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PROPERTIES FOR SUBCLASSES OF STARLIKE FUNCTION ASSOCIATED WITH Q-ANALOGUE OPERATOR AND COMPLEX ORDER

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Abstract. In this paper using a $q-$analogue operator, we define subclasses of univalent functions of complex order and fined coefficient bounds, distortion inequalities for functions in it and also obtaind inclusion relations associated with the $N_{q,\delta}-$neighborhood and some Hadamard results..

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1. Introduction

Quantum calculus, occasionally named calculus without limits. It is known as $q-$calculus which has infuenced many scientific fields due to its importants. Geometric function theory is no exception in this regard and many authors have already made a substantial research in this field. The generalization of derivative and integral in $q-$calculus are known as $q-$derivative and $q-$integral, were introduced and studied by Jackson [15]. Recently, many authors used the $q-$derivative and $q-$integral to generalize many classes and many operators in

geometric function theory see for example [3, 4, 6, 7, 10, 23, 24].

The class of univalent analytic functions of the form

$$\mathcal{F}(z) = z - \sum_{k=2}^{\infty} a_k z^k, (a_k \geq 0),\ z \in \mathcal{D} = \{z \in \mathbb{C} :| z |< 1\}, \tag{1}$$

is denoted by $\mathcal{T}$.

Given $0 \leq \alpha < 1$, a function $\mathcal{F} \in \mathcal{T}$ is said to be in the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathcal{D}$ if

$$\text{Re}\left\{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right\} > \alpha.$$

For $\mathcal{F} \in \mathcal{T},\ 0 < q < 1$, the $q-$difference operator ∇_q is given by [15] (see also [2, 4],[7]6,[12],[22, 23,24]);

$$\nabla_q \mathcal{F}(z) = \begin{cases} \frac{\mathcal{F}(z)-\mathcal{F}(qz)}{(1-q)z} & ,z\neq 0 \\ \mathcal{F}'(0) & ,z=0 \end{cases},$$

that is

$$\nabla_q \mathcal{F}(z) = 1 - \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \tag{2}$$

where

$$[j]_q = \frac{1-q^j}{1-q},\ [0]_q = 0. \tag{3}$$

As $q \to 1^-$, $[k]_q = k$ and $\nabla_q \mathcal{F}(z) = \mathcal{F}'(z)$.

For $\lambda \geq \mu \geq 0$, $0 < q < 1$, let

$$\mathcal{H}^0_{\lambda,\mu,q} \mathcal{F}(z) = \mathcal{F}(z),$$

$$\mathcal{H}^1_{\lambda,\mu,q} \mathcal{F}(z) = \mathcal{H}^m_{\lambda,\mu,q} \mathcal{F}(z) = (1-\lambda+\mu)\mathcal{F}(z) + (\lambda-\mu)z\nabla_q \mathcal{F}(z) + \lambda\mu z^2 \nabla_q^2 \mathcal{F}(z),$$

$$\mathcal{H}^2_{\lambda,\mu,q} \mathcal{F}(z) = \mathcal{H}_{\lambda,\mu,q}(\mathcal{H}_{\lambda,\mu,q} \mathcal{F}(z)),$$

and

$$\begin{aligned} \mathcal{H}^m_{\lambda,\mu,q} \mathcal{F}(z) &= \mathcal{H}_{\lambda,\mu,q}(\mathcal{H}^{m-1}_{\lambda,\mu,q} \mathcal{F}(z)) \\ &= z - \sum_{k=2}^{\infty} [1-\lambda+\mu+[k]_q (\lambda-\mu+\lambda\mu[k-1]_q)]^m a_k z^k, m \in \mathbb{N}. \end{aligned} \tag{4}$$

Note that

(i) $\lim_{q\to 1-} \mathcal{H}^m_{\lambda,\mu,q} \mathcal{F}(z) = \mathcal{H}^m_{\lambda,\mu} \mathcal{F}(z)$ see Orhan et al. [18] (see also [11], [17] and Răducanu and Orhan [19]);

(ii) $\mathcal{H}^m_{1,0,q} \mathcal{F}(z) = \mathcal{H}^m_q \mathcal{F}(z)$(see [14], [25] and [8]);

(iii) $\mathcal{H}^m_{\lambda,0,q} \mathcal{F}(z) = \mathcal{H}^m_{\lambda,q} \mathcal{F}(z)$ (see Aouf et al. [9]);

(iv) $\lim_{q\to 1-} \mathcal{H}^m_{\lambda,0,q} \mathcal{F}(z) = \mathcal{H}^m_{\lambda} \mathcal{F}(z)$ (see Al-Oboudi [1]).

Now, by making use of the operator $\mathcal{H}^m_{\lambda,\mu,q}$, we have

Definition 1. *Let* $\tau \in \mathbb{C}^* = \mathbb{C}/\{0\}$, $\lambda \geq \mu \geq 0$, $0 < q < 1$, $m \in \mathbb{N}_0$, $0 < \eta \leq 1$ *and* $\mathcal{F} \in \mathcal{T}$, *such that* $\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z) \neq 0$ *for* $z \in \mathcal{D}/\{0\}$. *We say that* $\mathcal{F} \in \mathbb{S}^m_q(\tau, \lambda, \mu, \eta)$ *if*

$$\left|\frac{1}{\tau}\left(\frac{z\nabla_q(\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z))}{\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z)} - 1\right)\right| < \eta. \tag{5}$$

Note that: For different values of $q, \tau, \lambda, \mu, \eta$, we have:

(i) $\lim_{q\to 1-} \mathbb{S}^m_q(\tau, \lambda, \mu, \eta) = \mathbb{S}^m(\tau, \lambda, \mu, \eta) = \left\{\mathcal{F}(z) : \left|\frac{1}{\tau}\left(\frac{z(\mathcal{H}^m_{\lambda,\mu}\mathcal{F}(z))'}{\mathcal{H}^m_{\lambda,\mu}\mathcal{F}(z)} - 1\right)\right| < \eta\right\}$;

(ii) $\mathbb{S}^m_q(\tau, 1, 0, \eta) = \mathbb{S}^m_q(\tau, \eta) = \left\{\mathcal{F}(z) : \left|\frac{1}{\tau}\left(\frac{z\nabla_q(\mathcal{H}^m_q\mathcal{F}(z))}{\mathcal{H}^m_q\mathcal{F}(z)} - 1\right)\right| < \eta\right\}$;

(iii) $\mathbb{S}^m_q(\tau, \lambda, 0, \eta) = \mathbb{S}^m_q(\tau, \lambda, \eta) = \left\{\mathcal{F}(z) : \left|\frac{1}{\tau}\left(\frac{z\nabla_q(\mathcal{H}^m_{\lambda,q}\mathcal{F}(z))}{\mathcal{H}^m_{\lambda,q}\mathcal{F}(z)} - 1\right)\right| < \eta\right\}$;

(iv) $\mathbb{S}^m_q(1-\gamma, \lambda, \mu, 1) = \mathbb{S}^m_q(\gamma, \lambda, \mu) = \left\{\mathcal{F}(z) : \mathrm{Re}\left\{\frac{z\nabla_q(\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z))}{\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z)}\right\} > \gamma, 0 \leq \gamma < 1\right\}$.

Goodman [13], Ruscheweyh [20] and Altintas et al.[2], Mostafa and Aouf [16](with $p = 1$), defined the $N_\delta-$neighborhood for $\mathcal{F}(z) \in \mathcal{T}$ by

$$N_\delta(\mathcal{F}, g) = \{g : g(z) \in \mathcal{T},\ g(z) = z - \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k\,|a_k - b_k| \leq \delta\}, \tag{6}$$

and for $e(z) = z$;

$$N_\delta(e, g) = \{g : g(z) \in \mathcal{T},\ g(z) = z - \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k\,|b_k| \leq \delta\}. \tag{7}$$

In [8] Aouf et al. (see also Madian and Aouf [5] (with $p = 1$)) defined the $N_{q,\delta}-$neighborhood for $\mathcal{F}(z) \in \mathcal{T}$ by

$$N_{q,\delta}(\mathcal{F}, g) = \{g : g(z) \in \mathcal{T},\ g(z) = z - \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} [k]_q\,|a_k - b_k| \leq \delta_q\}, \tag{8}$$

and for $e(z) = z$;

$$N_{q,\delta}(e, g) = \{g : g(z) \in \mathcal{T},\ g(z) = z - \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} [k]_q\,|b_k| \leq \delta_q\}. \tag{9}$$

2. Main Results

Unless indicated, we assume that $\tau \in \mathbb{C}^*$, $\lambda \geq \mu \geq 0$, $0 < q < 1$, $m \in \mathbb{N}_0$, $0 < \eta \leq 1$ and $\mathcal{F}(z)$ given by (1).

Theorem 1. *The function* $\mathcal{F} \in \mathbb{S}_q^m(\tau, \lambda, \mu, \eta)$ *if and only if*

$$\sum_{k=2}^{\infty}([k]_q + \eta\,|\tau| - 1)[1 - \lambda + \mu + [k]_q\,(\lambda - \mu + \lambda\mu[k-1]_q)]^m a_k \leq \eta\,|\tau|\,. \tag{10}$$

Proof. Assume that (10) holds true. Then we have

$$\begin{aligned} &(q + \eta\,|\tau|)\sum_{k=2}^{\infty}[1 - \lambda + \mu + [k]_q\,(\lambda - \mu + \lambda\mu[k-1]_q)]^m a_k \\ \leq\ & \sum_{k=2}^{\infty}([k]_q + \eta\,|\tau| - 1)[1 - \lambda + \mu + [k]_q\,(\lambda - \mu + \lambda\mu[k-1]_q)]^m a_k \\ \leq\ & \eta\,|\tau|\,, \end{aligned}$$

that is,

$$\sum_{k=2}^{\infty}[1 - \lambda + \mu + [k]_q\,(\lambda - \mu + \lambda\mu[k-1]_q)]^m a_k \leq \frac{\eta\,|\tau|}{(q + \eta\,|\tau|)}.$$

Since,

$$\begin{aligned} &\left|1 - \sum_{k=2}^{\infty}[1 - \lambda + \mu + [k]_q\,(\lambda - \mu + \lambda\mu[k-1]_q)]^m a_k z^{k-1}\right| \\ \geq\ & 1 - \sum_{k=2}^{\infty}[1 - \lambda + \mu + [k]_q\,(\lambda - \mu + \lambda\mu[k-1]_q)]^m a_k\,|z|^{k-1} \\ \geq\ & 1 - \sum_{k=2}^{\infty}[1 - \lambda + \mu + [k]_q\,(\lambda - \mu + \lambda\mu[k-1]_q)]^m a_k \\ >\ & 1 - \frac{\eta\,|\tau|}{(q + \eta\,|\tau|)} = \frac{q}{(q + \eta\,|\tau|)} > 0, \end{aligned}$$

then, we find that

$$\begin{aligned}\left|\frac{z\nabla_q(\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z))}{\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z)}-1\right| &= \left|\frac{\sum_{k=2}^{\infty}(1-[k]_q)[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m a_k z^{k-1}}{1-\sum_{k=2}^{\infty}[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m a_k z^{k-1}}\right|\\ &\le \frac{\sum_{k=2}^{\infty}([k]_q-1)[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m a_k\left|z\right|^{k-1}}{1-\sum_{k=2}^{\infty}[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m a_k\left|z\right|^{k-1}}\\ &\le \frac{\sum_{k=2}^{\infty}([k]_q-1)[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m a_k}{1-\sum_{k=2}^{\infty}[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m a_k}\\ &< \eta\left|\tau\right|.\end{aligned}$$

Hence $\mathcal{F}(z)$ satisfies the condition (5).

Assume $\mathcal{F}\in\mathbb{S}^m_q(\tau,\lambda,\mu,\eta)$, then

$$\begin{aligned}\mathrm{Re}\left\{\frac{z\nabla_q(\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z))}{\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(z)}\right\} &= \mathrm{Re}\left\{\frac{1-\sum_{k=2}^{\infty}[k]_q[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m a_k z^{k-1}}{1-\sum_{k=2}^{\infty}[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m a_k z^{k-1}}\right\}\\ &> 1-\eta\left|\tau\right|,\end{aligned}$$

as $z\to 1^-$, we can see that

$$\begin{aligned}&1-\sum_{k=2}^{\infty}[k]_q[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m a_k\\ >\ &(1-\eta\left|\tau\right|)(1-\sum_{k=2}^{\infty}[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m a_k).\end{aligned}$$

Thus, we have the inequality (10).

Corollary 2. *The function* $\mathcal{F}\in\mathbb{S}^m_q(\gamma,\lambda,\mu)$ *if and only if*

$$\sum_{k=2}^{\infty}([k]_q-\gamma)[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m a_k\le 1-\gamma. \tag{11}$$

Theorem 3. *If the function* $\mathcal{F}\in\mathbb{S}^m_q(\tau,\lambda,\mu,\eta)$, *then*

$$\sum_{k=2}^{\infty}a_k\le\frac{\eta\left|\tau\right|}{(q+\eta\left|\tau\right|)[1-\lambda+\mu+[2]_q\left(\lambda-\mu+\lambda\mu\right)]^m}, \tag{12}$$

and

$$\sum_{k=2}^{\infty}[k]_q a_k\le\frac{[2]_q\eta\left|\tau\right|}{(q+\eta\left|\tau\right|)[1-\lambda+\mu+[2]_q\left(\lambda-\mu+\lambda\mu\right)]^m}. \tag{13}$$

Proof. Let $\mathcal{F} \in \mathbb{S}_q^m(\tau, \lambda, \mu, \eta)$. Then, in view of the assertion (10) of Theorem 1, we have

$$(q+\eta|\tau|)[1-\lambda+\mu+[2]_q(\lambda-\mu+\lambda\mu)]^m \sum_{k=2}^{\infty} a_k \leq \eta|\tau|, \tag{14}$$

which immediately yields the first assertion of Theorem 2.

For the proof of the second assertion, by appealing to (10), we have

$$[1-\lambda+\mu+[2]_q(\lambda-\mu+\lambda\mu)]^m \sum_{k=2}^{\infty}[k]_q a_k \leq \eta|\tau|+(1-\eta|\tau|)[1-\lambda+\mu+[2]_q(\lambda-\mu+\lambda\mu)]^m \sum_{k=2}^{\infty} a_k, \tag{15}$$

which in view of (12), can be putten in the form:

$$[1-\lambda+\mu+[2]_q(\lambda-\mu+\lambda\mu)]^m \sum_{k=2}^{\infty}[k]_q a_k \leq \eta|\tau|+(1-\eta|\tau|)\frac{\eta|\tau|}{(q+\eta|\tau|)}. \tag{16}$$

Upon simplifying the right hand side of (16), we have the assertion (13).

Theorem 4. *If* $\mathcal{F} \in \mathbb{S}_q^m(\tau, \lambda, \mu, \eta)$. *Then,*

$$|\mathcal{F}(z)| \leq |z| + \frac{\eta|\tau|}{(q+\eta|\tau|)[1-\lambda+\mu+[2]_q(\lambda-\mu+\lambda\mu)]^m}|z|^2, \tag{17}$$

$$|\mathcal{F}(z)| \geq |z| - \frac{\eta|\tau|}{(q+\eta|\tau|)[1-\lambda+\mu+[2]_q(\lambda-\mu+\lambda\mu)]^m}|z|^2. \tag{18}$$

Equality holds for

$$\mathcal{F}(z) = z - \frac{\eta|\tau|}{(q+\eta|\tau|)[1-\lambda+\mu+[2]_q(\lambda-\mu+\lambda\mu)]^m}z^2.$$

Proof. Suppose $\mathcal{F}(z) \in \mathcal{T}$. Then

$$|\mathcal{F}(z)| = \left|z - \sum_{k=2}^{\infty} a_k z^k\right| \leq |z| + |z|^2 \sum_{k=2}^{\infty} a_k,$$

and

$$|\mathcal{F}(z)| = \left|z - \sum_{k=2}^{\infty} a_k z^k\right| \geq |z| - |z|^2 \sum_{k=2}^{\infty} a_k,$$

Since $\mathcal{F} \in \mathbb{S}_q^m(\tau, \lambda, \mu, \eta)$, then in view of (12), we have the assertions (17) and (18).

Theorem 5. *For* $\mathcal{F} \in \mathbb{S}_q^m(\tau, \lambda, \mu, \eta)$*. Then,*

$$|\nabla_q \mathcal{F}(z)| \leq 1 + \frac{[2]_q \eta |\tau|}{(q + \eta |\tau|)[1 - \lambda + \mu + [2]_q (\lambda - \mu + \lambda\mu)]^m} |z|, \tag{19}$$

$$|\nabla_q \mathcal{F}(z)| \geq 1 - \frac{[2]_q \eta |\tau|}{(q + \eta |\tau|)[1 - \lambda + \mu + [2]_q (\lambda - \mu + \lambda\mu)]^m} |z|. \tag{20}$$

Equality holds for

$$\nabla_q \mathcal{F}(z) = 1 - \frac{[2]_q \eta |\tau|}{(q + \eta |\tau|)[1 - \lambda + \mu + [2]_q (\lambda - \mu + \lambda\mu)]^m} z.$$

Proof. Suppose $\mathcal{F}(z) \in \mathcal{T}$. Then

$$|\nabla_q \mathcal{F}(z)| = \left|1 - \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}\right| \leq 1 + |z| \sum_{k=2}^{\infty} [k]_q a_k,$$

and

$$|\nabla_q \mathcal{F}(z)| = \left|1 - \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}\right| \geq 1 - |z| \sum_{k=2}^{\infty} [k]_q a_k,$$

which in view of (13), we have the assertions (19), (20).

We determine inclusion relations for the class $\mathbb{S}_q^m(\tau, \lambda, \mu, \eta)$ involving $q, \delta-$neighborhoods defined by (9).

Theorem 6. *If* $\mathcal{F}(z) \in \mathbb{S}_q^m(\tau, \lambda, \mu, \eta)$*, then*

$$\mathbb{S}_q^m(\tau, \lambda, \mu, \eta) \subset N_{q,\delta}(\mathcal{F}; q) \tag{21}$$

where the parameter δ_q *is given by*

$$\delta_q = \frac{[2]_q \eta |\tau|}{(q + \eta |\tau|)[1 - \lambda + \mu + [2]_q (\lambda - \mu + \lambda\mu)]^m} \tag{22}$$

Proof. For $\mathcal{F}(z) \in \mathbb{S}_q^m(\tau, \lambda, \mu, \eta)$, from Theorem 2, then (13) holds and in view of the (9), we get (21).

Now, we will obtain the modified Hadamard products for the subclass $\mathbb{S}_q^m(\gamma, \lambda, \mu)$.

For $\mathcal{F}_j(z); j = 1, 2$ defined by

$$\mathcal{F}_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k, \ (a_{k,j} \geq 0, \ j = 1, 2), \tag{23}$$

the modified Hadamard product is

$$(\mathcal{F}_1 * \mathcal{F}_2)(z) = z - \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k = (\mathcal{F}_2 * \mathcal{F}_1)(z). \tag{24}$$

Theorem 7. *If* $\mathcal{F}_j(z) \in \mathbb{S}_q^m(\gamma, \lambda, \mu)$, $j = 1, 2$. *Then*

$$(\mathcal{F}_1 * \mathcal{F}_2)(z) \in \mathbb{S}_q^m(\varsigma, \lambda, \mu), \tag{25}$$

where

$$\varsigma = 1 - \frac{[2]_q (1-\gamma)^2}{([2]_q - \gamma)^2 [1 - \lambda + \mu + [2]_q (\lambda - \mu + \lambda\mu)]^m - (1-\gamma)^2}. \tag{26}$$

The result is sharp.

Proof. Employing the techniques used by Schild and Silverman [21], we need to find the largest ς such that

$$\sum_{k=2}^{\infty} \frac{([k]_q - \varsigma)[1 - \lambda + \mu + [k]_q (\lambda - \mu + \lambda\mu[k-1]_q)]^m}{1-\varsigma} a_{k,1} a_{k,2} \leq 1. \tag{27}$$

Since

$$\sum_{k=2}^{\infty} \frac{([k]_q - \gamma)[1 - \lambda + \mu + [k]_q (\lambda - \mu + \lambda\mu[k-1]_q)]^m}{1-\gamma} a_{k,j} \leq 1 \ (j = 1, 2), \tag{28}$$

then Cauchy-Schwarz inequality yields

$$\sum_{k=2}^{\infty} \frac{([k]_q - \gamma)[1 - \lambda + \mu + [k]_q (\lambda - \mu + \lambda\mu[k-1]_q)]^m}{1-\gamma} \sqrt{a_{k,1} a_{k,2}} \leq 1. \tag{29}$$

Thus it suffices to show that

$$\begin{aligned} & \frac{([k]_q - \varsigma)[1 - \lambda + \mu + [k]_q (\lambda - \mu + \lambda\mu[k-1]_q)]^m}{1-\varsigma} a_{k,1} a_{k,2} \\ \leq \ & \frac{([k]_q - \gamma)[1 - \lambda + \mu + [k]_q (\lambda - \mu + \lambda\mu[k-1]_q)]^m}{1-\gamma} \sqrt{a_{k,1} a_{k,2}}, \end{aligned} \tag{30}$$

that is,

$$\sqrt{a_{k,1} a_{k,2}} \leq \frac{([k]_q - \gamma)(1-\varsigma)}{([k]_q - \varsigma)(1-\gamma)} (k \geq 2). \tag{31}$$

From (29) and (31), we need to prove that

$$\begin{aligned} & \frac{1-\gamma}{([k]_q-\gamma)[1-\lambda+\mu+[k]_q(\lambda-\mu+\lambda\mu[k-1]_q)]^m} \\ \leq \quad & \frac{([k]_q-\gamma)(1-\varsigma)}{([k]_q-\varsigma)(1-\gamma)}, \end{aligned} \tag{32}$$

which leads to

$$\varsigma \leq 1-\frac{[k]_q(1-\gamma)^2}{([k]_q-\gamma)^2[1-\lambda+\mu+[k]_q(\lambda-\mu+\lambda\mu[k-1]_q)]^m-(1-\gamma)^2}. \tag{33}$$

Since

$$\Phi_q(k)=1-\frac{[k]_q(1-\gamma)^2}{([k]_q-\gamma)^2[1-\lambda+\mu+[k]_q(\lambda-\mu+\lambda\mu[k-1]_q)]^m-(1-\gamma)^2}, \tag{34}$$

is an increasing function of k $(k \geq 2)$, letting $k=2$ in (34), we obtain

$$\varsigma \leq \Phi_q(2)=1-\frac{[2]_q(1-\gamma)^2}{([2]_q-\gamma)^2[1-\lambda+\mu+[2]_q(\lambda-\mu+\lambda\mu)]^m-(1-\gamma)^2}, \tag{35}$$

which proves the main assertion of Theorem 6.

Finally,

$$\mathcal{F}_j(z)=z-\frac{1-\gamma}{([2]_q-\gamma)[1-\lambda+\mu+[2]_q(\lambda-\mu+\lambda\mu)]^m}z^2 \quad (j=1,2), \tag{36}$$

give the sharpness.

Theorem 8. *If* $\mathcal{F}_1(z) \in \mathbb{S}_q^m(\gamma,\lambda,\mu)$ *and* $\mathcal{F}_2(z) \in \mathbb{S}_q^m(\rho,\lambda,\mu)$. *Then*

$$(\mathcal{F}_1 * \mathcal{F}_2)(z) \in \mathbb{S}_q^m(\xi,\lambda,\mu), \tag{37}$$

where

$$\xi=1-\frac{[2]_q(1-\gamma)(1-\rho)}{([2]_q-\gamma)([2]_q-\rho)[1-\lambda+\mu+[2]_q(\lambda-\mu+\lambda\mu)]^m-(1-\gamma)(1-\rho)}. \tag{38}$$

The result is the best possible for

$$\begin{aligned} \mathcal{F}_1(z) &= z-\frac{1-\gamma}{([2]_q-\gamma)[1-\lambda+\mu+[2]_q(\lambda-\mu+\lambda\mu)]^m}z^2, \\ \mathcal{F}_2(z) &= z-\frac{1-\rho}{([2]_q-\rho)[1-\lambda+\mu+[2]_q(\lambda-\mu+\lambda\mu)]^m}z^2. \end{aligned} \tag{39}$$

Proof. Proceeding as in the proof of Theorem 6, we get

$$\xi \leq \Psi_q(k) = 1 - \frac{[k]_q(1-\gamma)(1-\rho)}{([k]_q-\gamma)([k]_q-\rho)[1-\lambda+\mu+[k]_q(\lambda-\mu+\lambda\mu[k-1]_q)]^m-(1-\gamma)(1-\rho)}, \tag{40}$$

since the function $\Psi_q(k)$ is an increasing function of k $(k \geq 2)$, setting $k = 2$ in (40), we get

$$\xi \leq \Psi_q(2) = 1 - \frac{[2]_q(1-\gamma)(1-\rho)}{([2]_q-\gamma)([2]_q-\rho)[1-\lambda+\mu+[2]_q(\lambda-\mu+\lambda\mu)]^m-(1-\gamma)(1-\rho)}. \tag{41}$$

This completes the proof.

Theorem 9. *If* $\mathcal{F}_j(z) \in \mathbb{S}_q^m(\gamma, \lambda, \mu)$ $(j = 1, 2, 3)$. *Then*

$$(\mathcal{F}_1 * \mathcal{F}_2 * \mathcal{F}_3)(z) \in \mathbb{S}_q^m(\omega, \lambda, \mu), \tag{42}$$

where

$$\omega = 1 - \frac{[2]_q(1-\gamma)^3}{([2]_q-\gamma)^3[1-\lambda+\mu+[2]_q(\lambda-\mu+\lambda\mu)]^{2m}-(1-\gamma)^3}. \tag{43}$$

The result is the best possible for $\mathcal{F}_j(z)$ given by (36), $j = 1, 2, 3$.

Proof. From Theorem 6, we have $(\mathcal{F}_1 * \mathcal{F}_2)(z) \in \mathbb{S}_q^m(\varsigma, \lambda, \mu)$, where ς is given by (26). By using Theorem 7, we get (42), where

$$\omega = 1 - \frac{[2]_q(1-\gamma)(1-\varsigma)}{([2]_q-\gamma)([2]_q-\varsigma)[1-\lambda+\mu+[2]_q(\lambda-\mu+\lambda\mu)]^m-(1-\gamma)(1-\varsigma)}. \tag{44}$$

Then we get (43).

This completes the proof.

Theorem 10. *If* $\mathcal{F}_j(z) \in \mathbb{S}_q^m(\gamma, \lambda, \mu)$ $(j = 1, 2, 3)$. *Then*

$$h(z) = z - \sum_{k=2}^{\infty}(a_{k,1}^2 + a_{k,2}^2)z^k, \tag{45}$$

belongs to the class $\mathbb{S}_q^m(\sigma, \lambda, \mu)$, *where*

$$\sigma = 1 - \frac{2[2]_q(1-\gamma)^2}{([2]_q-\gamma)^2[1-\lambda+\mu+[2]_q(\lambda-\mu+\lambda\mu)]^m-2(1-\gamma)^2} \tag{46}$$

Proof. By virtue of Corollary 1, we obtain

$$\sum_{k=2}^{\infty}\left[\frac{([k]_q-\gamma)[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m}{1-\gamma}\right]^2 a_{k,j}^2$$
$$\leq \sum_{k=2}^{\infty}\left[\frac{([k]_q-\gamma)[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m}{1-\gamma}a_{k,j}\right]^2 \leq 1 \ (j=1,2). \tag{47}$$

It follows that

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{([k]_q-\gamma)[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m}{1-\gamma}\right]^2 (a_{k,1}^2+a_{k,2}^2)\leq 1. \tag{48}$$

Therefore, we need to find the largest σ such that

$$\frac{([k]_q-\sigma)[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m}{1-\sigma}$$
$$\leq \frac{1}{2}\left[\frac{([k]_q-\gamma)[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m}{1-\gamma}\right]^2 \ (k\geq 2), \tag{49}$$

that is,

$$\sigma\leq 1-\frac{2[k]_q(1-\gamma)^2}{([k]_q-\gamma)^2[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m-2(1-\gamma)^2} \ (k\geq 2). \tag{50}$$

Since

$$\varphi_q(k)=1-\frac{2[k]_q(1-\gamma)^2}{([k]_q-\gamma)^2[1-\lambda+\mu+[k]_q\left(\lambda-\mu+\lambda\mu[k-1]_q\right)]^m-2(1-\gamma)^2}, \tag{51}$$

is an increasing function of k ($k\geq 2$), setting $k=2$ we readily have

$$\sigma\leq\varphi_q(2)=1-\frac{2[2]_q(1-\gamma)^2}{([2]_q-\gamma)^2[1-\lambda+\mu+[2]_q\left(\lambda-\mu+\lambda\mu\right)]^m-2(1-\gamma)^2}. \tag{52}$$

The functions $\mathcal{F}_j(z)$ given by (36) gives the sharpness.

Remark 1. *For different values of τ,λ,μ,q and η in our results, we have results for the special classes defined in the introduction.*

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