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IDEALS IN THE BANACH ALGEBRAS OF α-LIPSCHITZ VECTOR-VALUED OPERATORS

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Abstract. We study an interesting class of Banach function algebras of vector-valued operators on compact metric spaces, and investigate certain ideals of the Lipschitz algebras. In this paper, we consider a nonempty compact metric space (X, d) and a commutative unital Banach algebra $(B, \| \,.\, \|)$ over the scalar field $\mathbb{F}(= \mathbb{R}$ *or* $\mathbb{C})$. At first, we define the B-valued α-Lipschitz operator algebras $Lip_\alpha(X, B)$ and $lip_\alpha(X, B)$, where $\alpha \in (0, 1]$. Then we characterize the norm closed ideals of $lip_\alpha(X, B)$, and primary ideals of $Lip_\alpha(X, B)$.

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1. Introduction

Throughout this paper, let (X, d) be a compact metric space which has at least two elements, $(B, \| \,.\, \|)$ be a commutative unital Banach algebra over the scalar field $\mathbb{F}(= \mathbb{R}$ *or* $\mathbb{C})$ with unit $\mathbf{e}$, $C(X, B)$ be the set of all B-valued continuous operators and $C_b(X, B)$ be the set of all bounded B-valued continuous operators on X, and also $\alpha \in \mathbb{R}$ with $0 < \alpha \leq 1$. When $B = \mathbb{F}$, we write $C(X)$ instead of $C(X, B)$.

The dual space of B is the vector space B^* whose elements are the continuous linear functionals on B. The set of all multiplicative functionals on B is called *spectrum* of B; we denote it by $\sigma(B)$. Suppose that throughout this article $\Lambda \in \sigma(B)$ is arbitrary and fixed. Since $\sigma(B)$ is a subset of the closed unit ball of B^*, $\| \Lambda \|$ is bounded, where

$$\| \Lambda \| = \sup\{ \ | \Lambda x | \ : \ x \in B \ , \ \| x \| \leq 1 \ \}.$$

When $B = \mathbb{F}$, take Λ as the identity function $\Lambda x = x$.

Consider the set Y as follows

$$Y := \{(x, y) \ : \ x, y \in X \ , \ x \neq y\}. \tag{1}$$

For an operator $f : X \to B$, and any $(x, y) \in Y$, define

$$L_f^\alpha(x,y) := \frac{\Big|\big(\Lambda o f\big)(x) - \big(\Lambda o f\big)(y)\Big|}{d^\alpha(x,y)}, \tag{2}$$

where $d^\alpha(x,y) = \big(d(x,y)\big)^\alpha$, and define

$$p_\alpha(f) := \sup_{x \neq y} \ L_f^\alpha(x,y),$$

which is called the *Lipschitz constant* of f. Also, for $0 < \alpha \leq 1$ define

$$Lip_\alpha(X,B) := \{f \in C_b(X,B) \ : \ p_\alpha(f) < +\infty\},$$

and for $0 < \alpha < 1$, define

$$lip_\alpha(X,B) := \{f \in Lip_\alpha(X,B) \ : \ \lim_{d(x,y)\to 0} L_f^\alpha(x,y) = 0\}.$$

The elements of $Lip_\alpha(X,B)$ and $lip_\alpha(X,B)$ are called *big* and *little* α-Lipschitz B-valued operators, respectively.

Now, for each $\lambda \in \mathbb{F}$ and $f, g \in C(X,B)$ define

$$\big(f+g\big)(x) := f(x) + f(x) \ , \ \big(\lambda f\big)(x) := \lambda f(x) \ , \ \forall x \in X,$$

$$\| \ f \ \|_\infty := \sup_{x \in X} \| \ f(x) \ \|,$$

and for any $f \in Lip_\alpha(X,B)$ define

$$\| \ f \ \|_\alpha := p_\alpha(f) + \| \ f \ \|_\infty \ .$$

It is easy to see that $\big(C(X,B), \| \ . \ \|_\infty\big)$ becomes a Banach algebra over $\mathbb{F}$.

Cao, Zhang and Xu in [9] proved that $\big(Lip_\alpha(X,B), \| \ . \ \|_\alpha\big)$ is a Banach space over $\mathbb{F}$ and $\big(lip_\alpha(X,B), \| \ . \ \|_\alpha\big)$ is a closed linear subspace of $\big(Lip_\alpha(X,B), \|, . \ \|_\alpha\big)$, when B is a Banach space.

We studied some of the properties of these algebras in [16, 17, 18, 19]. Also some properties of these algebras were studied by certain mathematicians including Abtahi [2], Ranjbary and Rejali [13].

Note that for $\alpha = 1$ and $B = \mathbb{F}$, the space $Lip_1(X, \mathbb{F})$ consisting of all Lipschitz functions from X into $\mathbb{F}(= \mathbb{R} \ or \ \mathbb{C})$ has a series of interesting and important properties, which has been studied by many mathematicians. Including the characterization of the ideals of these algebras in [1, 3 - 8, 11, 12, 14, 15] were researched and studied. In [10, 20] some properties of Lipschitz scalar-valued functions are mentioned.

Finally, in this paper we study the algebras of α-Lipschitz B-valued operators, and we will characterize the norm closed ideals of $lip_\alpha(X,B)$, and primary ideals of $Lip_\alpha(X,B)$.

2. Norm closed ideals

In this section, we characterize the norm closed ideals of little α-Lipschitz operator algebras $lip_\alpha(X,B)$. So suppose that $\alpha \in \mathbb{R}$ with $0<\alpha<1$.

In the complex plan $\mathbb{C}$, let $D(0,r)$ be the closed disk with center at the origin and radius $r>0$. Define the map $\Pi_r : \mathbb{C} \to D(0,r)$ by

$$\Pi_r(z) = \begin{cases} z & ; \quad |z| \leq r \\ \frac{rz}{|z|} & ; \quad |z| > r. \end{cases} \tag{3}$$

Lemma 1. *Let $f \in lip_\alpha(X,B)$, and define $\Lambda o f_n := \Pi_{\frac{1}{n}}(\Lambda o f)$; $n \in \mathbb{N}$. Then $\Lambda o f_n \in lip_\alpha(X,B)$ for any $n \in \mathbb{N}$.*

Proof. Since $f \in lip_\alpha(X,B)$, for any $(x,y) \in Y$ (Y is defined in (1)) we have

$$\lim_{d(x,y)\to 0} \frac{\left|\big(\Lambda o f\big)(x) - \big(\Lambda o f\big)(y)\right|}{d^\alpha(x,y)} = 0.$$

Then for each $n \geq 1$ and $(x,y) \in Y$, we have

$$\lim_{d(x,y)\to 0} \frac{\left|\big(\Lambda o f_n\big)(x) - \big(\Lambda o f_n\big)(y)\right|}{d^\alpha(x,y)} = \lim_{d(x,y)\to 0} \frac{\left|\Pi_{\frac{1}{n}}\Big(\big(\Lambda o f\big)(x)\Big) - \Pi_{\frac{1}{n}}\Big(\big(\Lambda o f\big)(y)\Big)\right|}{d^\alpha(x,y)}. \tag{4}$$

Now we have three case:
Case 1. Suppose $\left|\big(\Lambda o f\big)(x)\right| \leq \frac{1}{n}$ and $\left|\big(\Lambda o f\big)(y)\right| \leq \frac{1}{n}$. Then

$$(4) = \lim_{d(x,y)\to 0} \frac{\left|\big(\Lambda o f\big)(x) - \big(\Lambda o f\big)(y)\right|}{d^\alpha(x,y)} = 0.$$

Case 2. Suppose $\left|\big(\Lambda o f\big)(x)\right| > \frac{1}{n}$ and $\left|\big(\Lambda o f\big)(y)\right| > \frac{1}{n}$. Then

$$(4) = \lim_{d(x,y)\to 0} \frac{\left|\frac{\frac{1}{n}(\Lambda o f)(x)}{\left|(\Lambda o f)(x)\right|} - \frac{\frac{1}{n}(\Lambda o f)(y)}{\left|(\Lambda o f)(y)\right|}\right|}{d^\alpha(X,B)}\ , \tag{5}$$

if $\left|(\Lambda o f)(x)\right| = \left|(\Lambda o f)(y)\right|$, then

$$(5) = \frac{1}{n\left|(\Lambda o f)(x)\right|} \times \lim_{d(x,y)\to 0} \frac{\left|\big(\Lambda o f\big)(x) - \big(\Lambda o f\big)(y)\right|}{d^\alpha(x,y)} = 0,$$

and so $(4) = 0$.

If $\left|(\Lambda of)(x)\right| \neq \left|(\Lambda of)(y)\right|$, then we can assumed that $\left|(\Lambda of)(x)\right| > \left|(\Lambda of)(y)\right|$. Therefore

$$(5) \leq \frac{1}{n\,\left|(\Lambda of)(y)\right|} \times \lim_{d(x,y)\to 0} \frac{\left|\big(\Lambda of\big)(x) - \big(\Lambda of\big)(y)\right|}{d^{\alpha}(x,y)} = 0,$$

and so $(4) = 0$.
Case 3. Suppose $\left|(\Lambda of)(x)\right| > \frac{1}{n}$, $\left|(\Lambda of)(y)\right| \leq \frac{1}{n}$. Then

$$(4) = \lim_{d(x,y)\to 0} \frac{\left|\frac{\frac{1}{n}(\Lambda of)(x)}{\left|(\Lambda of)(x)\right|} - \big(\Lambda of\big)(y)\right|}{d^{\alpha}(X,B)} \leq \lim_{\to 0} \frac{\left|\big(\Lambda of\big)(x) - \big(\Lambda of\big)(y)\right|}{d^{\alpha}(x,y)} = 0,$$

and so $(4) = 0$.
Consequently, in any case we have

$$\lim_{d(x,y)\to 0} \frac{\left|\big(\Lambda of_n\big)(x) - \big(\Lambda of_n\big)(y)\right|}{d^{\alpha}(x,y)} = 0 \ ; \quad n \in \mathbf{N}.$$

This means for any $n \in \mathbb{N}$, $\Lambda of_n \in lip_{\alpha}(X,B)$. $\triangle$

Let H be a non-empty closed subset of X. Put

$$i(H) := \{f \in lip_{\alpha}(X,B) \ : \ (\Lambda of)\big|_H = 0\},$$

where $(\Lambda of)\big|_H$ is the restriction of Λof to H. It is easy to see that, $i(H)$ is an ideal of $lip_{\alpha}(X,B)$.

Lemma 2. *Suppose H is a closed subset of X, and $f \in i(H)$. Then there is a sequence $\{f_n\} \subset lip_{\alpha}(X,B)$ such that each f_n is equal to f on a neighborhood of H, and $\lim_{n\to+\infty} p_{\alpha}(\Lambda of_n) = 0$.*

Proof. For any $n \in \mathbb{N}$, define $\Lambda of_n := \Pi_{\frac{1}{n}}(\Lambda of)$, where the map Π_r is defined in (3). Then for each $n \in \mathbb{N}$, $\Lambda of_n \in lip_{\alpha}(X,B)$ by Lemma 1. Since $f \in i(H)$, $(\Lambda of)\big|_H = 0$. So for any $n \in \mathbb{N}$ and $x \in H$, $\left|(\Lambda of_n)(x)\right| < \frac{1}{n}$. Therefor on a neighborhood of H, we have

$$\Lambda(f_n(x)) = (\Lambda of_n)(x) = \Pi_{\frac{1}{n}}\big((\Lambda of)(x)\big) = (\Lambda of)(x) = \Lambda(f(x)).$$

Since $\Lambda \in \sigma(B)$ is arbitrary, $f_n(x) = f(x)$ on a neighborhood of H, where $n \in \mathbb{N}$.

Now, since for any $n \in \mathbb{N}$ we have $\Lambda o f_n \in lip_\alpha(X,B)$, for each $\epsilon > 0$ there exists $\delta > 0$ such that for any $(x,y) \in Y$ (Y is defined in (1)) with $d(x,y) < \delta$ we have

$$\frac{\big|(\Lambda o f_n)(x) - (\Lambda o f_n)(y)\big|}{d^\alpha(x,y)} < \epsilon.$$

Especially for $\epsilon = \frac{1}{n}$ (to large enough n) we have

$$\frac{\big|(\Lambda o f_n)(x) - (\Lambda o f_n)(y)\big|}{d^\alpha(x,y)} < \frac{1}{n}.$$

So, for to large enough n, $p_\alpha(\Lambda o f_n) < \frac{1}{n}$. Therefore $\lim_{n\to+\infty} p_\alpha(\Lambda o f_n) = 0$. △

For each subset $E \subset lip_\alpha(X,B)$, let its *hull* be the set

$$hull(E) := \{x \in X \ : \ (\Lambda o f)(x) = 0 \ , \ \ \forall f \in E\}.$$

A subset E of $lip_\alpha(X,B)$ is a *norm closed ideal*, if it is an ideal and it is closed in the topology induced by the norm on $lip_\alpha(X,B)$.

Lemma 3. *Let E be a norm closed ideal of $lip_\alpha(X,B)$, and suppose $f \in lip_\alpha(X,B)$ such that $\Lambda o f$ vanishes in a neighborhood of $hull(E)$. Then $f \in E$.*

Proof. Let $H := hull(E)$, $\epsilon > 0$, and $(\Lambda o f)(x) = 0$ for any $x \in X$ such that $d(x,H) < \epsilon$, where $d(x,H) := \inf\{d(x,y) \ : \ y \in H\}$. Suppose that $G := \{x \in X \ : \ d(x,H) \geq \frac{\epsilon}{2}\}$. It is obvious that G is a compact subset of X, and for any $x \in G$ there is a function $f_x \in E$ that $\Lambda o f_x$ is nonzero on an open neighborhood of x. As these neighborhoods cover G, by compactness. So we can find a finite set of points $x_1, x_2, ..., x_n \in G$ such that $\Lambda o g$ is nowhere zero on G, where $g := f_{x_1} + f_{x_2} + ... + f_{x_n}$. Then $g \in E$ and $g(x)$ is invertible for any $x \in G$. Define the function $h \in lip_\alpha(X,B)$ such that $(\Lambda o h)(x) := 0$ for $x \notin G$, and $h(x) := \big(g(x)\big)^{-1} f(x)$ for $x \in G$. Then $f = gh$ on G. By ideal properties, we have $f \in E$. △

Now we prove one of the main results of the article.

Theorem 4. *Let E be a norm closed ideal of $lip_\alpha(X,B)$. Then $E = i(H)$, where $H = hull(E)$.*

Proof. It is obvious that $E \subseteq i(H)$. We prove that $i(H) \subseteq E$. For this purpose, let $f \in i(H)$ be arbitrary, so we will show that $f \in E$.

It is clear that $hull(E)$ is a closed subset of X. So by Lemma 2, there is a sequence $\{f_n\} \subset lip_\alpha(X,B)$ such that $f_n = f$ on a neighborhood of H ($n \geq 1$), and

$\lim_{n\to+\infty} p_\alpha(\Lambda o f_n) = 0$. So $\Lambda o(f - f_n) = 0$ on a neighborhood of H $(n \geq 1)$. Then $f - f_n \in E$ $(n \geq 1)$ by Lemma 3. Since $\lim_{n\to+\infty} p_\alpha(\Lambda o f_n) = 0$ on a neighborhood of H,

$$\lim_{n\to+\infty} \frac{\left|(\Lambda o f_n)(x) - (\Lambda o f_n)(y)\right|}{d^\alpha(x,y)} = 0 \; ; \quad (x \neq y),$$

$$\Longrightarrow \lim_{n\to+\infty} \left|(\Lambda o f_n)(x) - (\Lambda o f_n)(y)\right| = 0 \; ; \quad (x \neq y),$$

$$\Longrightarrow \lim_{n\to+\infty} (\Lambda o f_n)(x) = \lim_{n\to+\infty} (\Lambda o f_n)(y) \; ; \quad (x \neq y),$$

on neighborhood of H. This relation shoes that f_n is a constant function on a neighborhood of H for each $n \geq 1$. So, by definition of $H = hull(E)$ and $f \in i(H)$, we have $\lim_{n\to+\infty}(\Lambda o f_n)(x) = 0$ in a neighborhood of H. Then $\sup \mid (\Lambda o f_n)(x) \mid \to 0$ on a neighborhood of H. Thus $\parallel \Lambda o f_n \parallel_\infty \to 0$ on a neighborhood of H. On the other hand we have $\lim_{n\to+\infty} p_\alpha(\Lambda o f_n) = 0$, so

$$\parallel \Lambda o f_n \parallel_\alpha = \parallel \Lambda o f_n \parallel_\infty + p_\alpha(\Lambda o f_n) \to 0$$

on a neighborhood of H.

Now define $g_n := f - f_n$ $(n \geq 1)$. Then $\{g_n\} \subset E$, and so we have

$$\parallel \Lambda o(f - g_n) \parallel_\alpha = \parallel \Lambda o f_n \parallel_\alpha \to 0$$

on a neighborhood of H. Since Λ is arbitrary, $\parallel f - g_n \parallel_\alpha \to 0$ on a neighborhood of H. Since $\{g_n\} \subset E$ and E is a norm closed ideal, $f \in E$. This completes the proof. $\triangle$

3. Primary ideals

In this section, we characterize the primary ideals of big α-Lipschitz operator algebras $Lip_\alpha(X, B)$. So suppose that $\alpha \in \mathbb{R}$ with $0 < \alpha \leq 1$.

Let H be a non-empty closed subset of X. Put

$$I(H) := \{f \in Lip_\alpha(X, B) \; : \; (\Lambda o f)\big|_H = 0\}.$$

Define the mapping λ as follows:

$$\lambda \; : \; Lip_\alpha(X, B) \to C(Y)$$

$$f \mapsto \lambda f$$

where Y is defined in (1), and $\lambda f \ : \ Y \mapsto \mathbb{F}$ with the criterion

$$\big(\lambda f\big)(x,y) := \frac{(\Lambda o f)(x) - (\Lambda o f)(y)}{d^{\alpha}(x,y)}.$$

Then $L_f^{\alpha}(x,y) = \big|(\lambda f)(x,y)\big|$ for all $(x,y) \in Y$, which $L_f^{\alpha}(x,y)$ is defined in (2). Also put

$$J(H) := \{f \in I(H) \ : \ \big|(\lambda f)(x,y)\big| \to 0 \quad as \quad d(x,H)\ ,\ d(y,H) \to 0\}.$$

Clearly for each ideal E in $Lip_{\alpha}(X,B)$ with $hull(E) = H$, we have:

Remark 1. *(i) $J(H)$ is the minimum ideal, and $\overline{J(H)}$ is the minimum closed ideal of $Lip_{\alpha}(X,B)$, where the norm closure $\overline{J(H)}$ of $J(H)$ is the intersection of all closed sets that contain $J(H)$.*
(ii) $I(H)$ is the maximum ideal of $Lip_{\alpha}(X,B)$, and
(iii) $J(H) \subset E \subset I(H)$.

Below we prove a theorem, which we need to prove the main result of the article.

Theorem 5. *Let H be a non-empty closed subset of X. Then $J(H) = \overline{I(H)^2}$, that by $\overline{I(H)^2}$ we mean the norm closure of the set of linear combinations of products fg where $f, g \in I(H)$.*

Proof. Since $J(H)$ and $\overline{I(H)^2}$ are ideals in $Lip_{\alpha}(X,B)$, Remark 1 implies that $J(H) \subseteq \overline{I(H)^2}$.

Now to prove the other side of the relationship, let $f, g \in I(H)$ be arbitrary such that for each $\epsilon > 0$ and any $(x,y) \in Y$

$$\big|(\Lambda o f)(x)\big| < \frac{\epsilon}{2\ L_g^{\alpha}(x,y)} \quad and \quad \big|(\Lambda o g)(y)\big| < \frac{\epsilon}{2\ L_f^{\alpha}(x,y)}$$

when $d(x,H),\ d(y,H) \to 0$. Then for any $(x,y) \in Y$ as $d(x,H),\ d(y,H) \to 0$ we have

$$
\begin{aligned}
\big|\big(\lambda(fg)\big)(x,y)\big| &= \frac{\big|\big(\Lambda o(fg)\big)(x)-\big(\Lambda o(fg)\big)(y)\big|}{d^{\alpha}(x,y)}\\
&= \frac{\big|(\Lambda of)(x)\ (\Lambda og)(x)-(\Lambda of)(y)\ (\Lambda og)(y)\big|}{d^{\alpha}(x,y)}\\
&\leq \frac{1}{d^{\alpha}(x,y)}\Big(\big|(\Lambda of)(x)\big|\ \big|(\Lambda og)(x)-(\Lambda og)(y)\big|\\
&\quad +\big|(\Lambda og)(y)\big|\ \big|(\Lambda of)(x)-(\Lambda of)(y)\big|\Big)\\
&\leq \big|(\Lambda of)(x)\big|\ L_g^{\alpha}(x,y)+\big|(\Lambda og)(y)\big|\ L_f^{\alpha}(x,y)\\
&< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}
$$

This implies that $fg \in J(H)$. It follows that $\overline{I(H)^2} \subseteq J(H)$, and the proof is complete. $\triangle$

Let E be an ideal in $Lip_{\alpha}(X,B)$. E is called *primary* if its hull contains exactly one point.

Now we prove the second main result of the article. The primary ideals of $Lip_{\alpha}(X,B)$ are characterized as follows.

Theorem 6. *Let $a \in X$, and take $H=\{a\}$. Suppose that E be a norm closed subspace of $Lip_{\alpha}(X,B)$ such that $J(H) \subset E \subset I(H)$. Then E is a primary ideal of $Lip_{\alpha}(X,B)$. Conversely, every primary ideal of $Lip_{\alpha}(X,B)$ is of this form.*

Proof. Let $f \in E$ and $g \in Lip_{\alpha}(X,B)$ be arbitrary. Then $g-(\Lambda og)(a) \in I(H)$. Hence, since $J(H)=\overline{I(H)^2}$ by Theorem 2,

$$\big(g-(\Lambda og)(a)\big)f \in I(H)E \subset I(H)^2 \subset J(H) \subset E.$$

Thus $\big(g-(\Lambda og)(a)\big)f \in E$. Since $(\Lambda og)(a)$ is a constant and $f \in E$, we have $(\Lambda og)(a)f \in E$. So $gf \in E$. As the same way, $fg \in E$. This shows that E is an ideal. Since

$$hull(E)=\{x \in X\ :\ (\Lambda of)(x)=0\ ,\ \forall f \in E\}=\{a\},$$

E is clearly primary.
The converse of theorem is true by Remark 1. $\triangle$

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References

[1] M. Abolfathi, A. Ebadian, *The Maximal ideal space of extended differential Lipschitz algebras*, Kyungpook Mathematica J., 60 (1) (2020), 117-125.

[2] M. Abtahi, *Vector-valued characters on vector-valued function algebras*, Banach J. Math. Anal., 10 (3) (2016), 608-620.

[3] D. Achour, E. Dahia, P. Turco, *The Lipschitz injective hull of Lipschitz operator ideals and applications*, Banach J. Math. Anal., 14 (2020) 1241-1257.

[4] C. Agrafeuil, M. Zarrabi, *Closed ideals with countable hull in algebras of analytic functions smooth up to the boundary*, Publ. Mat., 52 (1) (2008), 19-56.

[5] B. Bouya, *Closed ideals in analytic weighted Lipschitz algebras*, Advances in Mathematics, 219 (2008), 1446-1468.

[6] B. Bouya, M. Zarrabi, *On closed ideals in the big Lipschits algebras of analytic functions*, Bull. Sci. Math., 137 (2013), 575-583.

[7] M.G. Cabera-Padilla, J.A. Chavez-Dominguez, A. Jimenez-Vargas, M. Villegas-Vallecillas, *Duality for ideals of Lipschitz maps*, Banach J. Math. Anal., 11 (1) (2017), 108-129.

[8] M.G. Cabera-Padilla, J.A. Chavez-Dominguez, A. Jimenez-Vargas, M. Villegas-Vallecillas, *Maximal Banach ideals of Lipschitz maps*, Ann. Funct. Anal., 7 (4) (2016), 593-608.

[9] H.X. Cao, J.H. Zhang, Z.B. Xu, *Characterizations and extensions of Lipschitz-α operators*, Acta Mathematica Sinica, English series, 22 (3) (2006), 671-678.

[10] H.G. Dales, *Banach Algebras and Automatic continuty*, Clarendon press, Oxford, 2000.

[11] M. Mayghani, D. Alimohammadi, *Closed ideals, point derivations and weak amenability of extended little Lipschitz algebras*, Caspian J. Math. Sci. CJMS., 5 (1) (2016), 23-35.

[12] T.V. Pedersen, *Ideals in the big Lipschitz algebras of analytic functions*, Studia Mathematica, 161 (1) (2004), 33-58.

[13] A. Ranjbari, A. Rejali, *Frechet α-Lipschitz vector-valued operator algebra*, U.P.B. Sci. Bull. series A, 80 (4) (2018), 141-152.

[14] K. Saadi, *On the composition ideals of Lipschitz mappings*, Banach J. Math. Anal., 11 (4) (2017), 825-840.

[15] D.R. Sherbert, *The structure of ideals and point derivations in Banach algebras of Lipschitz functions*, Trans. Amer. Math. Soc., 111 (1964), 240-272.

[16] A.A. Shokri, A. Ebadian, A.R. Medghalchi, *Amenability and weak amenability of Lipschitz operator algebras*, Acta Universitatis Apulensis, 18 (2009), 87-96.

[17] A.A. Shokri, A. Shokri, *Homomorphisms of certain α-Lipschitz operator Algebras*, Acta Universitatis Apulensis, 27 (2011), 9-13.

[18] A.A. Shokri, A. Shokri, *Maximal ideal space of certain α-Lipschitz operator algebras*, J. Math. and App., 35 (2012), 141-147.

[19] A. Shokri, *Second dual space of little α-Lipschitz vector-valued operator algebras*, Sahand commun. Math. Anal. (SCMA), 8 (1) (2017), 33-41.

[20] N. Weaver, *Lipschitz algebras*, World scientific publishing Co., Inc., River Edge, NJ, 1999.

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