Join spaces, soft join spaces and lattices

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In the honour of Professor Mirela Ştefănescu

Abstract

The aim of this paper is to initiate and investigate new (soft) hyperstructures, particularly (soft) join spaces, using important classes of lattices: modular and distributive. They are used in order to study (soft) hyperstructures constructed on the set of all convex sublattices of a lattice.

1 Introduction

There are several theories, such as probabilities, fuzzy sets, rough sets, vague sets, interval mathematics, which can be considered as mathematical tools for dealing with uncertainties. In [24], Molodtsov pointed out the difficulties of these theories, that can be due to the inadequacy of the parametrization tool of each theory and he introduced a new tool, called *soft set*, in order to deal with uncertainties, which is free from the difficulties of the above mentioned theories. Several applications of soft sets have been established for instance in decision making problem [22, 3, 12, 9]. To address decision making problems based on fuzzy soft sets, Feng et al. introduced the concept of level soft sets of fuzzy soft sets and initiated an adjustable decision making scheme using fuzzy soft sets [10]. It is also interesting to see that soft sets are closely related to many other soft computing models such as rough sets and fuzzy

sets. Using soft sets as the granulation structures, Feng et al. [11] defined soft approximation spaces, soft rough approximations and soft rough sets, which are generalizations of Pawlak's rough set model based on soft sets and in some cases might provide better approximations than classical rough sets.

On the other hand, hypergroups were introduced in 1934 at the VIII-th Congress of Scandinavian Mathematicians by a French mathematician Marty [21]. Nowadays, hypergroup theory is a widely applied theory [5, 6].

A *hypergroupoid* is a nonempty set H endowed with a map $\cdot : H \times H \longrightarrow \mathcal{P}^*(H)$ called hyperoperation, where $\mathcal{P}^*(H)$ denotes the set of all non-empty subsets of H. A *hypergroup* is an associative hypergroupoid $(H, \cdot)$ in which $x \cdot H = H \cdot x = H$ for all $x \in H$, where for all $A, B \subseteq H$ and $x \in H$, $A \cdot B = \bigcup\limits_{a \in A, b \in B} a \cdot b$, $A \cdot x = A \cdot \{x\}$ and $x \cdot B = \{x\} \cdot B$. Many interesting examples of hypergroups are given in [5, 6].

A nonempty subset S of a hypergroup $(H, \cdot)$ is called a *subhypergroup* if $\forall x \in S$, $x \cdot S = S \cdot x = S$. For any x, y of H, we denote $x/y = \{u \mid x \in u \cdot y\}$.

Prenowitz introduced a particular type of hypergroups, called *join spaces*, and then founded, together with Jantosciak [28], geometries on join spaces, which became a useful instrument in the study of several matters: graphs, hypergraphs, binary relations, fuzzy sets and rough sets (see [6] for details). If $(H_1, \cdot)$ and $(H_2, \cdot)$ are hypergroups, then a map $f : H_1 \to H_2$ is called a *homomorphism* if for all x, y of H_1, $f(x \cdot y) \subseteq f(x) \cdot f(y)$.

A commutative hypergroup $(H, \cdot)$ is called *a join space* if for all x, y, z, v of H, the following implication holds: $x/y \cap z/v \neq \emptyset \Rightarrow x \cdot v \cap z \cdot y \neq \emptyset$.

On the other hand, in order to introduce the soft set notion, we consider a universe set denoted by U and a set of parameters denoted by E. Let $\mathcal{P}(U)$ be the power set of U and $A \subseteq E$.

A pair (f, A) is called a *soft set* over U, where $f : A \to \mathcal{P}(U)$ is a map. Hence, a soft set over U is a parameterized family of subsets of U. For all $a \in A$, the subset $f(a)$ can be considered as the set of *a-approximate elements* of (f, A).

Thus, for a certain element $a \in A$, the subset $f(a)$ of U is composed by all the elements of U, which correspond to the parameter a.

We consider now that the universe set U is a hypergroup $(H, \cdot)$ and, as above, A is a nonempty set and $f : A \to \mathcal{P}(H)$ be a map.

1.1. Definition [17] A pair (f, A) is called a *soft hypergroup* over H if:

$$\forall a \in A, f(a) \neq \emptyset \Rightarrow f(a) \text{ is a subhypergroup of } H.$$

Every fuzzy subhypergroup can be interpreted as a soft hypergroup. In-

deed, suppose that μ is a fuzzy subhypergroup of a hypergroup $(H, \circ)$ (see [6], page 212), which means that μ satisfies the following axioms:

i) $min\{\mu(x), \mu(y)\} \leq inf\{\mu(z): \ z \in x \circ y\}$ for all $x, y \in H$;

ii) for all $x, a \in H$, there exists $y \in H$ such that $x \in a \circ y$ and
$$min\{\mu(a), \mu(x)\} \leq \mu(y);$$

iii) for all $x, a \in H$, there exists $z \in H$ such that $x \in z \circ a$ and
$$min\{\mu(a), \mu(x)\} \leq \mu(z).$$

If we consider the family of α-level sets for μ, given by

$$f(\alpha) = \{x \in H: \ \mu(x) \geq \alpha\},$$

where $\alpha \in [0, 1]$, then for all $\alpha \in [0, 1]$, $f(\alpha)$ is a subhypergroup of H. Hence $(f, [0, 1])$ is a soft hypergroup over H.

Indeed, if μ is a fuzzy subhypergroup of a hypergroup $(H, \cdot)$ (see [6], page 212) and for all $\alpha \in [0, 1]$, $f(\alpha)$ is the α-level set for μ, then $(f, [0, 1])$ is a soft hypergroup over H.

1.2. Example If $(L, \vee, \wedge)$ is a modular lattice and we define

$$\forall a, b \in L, a \circ b = \{x \in L: \ a \vee x = b \vee x = a \vee b\},$$

then $(L, \circ)$ is a join space (see [6], page 128). For all $a \in L$, $I(a) = \{x \in L: x \leq a\}$ is a subhypergroup of $(L, \circ)$. We define a map $f : L \to \mathcal{P}(L)$ as follows $f(a) = I(a)$ and (f, L) is a soft hypergroup over L.

In what follows we need the following two notions:

1.3. Definition $(H, \cdot)$ be a hypergroup. A pair (f, A) is called a *soft join space* over H if:

$$\forall a \in A, f(a) \neq \emptyset \Rightarrow f(a) \ \text{is a join space of } H.$$

The soft hypergroup of Example 1.2 is a soft join space.

1.4. Definition Let (f, A) be a soft hypergroup over a hypergroup $(H, \cdot)$. A soft set (g, B) over H is called a *soft subhypergroup* of (f, A) if $B \subseteq A$ and for all $b \in B$, if $g(b)$ is nonempty, then it is a subhypergroup of $f(b)$.

2   Join spaces and soft join spaces associated with lattices

Now, we consider the following two hyperoperations on a lattice $(L, \vee, \wedge)$:

$$x \circ y = \{z \in L : \ x \vee z = y \vee z = x \vee y\}, \ x \diamond y = \{z \in L : \ x \wedge y \leq z \leq x \vee y\}.$$

The hyperoperation " $\circ$ " was introduced by T. Nakano [26] and analysed by St. Comer [5], J. Mittas, M. Konstantinidou [23], I. G. Rosenberg [19, 20], B. Davvaz [18] and others. The hyperoperation " $\diamond$ " was introduced by J.C. Varlet [31] and analysed by M. Konstantinidou, S. Serafimidis , Ath. Kehagias ciiteKS, KSK, SK, V. Leoreanu-Fotea, I. G. Rosenberg [19, 20], B. Davvaz [18] and others. It is frequently used in machine learning applications. St. Comer and respectively J.C. Varlet characterized modular, respectively distributive lattices, using the above hyperoperations. These characterizations are presented in [6], Chapter 4, paragraph 3, respectively paragraph 1.

2.1. Theorem

i) A lattice $(L, \vee, \wedge)$ is modular if and only if $(L, \circ)$ is a join space.

ii) A lattice $(L, \vee, \wedge)$ is distributive if and only if $(L, \diamond)$ is a join space.

In order to characterize some soft join spaces, we need the following notions:

2.2. Definition [27] Let $(L, \leq)$ be a lattice. A pair (f, A) is called a *soft lattice* over L if:

$$\forall a \in A, f(a) \neq \emptyset \Rightarrow f(a) \ \text{is a sublattice of } L.$$

A soft lattice (f, A) is called *soft modular (distributive)* over a lattice L if:

$$\forall a \in A, f(a) \neq \emptyset \Rightarrow f(a) \ \text{is a modular (distributive) sublattice of } L.$$

A soft set (g, B) over a lattice is called a *soft sublattice* of (f, A) if $B \subseteq A$ and for all $b \in B$, if $g(b)$ is nonempty, then it is a sublattice of $f(b)$.

There are many interesting examples of soft lattices, soft distributive (modular) lattices in [27]. We give here other examples.

2.3. Examples

i) Let S be a semigroup and H any sublattice of the lattice L of all fuzzy congruences on S such that $fg = gf$ for all $f, g \in H$. Then H is a modular lattice, see [1]. Hence, if T is a cardinally equivalent set to the set of all sublattices of L, then (φ, T) is a soft modular lattice over L, where φ is a bijective map from T to the set of all sublattices of L.

ii) According to [2], if C_0, C_1 are chains of a modular lattice L, then the sublattice L' of L, generated by $C_0 \cup C_1$ is distributive. Hence if $T = \{t\}$ is a singleton set and $\varphi : T \to \{L'\}$, we obtain a distributive soft lattice (φ, T) over L.

2.4. Corollary *A soft lattice (f, A) over a lattice L is modular if and only if (f, A) is a soft join space over the hypergroupoid $(L, \circ)$.*

Proof. A soft lattice (f, A) over a lattice L is modular if and only if for all $a \in A$ for which $f(a) \neq \emptyset$, $f(a)$ is a modular sublattice of L. According to Theorem 2.1 i), this happens if and only if $(f(a), \circ)$ is a join space, which means that (f, A) is a soft join space over the hypergroupoid $(L, \circ)$. ∎

The following notion helps us to characterize soft subhypergroup of the associated soft join space (f, A) over the hypergroupoid $(L, \circ)$.

2.5. Definition [27] Let (f, A) be a soft lattice over a lattice L. A soft set (g, B) over L is called a *soft ideal* of (f, A) if $B \subseteq A$ and for all $b \in B$, if $g(b)$ is nonempty, then it is an ideal of $f(b)$.

2.6. Theorem *Let (f, A) be a soft modular lattice over a lattice L. A soft set (g, B) over L is a soft ideal of (f, A) if and only if (g, B) is a soft subhypergroup of (f, A), with respect to the hyperoperation " $\circ$ ".*

Proof. Suppose that (g, B) is a soft ideal of (f, A). Then for all $b \in B$, if $g(b)$ is nonempty, then it is an ideal of $f(b)$. According to the above corollary, $(f(b), \circ)$ is a join space. Set x, y be elements of $g(b)$. If $u \in x \circ y$, then $u \leq x \vee y$, and so $u \in g(b)$. Hence $g(b) \circ g(b) \subseteq g(b)$. On the other hand, for all x, y elements of $g(b)$, there exists $z = x \vee y$, such that $x \in y \circ z \cap z \circ y$. Therefore, $(g(b), \circ)$ is a subhypergroup of the hypergroup $(f(b), \circ)$, which means that (g, B) is a soft subhypergroup of (f, A).

Conversely, suppose that (g, B) is a soft subhypergroup of (f, A). We check that for all $b \in B$, if $g(b)$ is nonempty, then it is an ideal of the modular lattice $f(b)$. Set x, y be elements of $g(b)$. We have $x \vee y \in x \circ y \subseteq g(b)$. Moreover, if $x \in g(b)$ and $u \leq z$, where $u \in f(b)$, then $u \in z \circ z \subseteq g(b)$. Hence $g(b)$ is an ideal of $f(b)$ and so (g, B) is a soft ideal of (f, A). ∎

2.7. Corollary *A soft lattice (f, A) over a lattice L is distributive if and only if (f, A) is a soft join space over the hypergroupoid $(L, \diamond)$.*

Proof. It follows from Theorem 2.1. ii). ∎

2.8. Theorem *Let (f, A) be a soft distributive lattice over a lattice L. If a soft set (g, B) over L is a soft ideal of (f, A), then it is a soft subhypergroup of (f, A), with respect to the hyperoperation " $\diamond$."*

Proof. Let $b \in B$, such that $g(b)$ is nonempty. According to the above corollary, $(f(b), \diamond)$ is a join space. Let $x, y \in g(b)$ and $z \in x \diamond y$. Hence $x \wedge y \leq z \leq x \vee y$. Since $g(b)$ is a ideal of $f(b)$, it follows that $x \vee y \in g(b)$ and then $z \in g(b)$. Thus, $x \diamond y \subseteq g(b)$. Moreover, $x \in x \diamond y \cap y \diamond x$, whence $x \in g(b) \diamond y \cap y \diamond g(b)$. Hence $g(b) = g(b) \diamond y = y \diamond g(b)$, for all $y \in g(b)$. Therefore $g(b)$ is a subhypergroup of $(f(b), \diamond)$, which means that $g(b)$ is a soft subhypergroup of the soft hypergroup (f, A). ∎

Notice that the converse is not true, as we can see from the following example:

2.9. Example Let $(L = \mathcal{P}(M), \subseteq)$ be the lattice of all parts of a nonempty set M, $A = B = \{a\}$, $f(a) = L$ and $g(a) = \{U \in \mathcal{P}(M) \mid C \cap D \leq U \leq C \cup D\}$, where C, D are nonempty subsets of M, which have a nonempty intersection. Then $(g(a), \circ)$ is a subhypergroup of $(\mathcal{P}(M), \circ)$, which means that (g, B) is a soft subhypergroup of (f, A). On the other hand, $g(a)$ is not an ideal of $f(a) = L$. Indeed, $\emptyset \subseteq C$, $C \in g(a)$, but $\emptyset \notin g(a)$. This means that (g, B) is not a soft ideal of (f, A).

2.10. Theorem *If (f, A) be a soft distributive lattice over a lattice L and if (g, B) is a soft subhypergroup of (f, A), with respect to the hyperoperation " $\diamond$ ", then (g, B) is a soft sublattice of (f, A).*

Proof. Indeed, if $b \in B$ such that $g(b)$ is nonempty. If $x, y \in g(b)$, then $x \diamond y \in g(b)$, whence for all $z \in f(b)$ such that $x \wedge y \leq z \leq x \vee y$, then $z \in g(b)$. We take $z_1 = x \vee y$ then $z_2 = x \wedge y$. Hence $x, y \in g(b)$ implies that $x \vee y \in g(b)$ and $x \wedge y \in g(b)$, which means that (g, B) is a soft subsemilattice of (f, A). ∎

We present now some results concerning soft lattice homomorphisms.

2.11. Definition [27] Let (f, A) and (h, B) be two soft lattices over L_1 and L_2 respectively. Let $\varphi : L_1 \to L_2$ and $\psi : A \to B$ be two maps. The pair (φ, ψ) is called a *soft lattice homomorphism* if φ is a lattice homomorphism and for all $a \in A$, $\varphi(f(a)) = h(\psi(a))$.

If φ is a lattice isomorphism and ψ is a bijection, then (φ, ψ) is called a *soft lattice isomorphism*.

2.12. **Definition** [27] Let (f, A) and (h, B) be two soft hypergroups over the hypergroups $(H_1, \cdot)$ and $(H_2, \cdot)$ respectively. Let $\varphi : H_1 \rightarrow H_2$ and $\psi : A \rightarrow B$ be two maps. The pair (φ, ψ) is called a *soft hypergroup homomorphism* if φ is a hypergroup homomorphism and for all $a \in A$, $\varphi(f(a)) = h(\psi(a))$.

Notice that if $x, y \in f(a)$, then $\varphi(x), \varphi(y) \in h(\psi(a))$ according to the above definition. In other words, if x, y belong to a certain subhypergroup of $(H_1, \cdot)$, then their images under φ belong both to the same subhypergroup of $(H_2, \cdot)$.

We need to introduce the dual hyperoperation " $*$ " of " $\circ$ " on a lattice L, defined as follows:

$$x * y = \{z \in L : \ x \wedge z = y \vee z = x \wedge y\}$$

2.13. Theorem *Using the above notations, (φ, ψ) is a homomorphism of soft modular lattices if and only if φ is a soft hypergroup homomorphism from (f, A) to (h, B), with respect to both hyperoperations " $\circ$ " and " $*$ ".*

Proof. First, notice that since (f, A) and (h, B) are soft modular lattices, it follows that (f, A) is a soft join space over both $(L_1, \circ)$ and $(L_1, *)$, while (h, B) is a soft join space over both $(L_2, \circ)$ and $(L_2, *)$. We have to check that φ is a lattice homomorphism from L_1 to L_2 if and only if φ is both a hypergroup homomorphism from $(L_1, \circ)$ to $(L_2, \circ)$ and a hypergroup homomorphism from $(L_1, *)$ to $(L_2, *)$. Suppose that φ is a lattice homomorphism. Since

$$\varphi(x \circ y) = \{\varphi(z) \mid z \in L_1, \ z \vee x = x \vee y = y \vee z\},$$

it follows that $\varphi(z) \in \varphi(x) \circ \varphi(y)$, whence $\varphi(x \circ y) \subseteq \varphi(x) \circ \varphi(y)$. Similarly, $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$, for all $x, y \in L_1$ implies that $\varphi(x * y) \subseteq \varphi(x) * \varphi(y)$. Hence φ is both a hypergroup homomorphism from $(L_1, \circ)$ to $(L_2, \circ)$ and a hypergroup homomorphism from $(L_1, *)$ to $(L_2, *)$.

Conversely, if $u \in x \circ y$, then $\varphi(u) \in \varphi(x) \circ \varphi(y)$, that is $\varphi(u) \vee \varphi(x) = \varphi(u) \vee \varphi(y) = \varphi(x) \vee \varphi(y) \geq \varphi(u)$. In particular, we take $u = x \vee y$ and so we obtain $\varphi(x) \vee \varphi(y) \geq \varphi(x \vee y)$. On the othar hand, notice that for all $u \leq x$, that is $u \in x \circ x$ we have $\varphi(u) \in \varphi(x) \circ \varphi(x)$, whence $\varphi(u) \leq \varphi(x)$. Thus, $\varphi(x) \leq \varphi(x \vee y)$, $\varphi(y) \leq \varphi(x \vee y)$, whence $\varphi(x) \vee \varphi(y) \leq \varphi(x \vee y)$. Therefore, for all $x, y \in L_1$, $\varphi(x) \vee \varphi(y) = \varphi(x \vee y)$. Similarly, from $\varphi(x * x) \subseteq \varphi(x) * \varphi(y)$, for all $x, y \in L_1$, it follows that $\varphi(x) \wedge \varphi(y) = \varphi(x \wedge y)$ and so, φ is a lattice homomorphism. ∎

A similar result can be obtained with respect to the hyperoperation " $\diamond$ " :

2.14. Theorem (φ, ψ) *is a homomorphism of soft distributive lattices if and only if* φ *is monotone and it is a soft hypergroup homomorphism from* (f, A) *to* (h, B), *with respect to the hyperoperation* " $\diamond$ ".

Proof. Notice that since (f, A) and (h, B) are soft distributive lattices, it follows that $(f, A), (h, B)$ are soft join spaces over $(L_1, \diamond)$ and $(L_2, \diamond)$ respectively. We have to check that φ is a lattice homomorphism from L_1 to L_2 if and only if φ is monotone and it is a hypergroup homomorphism from $(L_1, \diamond)$ to $(L_2, \diamond)$. Suppose that φ is a lattice homomorphism, so it is monotone. Since

$$\varphi(x \diamond y) = \{\varphi(z) \mid z \in L_1, \ x \wedge y \leq z \leq x \vee y\},$$

it follows that $\varphi(z) \in \varphi(x) \diamond \varphi(y)$, whence $\varphi(x \diamond y) \subseteq \varphi(x) \diamond \varphi(y)$.

Conversely, if $u \in x \diamond y$, then $\varphi(u) \in \varphi(x) \diamond \varphi(y)$, that is $\varphi(x) \wedge \varphi(y) \leq \varphi(u) \leq \varphi(x) \vee \varphi(y)$. In particular, we take $u = x \wedge y$ and $u = x \vee y$ and so we obtain $\varphi(x) \wedge \varphi(y) \leq \varphi(x \wedge y)$, $\varphi(x \vee y) \leq \varphi(x) \vee \varphi(y)$. On the other hand, φ is monotone and so $\varphi(x \wedge y) \leq \varphi(x) \wedge \varphi(y)$ and $\varphi(x) \vee \varphi(y) \leq \varphi(x \vee y)$. Hence φ is a lattice homomorphism from L_1 to L_2. ∎

3   Join spaces associated with convex sublattices of a lattice

We denote by $CS(L)$ the set of all convex sublattices of a lattice L. In [16] Lavanya and Parameswara Bhatta defined a partial order $\leq$ on $CS(L)$ as follows: for all $X, Y \in CS(L)$, $X \leq Y$ if and only if

$(\forall x \in X, \ \exists y \in Y : \ x \leq y)$ and $(\forall y' \in Y, \ \exists x' \in X : \ x' \leq y')$.

Then $(CS(L), \leq)$ is a lattice, in which

$$inf\{X, Y\} = < \{x \wedge y \mid x \in X, y \in Y\} >$$

$$= \{u \in L \mid x \wedge y \leq u \leq x_1 \wedge y_1 \text{ for some } x, x_1 \in X \text{ and } y, y_1 \in Y\},$$

$$sup\{X, Y\} = < \{x \vee y \mid x \in X, y \in Y\} >$$

$$= \{u \in L \mid x \vee y \leq u \leq x_1 \vee y_1 \text{ for some } x, x_1 \in X \text{ and } y, y_1 \in Y\},$$

where for all nonempty subset H of L, $< H >$ denotes the convex sublattice of L generated by H. For all $X, Y \in CS(L)$, denote

$$X \bar{\diamond} Y = \{Z \in CS(L) : \ inf\{X, Y\} \leq Z \leq sup\{X, Y\}\}.$$

3.1. Theorem $(L, \diamond)$ *is a join space if and only if* $(CS(L), \bar{\diamond})$ *is a join space.*

Proof. According to Theorem 2.1., (ii), $(L, \diamond)$ is a join space if and only if the lattice $(L, \vee, \wedge)$ is distributive. First, we check that if $(L, \vee, \wedge)$ is distributive, then the lattice $(CS(L), sup, inf)$ is distributive, too.

Set $X, Y, Z \in CS(L)$. We obtain

$$(X \ inf \ Y) \ sup \ Z = \{u \in L \mid \exists x_i \in X, \ \exists y_i \in Y, \ \exists z_i \in Z, i = 1, 2, :$$

$$t_1 \vee z_1 \leq u \leq t_2 \vee z_2, \ \text{where} \ x_1 \wedge y_1 \leq t_i \leq x_2 \wedge y_2\}$$

$$= \{u \in L \mid \exists x_i \in X, \ \exists y_i \in Y, \ \exists z_i \in Z, \ i \in \{1, 2\} :$$

$$(x_1 \wedge y_1) \vee z_1 \leq u \leq (x_2 \wedge y_2) \vee z_2\} \ \text{ and}$$

$$(X \ sup \ Z) \ inf \ (Y \ sup \ Z) = \{v \in L \mid \exists x_i \in X, \ \exists y_i \in Y, \ \exists z_i \in Z, \ i \in \{1, 2\} :$$

$$s_1 \wedge p_1 \leq v \leq s_2 \wedge p_2, \ \text{where} \ x_1 \vee z_1 \leq s_i \leq x_2 \vee z_2, \ y_1 \vee z_1 \leq p_i \leq y_2 \vee z_2, \ i = 1, 2\}$$

$$= \{v \in L \mid \ \exists x_i \in X, \ \exists y_i \in Y, \ \exists z_i \in Z, \ i \in \{1, 2\} :$$

$$(x_1 \vee z_1) \wedge (y_1 \vee z_1) \leq v \leq (x_2 \vee z_2) \wedge (y_2 \vee z_2)\}.$$

Since $(L, \vee, \wedge)$ is distributive, it follows that $(CS(L), \bar{\diamond})$ is a join space. Conversely, for all $x, y, z \in L$ it is sufficient to set $X = \{x\}$, $Y = \{y\}$, $Z = \{z\}$. ∎

Now, let (f, A) be a soft join space over $(L.\diamond)$. We define $\bar{f} : A \to CS(L)$ by $\bar{f}(a) = CS(f(a))$. By the above theorem and Corollary 2.7., we obtain

3.2. Corollary (f, A) *is a soft join space over* $(L, \diamond)$ *if and only if* $(\bar{f}, A)$ *is a soft join space over* $(CS(L), \bar{\diamond})$.

Now, for all $X, Y \in CS(L)$, denote

$$X \bar{\diamond} Y = \{Z \in CS(L) : \ sup\{X, Y\} = sup\{Y, Z\} = sup\{X, Z\}\}.$$

3.3. Theorem $(L, \circ)$ *is a join space if and only if* $(CS(L), \bar{\circ})$ *is a join space.*

Proof. According to Theorem 2.1., (i), $(L, \circ)$ is a join space if and only if the lattice $(L, \vee, \wedge)$ is modular. First, we check that if $(L, \vee, \wedge)$ is modular, then the lattice $(CS(L), sup, inf)$ is modular.

Set $X, Y, Z \in CS(L)$, $X \leq Z$. It is sufficient to check that

$$(*) \qquad (X \ sup \ Y) \ inf \ Z \leq X \ sup \ (Y \ inf \ Z),$$

which means that $\forall u \in (X \ sup \ Y) \ inf \ Z, \ \exists v \in X \ sup \ (Y \ inf \ Z)$ such that $u \leq v$ and $\forall \bar{v} \in X \ sup \ (Y \ inf \ Z), \ \exists \bar{u} \in (X \ sup \ Y) \ inf \ Z$, such that $\bar{u} \leq \bar{v}$.

Set $u \in (X \ sup \ Y) \ inf \ Z$. Hence, there exist $x_i \in X, y_i \in Y, z_i \in Z, i \in \{1, 2\}$, such that $t_1 \wedge z_1 \leq u \leq t_2 \wedge z_2$, where $t_i \in X \ sup \ Y$. Hence for $i \in \{1, 2\}$, $x_1 \vee y_1 \leq t_i \leq x_2 \vee y_2$. It follows that $(x_1 \vee y_1) \wedge z_1 \leq u \leq t_2 \wedge z_2 \leq (x_2 \vee y_2) \wedge z_2$. Since $X \leq Z$, it follows that $\forall x \in X, \exists \bar{z} \in Z : \ x \leq \bar{z}$ and $\forall z \in Z, \exists \bar{x} : \ \bar{x} \leq z$. So, for $x_1 \in X, \exists \bar{z}_1 \in Z : \ x_1 \leq \bar{z}_1$ and for $z_1 \in Z_1, \exists \bar{x}_1 : \ \bar{x}_1 \leq z_1$. Hence $x_1 \wedge \bar{x}_1 \leq z_1 \wedge \bar{z}_1$.

By the modularity of L, it follows that

$$(x_1 \vee y_1) \wedge z_1 \geq [(x_1 \wedge \bar{x}_1) \vee y_1] \wedge (z_1 \wedge \bar{z}_1) = (x_1 \wedge \bar{x}_1) \vee (y_1 \wedge z_1 \wedge \bar{z}_1).$$

Denote $x_1 \wedge \bar{x}_1 = x_3 \in X, \ z_1 \wedge \bar{z}_1 = z_3 \in Z$. Hence $u \geq x_3 \vee (y_1 \wedge z_3)$.

On the other hand, for $x_2, \exists \bar{z}_2 : x_2 \leq \bar{z}_2$ and for $z_2, \exists \bar{x}_2 : \bar{x}_2 \leq z_2$. Hence $x_2 \vee \bar{x}_2 \leq z_2 \vee \bar{z}_2$. By the modularity of L it follows that

$$(x_2 \vee y_2) \wedge z_2 \leq [(x_2 \vee \bar{x}_2) \vee y_2] \wedge (z_2 \vee \bar{z}_2) = (x_2 \vee \bar{x}_2) \vee (y_2 \wedge (z_2 \vee \bar{z}_2)).$$

Denote $x_2 \vee \bar{x}_2 = x_4 \in X, \ z_2 \vee \bar{z}_2 = z_4 \in Z$. We obtain $u \leq x_4 \vee (y_2 \wedge z_4)$. Therefore, $\exists x_3, x_4 \in X, \ \exists y_1, y_2 \in Y, \ \exists z_3, z_4 \in Z$ such that $x_3 \vee (y_1 \wedge z_3) \leq u \leq x_4 \vee (y_2 \wedge z_4)$, whence

$$X \ sup \ (Y \ inf \ Z) = \{v \in L \mid \exists x_i \in X, \exists t_i \in Y \ inf \ Z :$$

$$x_1 \vee t_1 \leq v \leq x_2 \vee t_2, \exists y_i \in Y, \exists z_i \in Z : \ y_1 \wedge z_1 \leq t_1, t_2 \leq y_2 \wedge z_2\}.$$

Hence $x_1 \vee (y_1 \wedge z_1) \leq x_1 \vee t_1 \leq v \leq x_2 \vee t_2 \leq x_2 \vee (y_2 \wedge z_2)$, whence it follows $(*)$. Conversely, for all $x, y, z \in L$ it is sufficient to set $X = \{x\}, \ Y = \{y\}, \ Z = \{z\}$.
∎

3.4.Corollary (f, A) *is a soft join space over* $(L, \circ)$ *if and only if* $(\bar{f}, A)$ *is a soft join space over* $(CS(L), \bar{\circ})$.

Proof. It follows from Theorem 3.3 and Corollary 2.4.   ∎

4   Conclusion

This paper continues the study of soft hyperstructures initiated in [17], by proposing and analysing new soft hyperstructures in connection with modular and distributive lattices. The results obtained in this context are used in the study of (soft) hyperstructures obtained on the set of all convex sublattices of a lattice.

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