{a}=\big[1_{\mathbf{K}}+\Phi_0(\mathbf{a})\big]\mathbf{a}= \qquad (2.1)\\ &= 2\sum_{n=0}^{\infty}\mathbf{a}(\mathbf{a}^*\mathbf{a})^n=\mathbf{a}\big[1_{\mathbf{H}}+\widetilde{\Phi}_0(\mathbf{a})\big]\in\mathbf{E}\quad\text{for any }\ \mathbf{a}\in\mathbf{B}.\end{aligned}$$

Let us finally define

$$\begin{aligned}\widehat{\mathbf{M}} := \Big\{&(t,\widetilde{t},\mathbf{x},\widetilde{\mathbf{x}})\in\widehat{\mathbf{E}}:\ t\in\mathcal{A}_+(\mathbf{E}),\ t^2-\mathbf{xx}^*=1_{\mathbf{K}},\ \widetilde{\mathbf{x}}=\mathbf{x}^*,\\ &\widetilde{t}\in\widetilde{\mathcal{A}}_+(\mathbf{E}),\ \widetilde{t}^2-\widetilde{\mathbf{x}}^*\widetilde{\mathbf{x}}=1_{\mathbf{H}},\ (1_{\mathbf{K}}+t)^{-1}\mathbf{x}=\mathbf{x}(1_{\mathbf{H}}+\widetilde{t})^{-1}\Big\}.\end{aligned}$$

Our main result reads as follows.

2.2 Theorem. *In the TRO-setting established above, we have* $\widehat{\Phi}:\mathbf{B}\leftrightarrow\widehat{\mathbf{M}}$ *with*

$$\widehat{\Phi}^{-1}(t,\widetilde{t},\mathbf{x},\widetilde{\mathbf{x}})=(1_{\mathbf{K}}+t)^{-1}\mathbf{x}=\widetilde{\mathbf{x}}^*(1_{\mathbf{H}}+\widetilde{t})^{-1},\qquad\Big((t,\widetilde{t},\mathbf{x},\widetilde{\mathbf{x}})\in\widehat{\mathbf{M}}\Big).$$

The vector fields $v_{\mathbf{u}}$ *of infinitesimal hyperbolic parallel shifts on* $\mathbf{B}$ *defined by* (1.2) *are lifted to restrictions of linear maps on* $\widehat{\mathbf{M}}$ *of the form* (1.4).

As it is well-known [3], the integration of a vector field $v_{\mathbf{u}}$ provides the flow $\big[M_{\mathbf{u}}^{\tau}:\ \tau\in\mathbb{R}\big]$ of Potapov-Möbius transformations

$$\begin{aligned}M_u^{\tau}(\mathbf{a}): &= (1_{\mathbf{K}}-\mathbf{u}_\tau\mathbf{u}_\tau^*)^{-1/2}(\mathbf{a}+\mathbf{u}_\tau)(1_{\mathbf{H}}+\mathbf{u}_\tau^*\mathbf{a})^{-1}(1_{\mathbf{H}}-\mathbf{u}_\tau^*\mathbf{u}_\tau)^{1/2}=\\ &= (1_{\mathbf{K}}-\mathbf{u}_\tau\mathbf{u}_\tau^*)^{-1/2}(1_{\mathbf{K}}+\mathbf{au}_\tau^*)^{-1}(\mathbf{a}+\mathbf{u}_\tau)(1_{\mathbf{H}}-\mathbf{u}_\tau^*\mathbf{u}_\tau)^{1/2},\quad(\mathbf{a}\in\mathbf{B})\end{aligned}$$

where, in terms of Kaup's odd functional calculus [1],

$$\mathbf{u}_\tau := \tanh(\tau\mathbf{u}) = \sum_{n=0}^{\infty} \alpha_n \tau^{2n+1}(\mathbf{u}\mathbf{u}^*)^n \mathbf{u} = \sum_{n=0}^{\infty} \alpha_n \tau^{2n+1} \mathbf{u}(\mathbf{u}^*\mathbf{u})^n$$

with the constants $\alpha_0, \alpha_1, \ldots \in \mathbb{R}$ of the expansion $\tanh(\xi) = \sum_{n=0}^{\infty} \alpha_n \xi^{2n+1}$.

On the other hand, linear vector fields are integrated simply by taking the exponentials of their multiples with the virtual time parameter τ. Taking into account that (1.4) can be written in the matrix form $\widehat{\Phi}^\# v_{\mathbf{u}} : \widehat{\mathbf{M}} \ni (t, \widetilde{t}, \mathbf{x}, \widetilde{\mathbf{x}}) \mapsto (t, \widetilde{t}, \mathbf{x}, \widetilde{\mathbf{x}})\mathbf{L}_{\mathbf{u}}$ with

$$\mathbf{L}_{\mathbf{u}} := \begin{bmatrix} 0 & 0 & R(\mathbf{u}) & L(\mathbf{u}^*) \\ 0 & 0 & L(\mathbf{u}) & R(\mathbf{u}^*) \\ R(\mathbf{u}^*) & L(\mathbf{u}^*) & 0 & 0 \\ L(\mathbf{u}) & R(\mathbf{u}) & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & S(\mathbf{u}) \\ S(\mathbf{u}^*) & 0 \end{bmatrix}$$

where $L(\cdot)$ and $R(\cdot)$ denote left and right multiplication as usually, we get the following.

2.3 Corollary. $M_{\mathbf{u}}^\tau(\mathbf{a}) = \widehat{\Phi}^{-1}\Big(\widehat{\Phi}(\mathbf{a}) \exp\big(\tau \mathbf{L}_{\mathbf{u}}\big)\Big), \quad (\tau \in \mathbb{R},\ \mathbf{a} \in \mathbf{B}).$

Let us restrict ourselves to the case $\mathbf{E} = \mathcal{L}(\mathbf{H})\big(= \mathcal{L}(\mathbf{H}, \mathbf{H})\big)$ and consider the behavior of $\widehat{\Phi}$ on the unit ball $\mathbf{B}^{(s)}$ of the self-adjoint part $\mathcal{L}^{(s)}(\mathbf{H}) := \{\mathbf{a} \in \mathcal{L}(\mathbf{H}) :\ \mathbf{a} = \mathbf{a}^*\}$. Then $\phi_0(\mathbf{a}) = \widetilde{\phi}_0(\mathbf{a}) = (1_{\mathbf{H}} + \mathbf{a}^2)(1_{\mathbf{H}} - \mathbf{a}^2)^{-1} \in \mathcal{L}^{(s)}(\mathbf{H})$ and $\phi_1(\mathbf{a}) = \widetilde{\phi}_1(\mathbf{a}) = 2a(1_{\mathbf{H}} - \mathbf{a}^2)^{-1} \in \mathcal{L}^{(s)}(\mathbf{H})$. From (1.4) we see also that

$$\big[\widehat{\Phi}^\# v_{\mathbf{u}}\big](t, \mathbf{x}, t, \mathbf{x}) = 2\big(\mathbf{x}\bullet\mathbf{u}, \mathbf{x}\bullet\mathbf{u}, t\bullet\mathbf{u}, t\bullet\mathbf{u}\big) \quad \text{if} \quad (t, \mathbf{x}, t, \mathbf{x}) = \widehat{\Phi}(\mathbf{a}), \quad \mathbf{a} \in \mathbf{B}^{(s)} \tag{2.4}$$

in terms of the Jordan product $\mathbf{x}\bullet\mathbf{y} := \frac{1}{2}(\mathbf{x}\mathbf{y} + \mathbf{y}\mathbf{x})$ on $\mathcal{L}^{(s)}(\mathbf{H})$. We get the following explicit linear representation for the Jordan manifold structure of the unit ball of $\mathcal{L}^{(s)}(\mathbf{H})$ discussed in Theorem 2.6 of our paper [4].

2.5 Corollary. *For the transformation* $\Phi := [\Phi_0, \Phi_1]$ *we have* $\Phi : \mathbf{B}^{(s)} \leftrightarrow \mathbf{M}^{(s)} := \{(t, \mathbf{x}) \in \mathcal{L}^{(s)}(\mathbf{H})^2 :\ t \geq 0,\ t^2 - \mathbf{x}^2 = 1_{\mathbf{H}}\}$. *The Möbius transformations* $M_{\mathbf{u}}^\tau$ ($\mathbf{u} \in \mathcal{L}^{(s)}(\mathbf{H})$ *map* $\mathbf{B}^{(s)}$ *onto itself and, in terms of the Jordan multiplication* $J(\mathbf{u}) := \frac{1}{2}[L(\mathbf{u}) + R(\mathbf{u})]$,

$$\begin{aligned} M_{\mathbf{u}}^\tau(\mathbf{a}) &= \Phi^{-1}\left(\Phi(\mathbf{a}) \exp\left(2\tau \begin{bmatrix} J(\mathbf{u}) & 0 \\ 0 & J(\mathbf{u}) \end{bmatrix}\right)\right) = \\ &= \Phi^{-1}\begin{pmatrix} \frac{1}{2}\Big(e^{\tau\mathbf{u}}[\phi_0(\mathbf{a}) + \phi_1(\mathbf{a})]e^{\tau\mathbf{u}} + e^{\tau\mathbf{u}}[\phi_0(\mathbf{a}) - \phi_1(\mathbf{a})]e^{\tau\mathbf{u}}, \\ e^{\tau\mathbf{u}}[\phi_0(\mathbf{a}) + \phi_1(\mathbf{a})]e^{\tau\mathbf{u}} - e^{\tau\mathbf{u}}[\phi_0(\mathbf{a}) - \phi_1(\mathbf{a})]e^{\tau\mathbf{u}}\Big) \end{pmatrix}. \end{aligned}$$

3 Proof of Theorem 2.2

Theorem 2.2 is an immediate consequence of the following substatements.

3.1 Lemma. *The component Φ_1 of Φ is injective. Moreover $\Phi_1 : \mathbf{B} \leftrightarrow \mathbf{E}$ with*

$$\Phi_1^{-1}(\mathbf{c}) = \Big[1_{\mathbf{K}} + \sqrt{1_{\mathbf{K}} + \mathbf{c}\mathbf{c}^*}\Big]^{-1}\mathbf{c} = \mathbf{c}\Big[1_{\mathbf{H}} + \sqrt{1_{\mathbf{H}} + \mathbf{c}^*\mathbf{c}}\Big]^{-1}, \qquad (\mathbf{c} \in \mathbf{E}).$$

3.2 Lemma. *For any $\mathbf{a} \in \mathbf{B}$, $\phi_0(\mathbf{a})^2 - \phi_1(\mathbf{a})\phi_1(\mathbf{a})^* = 1_{\mathbf{K}}$ and $\widetilde{\phi}_0(\mathbf{a})^2 - \phi_1(\mathbf{a})^*\phi_1(\mathbf{a}) = 1_{\mathbf{H}}$.*

3.3 Lemma. *Let $\mathbf{x} \in \mathbf{E}$, $t \in \mathcal{L}_+(\mathbf{K})$ and $\widetilde{t} \in \widetilde{\mathcal{L}}_+(\mathbf{H})$ be so given that $t^2 - \mathbf{x}\mathbf{x}^* = 1_{\mathbf{K}}$ and $\widetilde{t}^2 - \mathbf{x}^*\mathbf{x} = 1_{\mathbf{H}}$. Then $t \in \mathcal{A}_+(\mathbf{E})$, $\widetilde{t} \in \mathcal{A}_+(\widetilde{\mathbf{E}}) = \big(\mathcal{A}_+(\widetilde{\mathbf{E}}) := \mathbf{E}^*\big)$ and $(1_{\mathbf{K}} + t)^{-1}\mathbf{x} = \mathbf{x}(1_{\mathbf{H}} + \widetilde{t})^{-1} \in \mathbf{B}$. By writing $\mathbf{a} := (1_{\mathbf{K}} + t)^{-1}\mathbf{x}$ for the common value, we have $t = \Phi_0(\mathbf{a})$, $\widetilde{t} = \widetilde{\Phi}_0(\mathbf{a})$, $\mathbf{x} = \phi_1(\mathbf{a})$, $\mathbf{x}^* = \widetilde{\phi}_1(\mathbf{a})$.*

3.4 Proposition. *Let $\mathbf{M} := \big\{(t,\mathbf{x}) \in \mathcal{A}_+(\mathbf{E}) \times \mathbf{E} : \ t^2 - \mathbf{x}\mathbf{x}^* = 1_{\mathbf{K}}\big\}$ and let $\mathbf{u} \in \mathbf{E}$ be fixed arbitrarily. Then the submap $\Phi := [\Phi_0, \Phi_1]$ of $\widehat{\Phi}\big(= [\Phi_0, \widetilde{\Phi}_0, \Phi_1, \widetilde{\Phi}_1]\big)$ lifts the vector field $v_{\mathbf{u}}$ to $(t,\mathbf{x}) \mapsto \Big(\mathbf{u}\mathbf{x}^* + \mathbf{x}\mathbf{u}^*, t\mathbf{u} + \mathbf{u}\widetilde{t}\Big)$ with $\widetilde{t} := \sqrt{1_{\mathbf{H}} + \mathbf{x}^*\mathbf{x}}$ on $\mathbf{M}$. That is, given $(t,\mathbf{x}) \in \mathbf{M}$ and, by setting $\mathbf{a} := \Phi^{-1}(t,\mathbf{x}) = (1_{\mathbf{K}} + t)^{-1}\mathbf{x}$, we have*

$$\big[\Phi^{\#} v_{\mathbf{u}}\big](t,\mathbf{x}) = \frac{d}{d\tau}\Big|_{\tau=0} \Phi\big(\mathbf{a} + \tau(\mathbf{u} - \mathbf{a}\mathbf{u}^*\mathbf{a})\big) = \Big(\mathbf{u}\mathbf{x}^* + \mathbf{x}\mathbf{u}^*, t\mathbf{u} + \mathbf{u}\widetilde{t}\Big).$$

3.5 Corollary. *If $\widetilde{\mathbf{M}} := \big\{(\widetilde{t}, \widetilde{\mathbf{x}}) \in \widetilde{\mathcal{A}}_+(\mathbf{E}^*) \times \mathbf{E}^* : \ \widetilde{t}^2 - \widetilde{\mathbf{x}}\widetilde{\mathbf{x}}^* = 1_{\mathbf{H}}\big\}$ and $\mathbf{u} \in \mathbf{E}$*

is arbitrarily fixed then the submap $\widetilde{\Phi} := [\widetilde{\Phi}_0, \widetilde{\Phi}_1]$ of $\widehat{\Phi}$ lifts the vector field $v_{\mathbf{u}}$ to $(\widetilde{t}, \widetilde{\mathbf{x}}) \mapsto \Big(\mathbf{u}^*\widetilde{\mathbf{x}}^* + \widetilde{\mathbf{x}}\mathbf{u}, \widetilde{t}\mathbf{u}^* + \mathbf{u}^* t\Big)$ with $t := \sqrt{1_K + \widetilde{\mathbf{x}}^*\widetilde{\mathbf{x}}}$ to $\widetilde{\mathbf{M}}$. That is, given $(\widetilde{t}, \widetilde{\mathbf{x}}) \in \widetilde{\mathbf{M}}$ and, by setting $\mathbf{a} := \widetilde{\Phi}^{-1}(\widetilde{t}, \widetilde{\mathbf{x}}) = \widetilde{\mathbf{x}}^*(1_{\mathbf{H}} + \widetilde{t})^{-1}$, we have

$$\big[\widetilde{\Phi}^{\#} v_{\mathbf{u}}\big](\widetilde{t}, \widetilde{\mathbf{x}}) = \frac{d}{d\tau}\Big|_{\tau=0} \widetilde{\Phi}\big(\mathbf{a} + \tau(\mathbf{u} - \mathbf{a}\mathbf{u}^*\mathbf{a})\big) = \Big(\mathbf{u}^*\widetilde{\mathbf{x}}^* + \widetilde{\mathbf{x}}\mathbf{u}, \widetilde{t}\mathbf{u}^* + \mathbf{u}^* t\Big).$$

Proof of 3.1. Given any $\mathbf{c} \in \mathbf{E}$ let $t_0(\mathbf{c}) := \psi_0(\mathbf{c}\mathbf{c}^*)$ with the continuous real function $\psi_0(\tau) := 1 + \sqrt{1 + \tau}$. By the Spectral Mapping Theorem, $\mathrm{Sp}(t_0(\mathbf{c})) = \psi_0\big(\mathrm{Sp}(\mathbf{c}\mathbf{c}^*)\big) > 0$. Hence $t(\mathbf{c}) : 2\big[1_{\mathbf{K}} + \sqrt{1 + \mathbf{c}\mathbf{c}^*}\big] = 2t_0(\mathbf{c})^{-1}$ is well-defined and, by Sinclair's Theorem*, $\|t(\mathbf{c})\mathbf{c}\|^2 = \|t(\mathbf{c})\mathbf{c}\mathbf{c}^* t(\mathbf{c})\| = \max\big\{\psi(\tau)^2\tau :\ \tau \in \mathrm{Sp}(\mathbf{c}\mathbf{c}^*)\big\} \leq 4\|\mathbf{c}\|^2/[1+\sqrt{1+\|\mathbf{c}\|^2}\,]^2 < 1$. To see that $t(\mathbf{c})\mathbf{c} \in \mathbf{E}$ and hence also $\in \mathbf{B}$, notice that, by Weierstrass' Approximation Theorem, there is a sequence $\pi_1, \pi_2, \ldots$ of real polynomials converging uniformly to ψ on $\mathrm{Sp}(\mathbf{c}\mathbf{c}^*)$. By Sinclair's Theorem again, $\pi_n(\mathbf{c}\mathbf{c}^*) \to t(\mathbf{c})$ in norm. However $\mathbf{c}\mathbf{c}^* \in \mathcal{A}(\mathbf{E})$, whence also $\pi_n(\mathbf{c}\mathbf{c}^*) \in \mathcal{A}(\mathbf{E})$ entailing $t(\mathbf{c}) \in \mathcal{A}(\mathbf{E})$ and $t(\mathbf{c})\mathbf{c} \in \mathbf{B}$.

To complete the proof, we show that $t\big(\phi_0(\mathbf{a})\big)\phi_0(\mathbf{a}) = \mathbf{a}$ for any $\mathbf{a} \in \mathbf{B}$. Given $\mathbf{a}\in\mathbf{B}$, we have $1_\mathbf{K} + \phi_0(\mathbf{a})\phi_0(\mathbf{a})^* = 1_\mathbf{K} + 4(1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-1}\mathbf{a}\mathbf{a}^*(1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-1} = (1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-2}\big[(1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^2 + 4\mathbf{a}\mathbf{a}^*\big] = (1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-2}(1_\mathbf{K} + \mathbf{a}\mathbf{a}^*)^2$. It follows $1_\mathbf{K} + \sqrt{1_\mathbf{K} + \phi_0(\mathbf{a})\phi_0(\mathbf{a})^*} = (1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-1}\big[(1_\mathbf{K} - \mathbf{a}\mathbf{a}^*) + 1_\mathbf{K} + \mathbf{a}\mathbf{a}^*\big] = 2(1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-1}$ entailing $t\big(\phi_0(\mathbf{a})\big)\phi_0(\mathbf{a}) = \frac{1}{2}(1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)\big[2(1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-1}\mathbf{a}\big] = \mathbf{a}$.

Proof of 3.2. Given any operator $\mathbf{a} \in \mathbf{B}$, we have

$$\begin{aligned}
&\phi_0(\mathbf{a})^2 - \phi_1(\mathbf{a})\phi_1(\mathbf{a})^* = \\
=\ & (1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-1}(1_\mathbf{K} + \mathbf{a}\mathbf{a}^*)^2(1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-1} - (1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-1}(4\mathbf{a}\mathbf{a}^*)(1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-1} = \\
=\ & (1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-1}\big[1_\mathbf{K} + 2\mathbf{a}\mathbf{a}^* + (\mathbf{a}\mathbf{a}^*)^2 - 4\mathbf{a}\mathbf{a}^*\big](1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-1} = \\
=\ & (1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-1}\big[1_\mathbf{K} - 2\mathbf{a}\mathbf{a}^* + (\mathbf{a}\mathbf{a}^*)^2\big](1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-1} = \\
=\ & (1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-1}\big[1_\mathbf{K} - 2\mathbf{a}\mathbf{a}^* + (\mathbf{a}\mathbf{a}^*)^2\big](1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-1} = \\
=\ & (1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-1}(1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^2(1_\mathbf{K} - \mathbf{a}\mathbf{a}^*)^{-1} = 1_\mathbf{K}\ .
\end{aligned}$$

The proof of the relationship $\widetilde{\phi}_0(\mathbf{a})^2 - \phi_1(\mathbf{a})^*\phi_1(\mathbf{a}) = 1_\mathbf{H}$ is analogous with terms $\mathbf{a}^*\mathbf{a}$ replacing $\mathbf{a}\mathbf{a}^*$ and $1_\mathbf{H}$ instead of $1_\mathbf{K}$.

Proof of 3.3. Since $\mathbf{x}\mathbf{x}^* \in \mathcal{A}_+(\mathbf{E}$, by Sinclair's and Weierstrass' Theorems (as in the proof of 3.1), $t = \sqrt{1_\mathbf{K} + \mathbf{x}\mathbf{x}^*} \in \mathcal{A}_+(\mathbf{E})$. Similarly $\widetilde{t} = \sqrt{1_\mathbf{H} + \mathbf{x}^*\mathbf{x}} \in \mathcal{A}_+(\widetilde{\mathbf{E}})$. By Lemma 3.1, $(1_\mathbf{K} + t)^{-1}\mathbf{x} = \big[1_\mathbf{K} + \sqrt{1_\mathbf{K} + \mathbf{x}\mathbf{x}^*}\big]^{-1}\mathbf{x} = \Phi_1^{-1}(\mathbf{x}) = \mathbf{x}\big[1_\mathbf{H} + \sqrt{1_\mathbf{H} + \mathbf{x}\mathbf{x}^*}\big]^{-1} = \mathbf{x}(1_\mathbf{H} + \widetilde{t})^{-1}$. Hence the definition of $\mathbf{a}$ ensures that $\mathbf{x} = \Phi_1(\mathbf{a}) = \widetilde{\Phi}_1(\mathbf{a})^*$. Thus, by Lemma 3.2, $t = \sqrt{1_\mathbf{K} + \mathbf{x}\mathbf{x}^*} = \sqrt{1_\mathbf{K} + \Phi_1(\mathbf{a})\Phi_1(\mathbf{a})^*} = \Phi_0(\mathbf{a})$ and $\widetilde{t} = \sqrt{1_\mathbf{H} + \mathbf{x}^*\mathbf{x}} = \sqrt{1_\mathbf{K} + \Phi_1(\mathbf{a})^*\Phi_1(\mathbf{a})} = \widetilde{\Phi}_0(\mathbf{a})$.

Proof of 3.4. Notice that, by 3.3 we have $\mathbf{M} = \mathrm{range}(\Phi)$. Let any point $(t, \mathbf{x}) \in \mathbf{M}$ and $\mathbf{u} \in \mathbf{E}$ be fixed arbitrarily and write

$$\mathbf{a} := \Phi^{-1}(t, \mathbf{x}) = (1_\mathbf{K} + t)^{-1}\mathbf{x}, \qquad \mathbf{v} := v_\mathbf{u}(\mathbf{a}) = \mathbf{a} - \mathbf{a}\mathbf{u}^*\mathbf{a}. \tag{3.6}$$

*The norm of a self-adjoint operator coincides with its spectral radius.

Then, in terms of the Möbius transformations $M_{\mathbf{u}}^\tau := \exp(\tau v_{\mathbf{u}})$ we have

$$\big[\Phi^\# v_{\mathbf{u}}\big](t,\mathbf{x}) = \frac{d}{d\tau}\Big|_{\tau=0} \Phi\circ M_{\mathbf{u}}^\tau \circ \Phi^{-1}(t,\mathbf{x}) = \frac{d}{d\tau}\Big|_{\tau=0} \Phi\circ M_{\mathbf{u}}^\tau(\mathbf{a}) = \Phi'(\mathbf{a})\mathbf{v}.$$

We calculate both $\Phi'(\mathbf{a})$ and $\mathbf{v}$ in terms of t,x. For the first component of Φ,

$$\begin{aligned}\Phi_0(\mathbf{a}) &= (1_{\mathbf{K}}-\mathbf{a}\mathbf{a}^*)^{-1}(1_{\mathbf{K}}+\mathbf{a}\mathbf{a}^*) = (1_{\mathbf{K}}-\mathbf{a}\mathbf{a}^*)^{-1}[(1_{\mathbf{K}}-\mathbf{a}\mathbf{a}^*)+2\mathbf{a}\mathbf{a}^*] = \\ &= 1_{\mathbf{K}}+2(1_{\mathbf{K}}-\mathbf{a}\mathbf{a}^*)^{-1}\mathbf{a}\mathbf{a}^* = 1_{\mathbf{K}}+\Phi_1(\mathbf{a})\mathbf{a}^* \ .\end{aligned}$$

Since, by definition $\Phi_1(\mathbf{a})=\mathbf{x}$, hence we get

$$\begin{aligned}\Phi_0'(\mathbf{a})\mathbf{v} &= \frac{d}{d\tau}\Big|_{\tau=0}\Phi_1(\mathbf{a}+\tau\mathbf{v})(\mathbf{a}+\tau\mathbf{v})^* = \phi_1(\mathbf{a})\mathbf{v}^* + \big[\phi_1'(\mathbf{a})\mathbf{v}\big]\mathbf{a}^* = \\ &= \mathbf{x}\mathbf{v}^* + \big[\phi_1'(\mathbf{a})\mathbf{v}\big]\mathbf{a}^* \ .\end{aligned}$$

We can express $\Phi_1'(\mathbf{a})\mathbf{v}$ in algebraic terms of $\mathbf{a},\mathbf{v}$ as follows:

$$\begin{aligned}\Phi_1'(\mathbf{a})\mathbf{v} &= \frac{d}{d\tau}\Big|_{\tau=0}\Phi_1(\mathbf{a}+\tau\mathbf{v}) = \frac{d}{d\tau}\Big|_{\tau=0} 2[1_{\mathbf{K}}+(\mathbf{a}+\tau\mathbf{v})(\mathbf{a}+\tau\mathbf{v})^*]^{-1}(\mathbf{a}+\tau\mathbf{v}) = \\ &= 2[1_{\mathbf{K}}-\mathbf{a}\mathbf{a}^*]^{-1}\mathbf{v} + 2[1_{\mathbf{K}}-\mathbf{a}\mathbf{a}^*]^{-1}(\mathbf{a}\mathbf{v}^*+\mathbf{v}\mathbf{a}^*)[1_{\mathbf{K}}-\mathbf{a}\mathbf{a}^*]^{-1}\mathbf{a} \ .\end{aligned}$$

Since $\mathbf{a} = \Phi^{-1}(t,\mathbf{x}) = (1_{\mathbf{K}}+t)^{-1}\mathbf{x}$ and since $\mathbf{x}\mathbf{x}^* = t^2 - 1_{\mathbf{K}} = (t-1_{\mathbf{K}})(1_{\mathbf{K}}+t)$, here we have

$$\begin{aligned}1_{\mathbf{K}}-\mathbf{a}\mathbf{a}^* &= 1_{\mathbf{K}}-(1_{\mathbf{K}}+t)^{-1}\mathbf{x}\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1} = 1_{\mathbf{K}}-(1_{\mathbf{K}}+t)^{-1}(t-1_{\mathbf{K}}) = \\ &= (1_{\mathbf{K}}+t)^{-1}[(1_{\mathbf{K}}+t)-(t-1_{\mathbf{K}})] = 2(1_{\mathbf{K}}+t)^{-1} \ , \\ [1_{\mathbf{K}}-\mathbf{a}\mathbf{a}^*]^{-1} &= \frac{1}{2}(1_{\mathbf{K}}+t) \ .\end{aligned}$$

Hence and with (3.6) we conclude

$$\begin{aligned}\Phi_1'(\mathbf{a})\mathbf{v} &= (1_{\mathbf{K}}+t)\mathbf{v} + 2[1_{\mathbf{K}}-\mathbf{a}\mathbf{a}^*]^{-1}\mathbf{a}\mathbf{v}^*[1_{\mathbf{K}}-\mathbf{a}\mathbf{a}^*]^{-1}\mathbf{a} + \\ &+ 2[1_{\mathbf{K}}-\mathbf{a}\mathbf{a}^*]^{-1}\mathbf{v}\mathbf{a}^*[1_{\mathbf{K}}-\mathbf{a}\mathbf{a}^*]^{-1}\mathbf{a} = \\ &= (1_{\mathbf{K}}+t)\mathbf{v} + \frac{1}{2}\mathbf{x}\mathbf{v}^*\mathbf{x} + \frac{1}{2}(1_{\mathbf{K}}+t)\mathbf{v}\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1}\mathbf{x} \ .\end{aligned}$$

We can express $\mathbf{v}$ in terms of $t,\mathbf{x}$ as

$$\mathbf{v} = \mathbf{u} - \mathbf{a}\mathbf{u}^*\mathbf{a} = \mathbf{u} - (1_{\mathbf{K}}+t)^{-1}\mathbf{x}\mathbf{u}^*(1_{\mathbf{K}}+t)^{-1}\mathbf{x} \ .$$

Thus

$$\begin{aligned}\Phi_1'(\mathbf{a})\mathbf{v} &= \overbrace{(1_{\mathbf{K}}+t)\mathbf{u}}^{(1)}\overbrace{-\mathbf{x}\mathbf{u}^*(1_{\mathbf{K}}+t)^{-1}\mathbf{x}}^{(2)}+\\ &\quad\overbrace{+\frac{1}{2}\mathbf{x}\mathbf{u}^*\mathbf{x}}^{(3)}\overbrace{-\frac{1}{2}\mathbf{x}\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1}\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1}\mathbf{x}}^{(4)}+\\ &\quad\overbrace{+\frac{1}{2}(1_{\mathbf{K}}+t)\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1}\mathbf{x}}^{(5)}\overbrace{-\frac{1}{2}\mathbf{x}\mathbf{u}^*(1_{\mathbf{K}}+t)^{-1}\mathbf{x}\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1}\mathbf{x}}^{(6)}\ .\end{aligned}$$

The sum (2)+(3)+(6) vanishes because $\mathbf{x}\mathbf{x}^* = (1_{\mathbf{K}}+t)(t-1_{\mathbf{K}})$ and hence

$$\begin{aligned}(2)+(3)+(6) &= \mathbf{x}\mathbf{u}^*\left[-(1_{\mathbf{K}}+t)^{-1}+\frac{1}{2}\cdot 1_{\mathbf{K}}-\frac{1}{2}(t_{\mathbf{K}}-1)(1_{\mathbf{K}}+t)^{-1}\right]\mathbf{x} =\\ &= \mathbf{x}\mathbf{u}^*\left[-1_{\mathbf{K}}+\frac{1}{2}\cdot 1_{\mathbf{K}}+\frac{1}{2}t-\frac{1}{2}t+\frac{1}{2}\cdot 1_{\mathbf{K}}\right](1_{\mathbf{K}}+t)^{-1}\mathbf{x} = 0\ .\end{aligned}$$

The sum (4)+(5) can also be simplified as

$$\begin{aligned}(4)+(5) &= \frac{1}{2}\Big[-(1_{\mathbf{K}}+t)^{-1}\overbrace{\mathbf{x}\mathbf{x}^*}^{(1_{\mathbf{K}}+t)(t_{\mathbf{K}}-1)}+(1_{\mathbf{K}}+t)\Big]\,\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1}\mathbf{x} =\\ &= \frac{1}{2}\left[-(t-1_{\mathbf{K}})+(1_{\mathbf{K}}+t)\right]\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1}\mathbf{x} =\\ &= \mathbf{u}\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1}\mathbf{x}\ .\end{aligned}$$

Summing up $(1)+\cdots+(6)$, we get

$$\begin{aligned}\Phi_1'(\mathbf{a})\mathbf{v} &= (1_{\mathbf{K}}+t)\mathbf{u}+\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1}\mathbf{x} = \mathbf{u}+t\mathbf{u}+\mathbf{u}\mathbf{x}^*\mathbf{x}(1_{\mathbf{H}}+\widetilde{t})^{-1} =\\ &= \mathbf{u}+t\mathbf{u}+\mathbf{u}(\widetilde{t}^2-1_{\mathbf{H}})(1_{\mathbf{H}}+\widetilde{t})^{-1} = \mathbf{u}+t\mathbf{u}+\mathbf{u}(\widetilde{t}-1_{\mathbf{H}}) =\\ &= t\mathbf{u}+\mathbf{u}\widetilde{t}\ ,\\ \Phi_0'(\mathbf{a})\mathbf{v} &= \mathbf{x}\mathbf{v}^*+\big[\Phi_1'(\mathbf{a})\mathbf{v}\big]\mathbf{a}^* =\\ &= \mathbf{x}\mathbf{u}^*-(1_{\mathbf{K}}+t)^{-1}\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1}+\\ &\quad+\big[(1_{\mathbf{K}}+t)\mathbf{u}+\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1}\mathbf{x}\big]\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1} =\\ &= \overbrace{\mathbf{x}\mathbf{u}^*}^{(1)}\overbrace{-\mathbf{x}\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1}\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1}}^{(2)}+\\ &\quad\overbrace{+(1_{\mathbf{K}}+t)\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1}}^{(3)}\overbrace{+\mathbf{u}\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1}\mathbf{x}\mathbf{x}^*(1_{\mathbf{K}}+t)^{-1}}^{(4)}\ .\end{aligned}$$

Using again the identity $\mathbf{x}\mathbf{x}^* = (1_\mathbf{K} + t)(t - 1_\mathbf{K})$, here we can write

$$\begin{aligned}
(2)+(3) &= \big[-(1_\mathbf{K}+t)^{-1}(1_\mathbf{K}+t)(t_\mathbf{K}-1)+(1_\mathbf{K}+t)\big]\mathbf{u}\mathbf{x}^*(1_\mathbf{K}+t)^{-1} = \\
&= 2\mathbf{u}\mathbf{x}^*(1_\mathbf{K}+t)^{-1}\ , \\
(4) &= \mathbf{u}\mathbf{x}^*(1_\mathbf{K}+t)^{-1}(1_\mathbf{K}+t)(t_\mathbf{K}-1)(1_\mathbf{K}+t)^{-1} = \mathbf{u}\mathbf{x}^*(t_\mathbf{K}-1)(1+t_\mathbf{K})^{-1} = \\
&= -\mathbf{u}\mathbf{x}^*(1_\mathbf{K}+t)^{-1}\big[(1_\mathbf{K}+t)-2t\big] = -\mathbf{u}\mathbf{x}^* + 2\mathbf{u}\mathbf{x}^* t(1_\mathbf{K}+t)^{-1}\ .
\end{aligned}$$

Therefore

$$\begin{aligned}
\Phi_0'(\mathbf{a})\mathbf{v} &= [(1)+(4)]+[(2)+(3)] = \\
&= \mathbf{x}\mathbf{u}^* - \mathbf{u}\mathbf{x}^* + 2\mathbf{u}\mathbf{x}^* t(1_\mathbf{K}+t)^{-1} + 2\mathbf{u}\mathbf{x}^*(1_\mathbf{K}+t)^{-1} = \\
&= \mathbf{x}\mathbf{u}^* + \mathbf{u}\mathbf{x}^*\big[-(1_\mathbf{K}+t)+2t+2\cdot 1_\mathbf{K}\big](1_\mathbf{K}+t)^{-1} = \\
&= \mathbf{x}\mathbf{u}^* + \mathbf{u}\mathbf{x}^*\ . \qquad \text{Qu.e.d.}
\end{aligned}$$

Proof of 3.5. By Lemmas 3.1-3 it suffices to see that we have $\big[\widetilde{\Phi}'(\mathbf{a})\big]v_\mathbf{u}(\mathbf{a}) = \big(\mathbf{u}^*\widetilde{\mathbf{x}}^* + \widetilde{\mathbf{x}}\mathbf{u}, \widetilde{t}\mathbf{u}^* + \mathbf{u}^* t\big)$ whenever $t = \Phi_0(\mathbf{a})$, $\widetilde{t} = \widetilde{\Phi}_0(\mathbf{a})$ and $\mathbf{x} := \widetilde{\mathbf{x}}^* = \Phi_1(\mathbf{a})$. Let $t := \Phi_0(\mathbf{a})$, $\widetilde{t} := \widetilde{\Phi}_0(\mathbf{a})$, $\mathbf{x} := \widetilde{\mathbf{x}}^* := \Phi_1(\mathbf{a})$. By 3.3 and since $\widetilde{\Phi}_1 = [\Phi_1]^*$, we have indeed

$$\big[\widetilde{\Phi}_1'(\mathbf{a})\big]v_\mathbf{u}(\mathbf{a}) = \left[\frac{d}{d\tau}\Big|_{\tau=0}\Phi_1\big(\mathbf{a}+\tau(\mathbf{u}-\mathbf{a}\mathbf{u}^*\mathbf{a})\big)\right]^* = \big[t\mathbf{u}+\mathbf{u}\widetilde{t}\big]^* = \mathbf{u}^* t + \widetilde{t}\mathbf{u}^*.$$

We can deduce the expression of $\big[\widetilde{\Phi}_0'(\mathbf{a})\big]v_\mathbf{u}(\mathbf{a})$ by reversing the order of operator multiplications during the proof of the relation $\big[\Phi_0'(\mathbf{a})\big]v_\mathbf{u}(\mathbf{a}) = \mathbf{u}\mathbf{x}^* + \mathbf{x}\mathbf{u}^*$. Hence we get

$$\big[\widetilde{\Phi}_0'(\mathbf{a})\big]v_\mathbf{u}(\mathbf{a}) = \mathbf{x}^*\mathbf{u} + \mathbf{u}^*\mathbf{x} = \widetilde{\mathbf{x}}\mathbf{u} + \mathbf{u}^*\mathbf{x}^*.$$

4 Proof of Corollary 2.5

Henceforth assume $\mathbf{H} = \mathbf{K}$ and consider any $\mathbf{a} \in \mathbf{B}^{(s)}$, $\mathbf{u} \in \mathbf{E}^{(s)} := \mathcal{L}^{(s)}(\mathbf{H})$. By definition $\mathbf{a} = \mathbf{a}^*$ and $\mathbf{u} = \mathbf{u}^*$ whence both the operators

$$\begin{aligned}
t &:= \Phi_0(\mathbf{a}) = (1_\mathbf{H}+\mathbf{a}^2)(1_\mathbf{H}-\mathbf{a}^2)^{-1} = (1_\mathbf{H}-\mathbf{a}^2)^{-1}(1_\mathbf{H}+\mathbf{a}^2)\big(=\widetilde{\Phi}_0(\mathbf{a})\big), \\
\mathbf{x} &:= \Phi_1(\mathbf{a} = 2(1_\mathbf{H}-\mathbf{a}^2)^{-1}\mathbf{a} = 2\mathbf{a}(1_\mathbf{H}-\mathbf{a}^2)^{-1}\big(=\widetilde{\Phi}_1(\mathbf{a})\big)
\end{aligned}$$

are self-adjoint. Thus, since $\Phi : \mathbf{B} \leftrightarrow \mathbf{M}$, also $\Phi : \mathbf{B}^{(s)} \leftrightarrow \mathbf{M}^{(s)}$. On the other hand, the vector field $v_\mathbf{u} : \mathbf{b} \mapsto \mathbf{u} - \mathbf{b}\mathbf{u}^*\mathbf{b} = \mathbf{u} - \mathbf{b}\mathbf{u}\mathbf{b}$ is complete in $\mathbf{B}$ and ranges in $\mathbf{E}^{(s)}$ when restricted to $\mathbf{B}^{(s)} = \mathbf{B} \cap \mathbf{E}^{(s)}$. That is for the

Möbius transformations $M_{\mathbf{u}}^\tau = \exp\big(\tau v_{\mathbf{u}}\big)$ we have $M_{\mathbf{u}}^\tau : \mathbf{B}^{(s)}) \leftrightarrow \mathbf{B}^{(s)}$ $(\tau \in \mathbb{R})$ and there lifting $\Phi^\# M_{\mathbf{u}}^\tau = \Phi \circ M_{\mathbf{u}}^\tau \Phi^{-1} : \mathbf{M}^{(s)} \leftrightarrow \mathbf{M}^{(s)}$ can be calculated by taking the exponentials of the vector fields $\tau \Phi^\# v_{\mathbf{u}}$ which are complete in $\mathbf{M}^{(s)} = \mathbf{E}^{(s)} \cap \mathbf{M}$. By 3.4 we have

$$\big[\Phi^\# v_{\mathbf{u}}\big](t,\mathbf{x}) = \Big(\mathbf{ux}+\mathbf{xu}, \mathbf{u}t + t\mathbf{u}\Big) = (t,\mathbf{x}) \begin{bmatrix} 0 & L(\mathbf{u})+R(\mathbf{u}) \\ L(\mathbf{u})+R(\mathbf{u}) & 0 \end{bmatrix} ,$$

$$\big[\Phi^\# M_{\mathbf{u}}^\tau\big](t,\mathbf{x}) = (t,\mathbf{x}) \exp\left(\tau \begin{bmatrix} 0 & L(\mathbf{u})+R(\mathbf{u}) \\ L(\mathbf{u})+R(\mathbf{u}) & 0 \end{bmatrix}\right) .$$

Straightforward calculations with the power series

$$\exp\big(\tau \Phi^\# v_{\mathbf{u}}\big) = \sum_{n=0}^\infty \tfrac{\tau^n}{n!} \begin{bmatrix} 0 & L(\mathbf{u})+R(\mathbf{u}) \\ L(\mathbf{u})+R(\mathbf{u}) & 0 \end{bmatrix}^n$$

yield the following:

$$\exp\left(\tau \begin{bmatrix} 0 & L(\mathbf{u})+R(\mathbf{u}) \\ L(\mathbf{u})+R(\mathbf{u}) & 0 \end{bmatrix}\right) = \sum_{n=0}^\infty \frac{\tau^n}{n!} \begin{bmatrix} 0 & L(\mathbf{u})+R(\mathbf{u}) \\ L(\mathbf{u})+R(\mathbf{u}) & 0 \end{bmatrix}^n =$$

$$= \sum_{k=0}^\infty \frac{\tau^{2k}}{(2k)!} \begin{bmatrix} 0 & L(\mathbf{u})+R(\mathbf{u}) \\ L(\mathbf{u})+R(\mathbf{u}) & 0 \end{bmatrix}^{2k} + \sum_{k=0}^\infty \frac{\tau^{2k+1}}{(2k+1)!} \begin{bmatrix} 0 & L(\mathbf{u})+R(\mathbf{u}) \\ L(\mathbf{u})+R(\mathbf{u}) & 0 \end{bmatrix}^{2k+1} =$$

$$= \sum_{k=0}^\infty \frac{\tau^{2k}}{(2k)!} \begin{bmatrix} [L(\mathbf{u})+R(\mathbf{u})]^{2k} & 0 \\ 0 & [L(\mathbf{u})+R(\mathbf{u})]^{2k} \end{bmatrix} + \sum_{k=0}^\infty \frac{\tau^{2k+1}}{(2k+1)!} \begin{bmatrix} 0 & [L(\mathbf{u})+R(\mathbf{u})]^{2k+1} \\ (L(\mathbf{u})+R(\mathbf{u}))^{2k+1} & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} \cosh\big(\tau[L(\mathbf{u})+R(\mathbf{u})]\big) & 0 \\ 0 & \cosh\big(\tau[L(\mathbf{u})+R(\mathbf{u})]\big) \end{bmatrix} + \begin{bmatrix} 0 & \sinh\big(\tau[L(\mathbf{u})+R(\mathbf{u})]\big) \\ \sinh\big(\tau[L(\mathbf{u})+R(\mathbf{u})]\big) & 0 \end{bmatrix}.$$

Since left and right multiplications commute $\big($that is $L(\mathbf{g})R(\mathbf{h})\mathbf{z} = \mathbf{g}(\mathbf{zh}) = (\mathbf{gz})\mathbf{h} = R(\mathbf{h})L(\mathbf{g})\mathbf{z}$ for $\mathbf{g},\mathbf{h},\mathbf{z} \in \mathbf{E}\big)$, it follows

$$\cosh\big(\tau[L(\mathbf{u})+R(\mathbf{u})]\big) = \frac{1}{2}\exp\big(\tau[L(\mathbf{u})+R(\mathbf{u})]\big) + \frac{1}{2}\exp\big(-\tau[L(\mathbf{u})+R(\mathbf{u})]\big) =$$

$$= \frac{1}{2}\exp\big(\tau L(\mathbf{u})\big)\exp\big(\tau R(\mathbf{u})\big) + \frac{1}{2}\exp\big(-\tau L(\mathbf{u})\big)\exp\big(-\tau R(\mathbf{u})\big) =$$

$$= \frac{1}{2}L\big(\exp(\tau\mathbf{u})\big)R\big(\exp(\tau\mathbf{u})\big) + \frac{1}{2}L\big(\exp(-\tau\mathbf{u})\big)R\big(\exp(-\tau\mathbf{u})\big)$$

with the effect $\cosh\big(\tau[L(\mathbf{u})+R(\mathbf{u})]\big) : \mathbf{z} \mapsto \frac{1}{2}\exp(\tau\mathbf{u})\mathbf{z}\exp(\tau\mathbf{u}) + \frac{1}{2}\exp(-\tau\mathbf{u})\mathbf{z}\exp(-\tau\mathbf{u})$. Similarly $\sinh\big(\tau[L(\mathbf{u})+R(\mathbf{u})]\big) : \mathbf{z} \mapsto \frac{1}{2}\exp(\tau\mathbf{u})\mathbf{z}\exp(\tau\mathbf{u}) - \frac{1}{2}\exp(-\tau\mathbf{u})\mathbf{z}\exp(-\tau\mathbf{u})$. Q.e.d.

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