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# ON THE SOME GENERALIZATION OF INEQUALITIES ASSOCIATED WITH BULLEN, SIMPSON, MIDPOINT AND TRAPEZOID TYPE

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Abstract. The main object of this paper is to present a new inequalities by using of the Hermite-Hadamard inequalities. Later, in some special cases, Bullen, Simpson, Midpoint and Trapezoid type inequalities and many new results related to these inequalities will be given.

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## 1. Introduction

It is obvious that the theory of inequalities has an important role in the fields of mathematics and engineering applications. In particular, the most important of these inequalities are Hermite-Hadamard inequality, Bullen inequality and Simpson inequality. In addition, many different types of inequalities have been proposed by many authors with the help of convex functions for these inequalities. The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard and Bullen inequalities. It gives an estimate from both sides of the mean value of a convex function and also ensure the integrability of convex function. It is also a matter of great interest and one has to note that some of the classical inequalities for means can be obtained from Hadamard's inequality under the utility of peculiar convex functions $f$. These inequalities for convex functions play a crucial role in analysis and as well as in other areas of pure and applied mathematics. Some of them can be viewed in the references section, [6], [15], [16]. Firstly, we can briefly state these general inequalities as follows.

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping defined on the interval $I$ of real numbers and $a, b \in I$, with $a < b$. The following double inequality is well known in the literature as the Hermite-Hadamard inequality [8]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Given a convex function $f:[a,b]\to\mathbb{R}$, the following inequality is well known in the literature as the Bullen inequality [15]:

$$\frac{2}{b-a}\int_a^b f(x)\,dx \leq f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}.$$

Bullen-type inequalities were obtained with the help of the different kinds of convexity of the first-order absolute value function in [2], [3]. Further, Noor et. al generalized the Bullen inequality for $h$-convex functions in [13].

The following inequality is well known in the literature as Simpson's inequality.

Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable mapping on $(a,b)$ and $\left\|f^{(4)}\right\|_\infty=\sup\left|f^{(4)}(x)\right|<\infty$. Then, the following inequality holds:

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see ([1], [5], [9], [10]-[12], [14], [17]-[23]).

Trapezoid and midpoint inequalities are another known inequalities related to the right and left sides of the Hermite-Hadamard inequality. In order to obtain these inequalities, as stated below, identities play an important role.

In [7], Dragomir and Agarwal proved the following results connected with the right part of (1).

**Lemma 1.** *Let $f:I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $I^\circ$, $a,b\in I^\circ$ ($I^\circ$ is the interior of $I$) with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \tag{2}$$

**Theorem 2.** *Let $f:I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $I^\circ$, $a,b\in I^\circ$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right). \tag{3}$$

In [11], Kirmaci proved the following results connected with the left part of (1). In [11] some inequalities of Hermite-Hadamard type for differentiable convex mappings were proved using the following lemma.

**Lemma 3.** *Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on $I^{\circ}$, $a, b \in I^{\circ}$ with $a < b$. If $f' \in L([a, b])$, then we have*

$$\begin{aligned} & \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \qquad (4) \\ = \; & (b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)\, f'(ta+(1-t)b)dt\right]. \end{aligned}$$

**Theorem 4.** *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$, $a, b \in I^{\circ}$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right). \qquad (5)$$

The purpose of this paper is established a new inequalities by using of the Hermite-Hadamard inequalities. Later, in some special cases, Bullen, Simpson, midpoint and trapezoid type inequalities and many new results related to these inequalities will be given. Using functions whose first derivatives absolute values are convex, we obtained new Bullen, Simpson, trapezoid and midpoint inequalities that are connected with the celebrated Bullen type.

## 2. Generalized Bullen type inequalities

In this section, we begin by the following theorem:

**Theorem 5.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$, then the following inequalities hold:*

$$\begin{aligned} & f\left(\frac{a+x}{2}\right) + f\left(\frac{b+x}{2}\right) \qquad (6) \\ \leq \; & \frac{1}{x-a}\int_a^x f(t)\,dt + \frac{1}{b-x}\int_x^b f(t)\,dt \\ \leq \; & f(x) + \frac{f(a)+f(b)}{2} \end{aligned}$$

*for $x \in [a, b]$.*

*Proof.* We splint interval $[a, b]$ into $[a, x]$ and $[x, b]$. Since $f$ is a convex function on $[a, x] \subset [a, b]$, by using inequalities (1) we get

$$f\left(\frac{a+x}{2}\right) \leq \frac{1}{x-a}\int_a^x f(t)\,dt \leq \frac{f(a)+f(x)}{2}. \qquad (7)$$

By similar way for $[x,b] \subset [a,b]$, it follows that

$$f\left(\frac{b+x}{2}\right) \leq \frac{1}{b-x}\int_x^b f(t)\,dt \leq \frac{f(b)+f(x)}{2}. \tag{8}$$

As consequence, by adding (7) and (8) we obtain

$$f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right) \leq \frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt \leq f(x)+\frac{f(a)+f(b)}{2}$$

which completes the proof of (1).

**Corollary 6.** *Under assumption of Theorem 5, if we choose $x=\frac{a+b}{2}$, we have the following Bullen type inequalities*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]. \end{aligned} \tag{9}$$

*Proof.* By convexity of $f$, we have

$$f\left(\frac{a+b}{2}\right)=f\left(\frac{1}{2}\frac{3a+b}{4}+\frac{1}{2}\frac{a+3b}{4}\right) \leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]$$

which completes the proof of (9).

**Remark 1.** *In Theorem 5, if we choose $x=a$ or $x=b$, then the inequalities (6) become the inequalities (1) by using the fact that*

$$\lim_{x\to a}\frac{1}{x-a}\int_a^x f(t)dt = f(a) \quad or \quad \lim_{x\to b}\frac{1}{b-x}\int_x^b f(t)dt = f(b).$$

## 3. Trapezoid type inequalities

In this section, we give an identity which use to assist us is proving our results as follows:

**Lemma 7.** *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$f(x) + \frac{f(a)+f(b)}{2} - \left[\frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt\right] \tag{10}$$

$$= \frac{(x-a)}{2}\int_0^1 (1-2\lambda) f'(\lambda a + (1-\lambda)x)\, d\lambda$$

$$+\frac{(b-x)}{2}\int_0^1 (1-2\lambda) f'(\lambda x + (1-\lambda)b)\, d\lambda$$

*for $x \in [a,b]$ and $\lambda \in [0,1]$.*

*Proof.* Here, we apply integration by parts in integrals of right part of (10), and by using the change of the variable $t = \lambda a + (1-\lambda)x$, then we have

$$F_1 = \int_0^1 (1-2\lambda) f'(\lambda a + (1-\lambda)x)\, d\lambda = \frac{f(x)}{x-a} + \frac{f(a)}{x-a} - \frac{2}{(x-a)^2}\int_a^x f(t)dt.$$

And similarly, we obtain

$$F_2 = \int_0^1 (1-2\lambda) f'(\lambda x + (1-\lambda)b)\, d\lambda = \frac{f(b)}{b-x} + \frac{f(x)}{b-x} - \frac{2}{(b-x)^2}\int_x^b f(t)dt.$$

If we add $F_1$ from $F_2$ and multiply by $\frac{(x-a)}{2}$ and $\frac{(b-x)}{2}$, respectively we obtain proof of the (10).

**Remark 2.** *If in Lemma 7, we get $x = a$ or $x = b$, then the identity (10) becomes the identity (2).*

**Corollary 8.** *Under assumption of Lemma 7, if we choose $x = \frac{a+b}{2}$, then the identity (10) reduces to*

$$\frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] - \frac{1}{b-a}\int_a^b f(t)dt$$

$$= \frac{(b-a)}{8}\int_0^1 (1-2\lambda) f'\left(\lambda a + (1-\lambda)\frac{a+b}{2}\right) d\lambda$$

$$+\frac{(b-a)}{8}\int_0^1 (1-2\lambda)\, f'\left(\lambda\frac{a+b}{2}+(1-\lambda)b\right) d\lambda.$$

Now, we extend some estimates of the right hand side of a Bullen type inequalities for functions whose first derivatives absolute values are convex as follows:

**Theorem 9.** *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left| f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \tag{11}$$
$$\leq \frac{(b-a)}{8}\left|f'(x)\right|+\frac{(x-a)\left|f'(a)\right|+(b-x)\left|f'(b)\right|}{8}$$

*for $x\in[a,b]$.*

*Proof.* Using Lemma 7 and the convexity of $|f'|$, we find that

$$\left| f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|$$
$$\leq \frac{(x-a)}{2}\int_0^1 |1-2\lambda|\left|f'(\lambda a+(1-\lambda)x)\right| d\lambda$$
$$+\frac{(b-x)}{2}\int_0^1 |1-2\lambda|\left|f'(\lambda x+(1-\lambda)b)\right| d\lambda$$
$$\leq \frac{(x-a)}{2}\int_0^1 |1-2\lambda|\left[\lambda\left|f'(a)\right|+(1-\lambda)\left|f'(x)\right|\right] d\lambda$$
$$+\frac{(b-x)}{2}\int_0^1 |1-2\lambda|\left[\lambda\left|f'(x)\right|+(1-\lambda)\left|f'(b)\right|\right] d\lambda$$
$$=\frac{(x-a)}{2}\int_0^{\frac{1}{2}} (1-2\lambda)\left[\lambda\left|f'(a)\right|+(1-\lambda)\left|f'(x)\right|\right] d\lambda$$
$$+\frac{(x-a)}{2}\int_{\frac{1}{2}}^1 (2\lambda-1)\left[\lambda\left|f'(a)\right|+(1-\lambda)\left|f'(x)\right|\right] d\lambda$$

$$
\begin{aligned}
&+\frac{(b-x)}{2}\int_{0}^{\frac{1}{2}}(1-2\lambda)\left[\lambda\left|f'(x)\right|+(1-\lambda)\left|f'(b)\right|\right]d\lambda \\
&+\frac{(b-x)}{2}\int_{\frac{1}{2}}^{1}(2\lambda-1)\left[\lambda\left|f'(x)\right|+(1-\lambda)\left|f'(b)\right|\right]d\lambda \\
=\ &\frac{(b-a)}{8}\left|f'(x)\right|+\frac{(x-a)\left|f'(a)\right|+(b-x)\left|f'(b)\right|}{8}
\end{aligned}
$$

which this completes the proof of the (11).

**Remark 3.** *If in Theorem 9, we get $x=a$ or $x=b$, then the inequality (11) becomes the inequality (3).*

**Corollary 10.** *Under assumption of Theorem 9, if we choose $x=\frac{a+b}{2}$, then the inequality (11) reduces to*

$$
\begin{aligned}
&\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_{a}^{b}f(t)dt\right| \\
\leq\ &\frac{(b-a)}{16}\left[\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}\right] \\
\leq\ &\frac{(b-a)}{8}\left(\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}\right).
\end{aligned}
$$

**Theorem 11.** *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a<b$. If $|f'|^{q}$ is convex on $[a,b]$ for some $q>1$, then the following inequality holds:*

$$
\begin{aligned}
&\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_{a}^{x}f(t)dt+\frac{1}{b-x}\int_{x}^{b}f(t)dt\right]\right| \\
\leq\ &\frac{1}{2(p+1)^{\frac{1}{p}}}\left[(x-a)\left(\frac{\left|f'(a)\right|^{q}+\left|f'(x)\right|^{q}}{2}\right)^{\frac{1}{q}}+(b-x)\left(\frac{\left|f'(b)\right|^{q}+\left|f'(x)\right|^{q}}{2}\right)^{\frac{1}{q}}\right]
\end{aligned}
\tag{12}
$$

*where $x\in[a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.*

*Proof.* Using Lemma 7, Hölder's inequality and the convexity of $|f'|^{q}$, we find that

$$
\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_{a}^{x}f(t)dt+\frac{1}{b-x}\int_{a}^{x}f(t)dt\right]\right|
$$

$$
\begin{aligned}
&\leq \frac{(x-a)}{2}\int_0^1 |1-2\lambda|\left|f'(\lambda a+(1-\lambda)x)\right| d\lambda+\frac{(b-x)}{2}\int_0^1 |1-2\lambda|\left|f'(\lambda x+(1-\lambda)b)\right| d\lambda \\
&\leq \frac{(x-a)}{2}\left(\int_0^1 |1-2\lambda|^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(\lambda a+(1-\lambda)x)\right|^q d\lambda\right)^{\frac{1}{q}} \\
&\quad +\frac{(b-x)}{2}\left(\int_0^1 |1-2\lambda|^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(\lambda x+(1-\lambda)b)\right|^q d\lambda\right)^{\frac{1}{q}} \\
&\leq \frac{(x-a)}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1 \left[\lambda\left|f'(a)\right|^q+(1-\lambda)\left|f'(x)\right|^q\right] d\lambda\right)^{\frac{1}{q}} \\
&\quad +\frac{(b-x)}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1 \left[\lambda\left|f'(x)\right|^q+(1-\lambda)\left|f'(b)\right|^q\right] d\lambda\right)^{\frac{1}{q}} \\
&= \frac{(x-a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}+\frac{(b-x)}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(b)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}
\end{aligned}
$$

which this completes the proof of the (12).

**Corollary 12.** *Under assumption of Theorem11, if we choose $x=\frac{a+b}{2}$, then the inequality (12) reduces to*

$$
\begin{aligned}
&\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\leq \frac{(b-a)}{8(p+1)^{\frac{1}{p}}}\left[\left(\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{(b-a)}{8(p+1)^{\frac{1}{p}}}\left[\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(b)|^q+|f'(a)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

**Remark 4.** *If in Theorem 11, we get $x=a$ or $x=b$, then, the inequality (12) becomes the inequality (2.4) of Theorem 2.3 in [7].*

## 4. Midpoint type inequalities

Before starting and proving our next result, we need the following lemma.

**Lemma 13.** *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$f\left(\frac{a+x}{2}\right) + f\left(\frac{b+x}{2}\right) - \left[\frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt\right] \tag{13}$$

$$= (x-a)\int_0^{\frac{1}{2}} \lambda\left[f'(\lambda x + (1-\lambda)a) - f'(\lambda a + (1-\lambda)x)\right] d\lambda$$

$$+ (b-x)\int_{\frac{1}{2}}^1 (1-\lambda)\left[f'(\lambda x + (1-\lambda)b) - f'(\lambda b + (1-\lambda)x)\right] d\lambda$$

*for $x \in [a,b]$.*

*Proof.* In the proof of (13), we apply integration by parts, then we have

$$\begin{aligned} T_1 &= \int_0^{\frac{1}{2}} \lambda\left[f'(\lambda x + (1-\lambda)a) - f'(\lambda a + (1-\lambda)x)\right] d\lambda \\ &= \frac{1}{(x-a)} f\left(\frac{a+x}{2}\right) - \frac{1}{(x-a)^2}\int_a^x f(t)dt \end{aligned}$$

$$\begin{aligned} T_2 &= \int_{\frac{1}{2}}^1 (1-\lambda)\left[f'(\lambda x + (1-\lambda)b) - f'(\lambda b + (1-\lambda)x)\right] d\lambda \\ &= \frac{1}{(b-x)} f\left(\frac{b+x}{2}\right) - \frac{1}{(b-x)^2}\int_x^b f(t)dt. \end{aligned}$$

If we add $T_1$ from $T_1$ and multiply by $(x-a)$ and $(b-x)$, respectively we obtain proof of the identity (13).

**Remark 5.** *If in Lemma 13, we take $x = a$ or $x = b$ then, the identity (13) becomes the identity (4).*

**Corollary 14.** *Under assumption of Lemma 13, if we choose $x = \frac{a+b}{2}$ the identity (13) reduces to*

$$\begin{aligned} &\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{3b+a}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt \\ = &\frac{b-a}{4}\left\{\int_0^{\frac{1}{2}}\lambda\left[f'\left(\lambda\frac{a+b}{2}+(1-\lambda)a\right)-f'\left(\lambda a+(1-\lambda)\frac{a+b}{2}\right)\right]d\lambda\right. \\ &\left.+\int_{\frac{1}{2}}^{1}(1-\lambda)\left[f'\left(\lambda\frac{a+b}{2}+(1-\lambda)b\right)-f'\left(\lambda b+(1-\lambda)\frac{a+b}{2}\right)\right]d\lambda\right\}. \end{aligned}$$

Finally, we extend some estimates of the left hand side of a Bullen type inequalities for functions whose first derivatives absolute values are convex as follows:

**Theorem 15.** *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on $(a, b)$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\begin{aligned} &\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\ \leq &\frac{(b-a)}{8}\left|f'(x)\right|+\frac{(x-a)\left|f'(a)\right|+(b-x)\left|f'(b)\right|}{8} \end{aligned} \tag{14}$$

*for $x \in [a, b]$.*

*Proof.* From Lemma 13 and using the convexity of $|f'|$, then we have

$$\begin{aligned} &\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\ \leq &(x-a)\int_0^{\frac{1}{2}}\lambda\left[\left|f'(\lambda x+(1-\lambda)a)\right|+\left|f'(\lambda a+(1-\lambda)x)\right|\right]d\lambda \end{aligned}$$

$$
\begin{aligned}
&+(b-x)\int_{\frac{1}{2}}^{1}(1-\lambda)\left[\left|f'(\lambda x+(1-\lambda)b)\right|+\left|f'(\lambda b+(1-\lambda)x)\right|\right]d\lambda \\
\leq\ & (x-a)\left[\left|f'(x)\right|+\left|f'(a)\right|\right]\int_{0}^{\frac{1}{2}}\lambda d\lambda+(b-x)\left[\left|f'(x)\right|+\left|f'(a)\right|\right]\int_{\frac{1}{2}}^{1}(1-\lambda)\,d\lambda \\
=\ & \frac{(b-a)}{8}\left|f'(x)\right|+\frac{(x-a)\left|f'(a)\right|+(b-x)\left|f'(b)\right|}{8}.
\end{aligned}
$$

This completes the proof.

**Remark 6.** *If in Theorem 15, we take $x=a$ or $x=b$ then, the inequality (14) reduces to the inequality (5).*

**Corollary 16.** *Under assumption of Theorem 15, if we choose $x=\frac{a+b}{2}$ the identity (14) reduces to*

$$
\begin{aligned}
&\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{3b+a}{4}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)dt\right| \\
\leq\ & \frac{(b-a)}{16}\left[\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}\right] \\
\leq\ & \frac{(b-a)}{8}\left(\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}\right).
\end{aligned}
$$

**Theorem 17.** *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a<b$. If $|f'|^{q}$ is convex on $[a,b]$ for some $q>1$, then the following inequality holds:*

$$
\begin{aligned}
&\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_{a}^{x}f(t)dt+\frac{1}{b-x}\int_{x}^{b}f(t)dt\right]\right| \qquad (15)\\
\leq\ & \frac{1}{(1+p)^{\frac{1}{p}}\,2^{1+\frac{1}{p}}}\left\{(x-a)\left[\left(\frac{\left|f'(x)\right|^{q}+\left|f'(a)\right|^{q}}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^{q}+\left|f'(x)\right|^{q}}{2}\right)^{\frac{1}{q}}\right]\right. \\
&\left.+(b-x)\left[\left(\frac{\left|f'(x)\right|^{q}+\left|f'(b)\right|^{q}}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(b)\right|^{q}+\left|f'(x)\right|^{q}}{2}\right)^{\frac{1}{q}}\right]\right\}
\end{aligned}
$$

*where* $x \in [a,b]$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* From Lemma 13, Hölder's inequality and the convexity of $|f'|^q$, we find that

$$\left| f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|$$

$$\begin{aligned}
\leq\ & (x-a)\left(\int_0^{\frac{1}{2}} \lambda^p d\lambda\right)^{\frac{1}{p}} \\
& \times\left[\left(\int_0^{\frac{1}{2}} \left|f'\left(\lambda x+(1-\lambda)a\right)\right|^q d\lambda\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}} \left|f'\left(\lambda a+(1-\lambda)x\right)\right|^q d\lambda\right)^{\frac{1}{q}}\right] \\
& +(b-x)\left(\int_{\frac{1}{2}}^1 (1-\lambda)^p d\lambda\right)^{\frac{1}{p}} \\
& \times\left[\left(\int_{\frac{1}{2}}^1 \left|f'\left(\lambda x+(1-\lambda)b\right)\right|^q d\lambda\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1 \left|f'\left(\lambda b+(1-\lambda)x\right)\right|^q d\lambda\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{(x-a)}{(1+p)^{\frac{1}{p}}\, 2^{1+\frac{1}{p}}}\left\{\left(\int_0^{\frac{1}{2}} \left[\lambda\left|f'(x)\right|^q+(1-\lambda)\left|f'(a)\right|^q\right] d\lambda\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\int_0^{\frac{1}{2}} \left[\lambda\left|f'(a)\right|^q+(1-\lambda)\left|f'(x)\right|^q\right] d\lambda\right)^{\frac{1}{q}}\right\} \\
& +\frac{(b-x)}{(1+p)^{\frac{1}{p}}\, 2^{1+\frac{1}{p}}}\left\{\left(\int_0^{\frac{1}{2}} \left[\lambda\left|f'(x)\right|^q+(1-\lambda)\left|f'(b)\right|^q\right] d\lambda\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$
\left.+\left(\int_{0}^{\frac{1}{2}}\left[\lambda\left|f'(b)\right|^{q}+(1-\lambda)\left|f'(x)\right|^{q}\right]d\lambda\right)^{\frac{1}{q}}\right\}
$$

$$
\begin{aligned}
\leq\ & \frac{(x-a)}{(1+p)^{\frac{1}{p}}\,2^{1+\frac{1}{p}}}\left[\left(\frac{|f'(x)|^{q}+|f'(a)|^{q}}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+|f'(x)|^{q}}{2}\right)^{\frac{1}{q}}\right] \\
& +\frac{(b-x)}{(1+p)^{\frac{1}{p}}\,2^{1+\frac{1}{p}}}\left[\left(\frac{|f'(x)|^{q}+|f'(b)|^{q}}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(b)|^{q}+|f'(x)|^{q}}{2}\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

This completes the proof.

**Remark 7.** *If in Theorem 17, we take $x = a$ or $x = b$ then, the inequality (15) reduces to the inequality (2.3) of Theorem 2.3 by Kirmaci in [11].*

**Corollary 18.** *Under assumption of Theorem 17, if we choose $x = \frac{a+b}{2}$ the inequality (15) reduces to*

$$
\begin{aligned}
& \left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{3b+a}{4}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)dt\right| \\
\leq\ & \frac{(b-a)}{(1+p)^{\frac{1}{p}}\,2^{3+\frac{1}{p}}}\left\{\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+|f'(a)|^{q}}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+\left|f'\left(\frac{a+b}{2}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}\right]\right. \\
& \left.+\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+|f'(b)|^{q}}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(b)|^{q}+\left|f'\left(\frac{a+b}{2}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}\right]\right\} \\
\leq\ & \frac{(b-a)}{(1+p)^{\frac{1}{p}}\,2^{2+\frac{1}{p}}}\left[\left(\frac{3|f'(a)|^{q}+|f'(b)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+3|f'(b)|^{q}}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

## 5. Simpson type inequalities

Before starting and proving our next result, we need the following lemma.

**Lemma 19.** *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on $(a, b)$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$
\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right] \tag{16}
$$

$$-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]$$

$$=(x-a)\int_0^{\frac{1}{2}}\left(\lambda-\frac{1}{6}\right)\left[f'\left(\lambda x+(1-\lambda)a\right)-f'\left(\lambda a+(1-\lambda)x\right)\right]d\lambda$$

$$+(b-x)\int_{\frac{1}{2}}^1\left(\frac{5}{6}-\lambda\right)\left[f'\left(\lambda x+(1-\lambda)b\right)-f'\left(\lambda b+(1-\lambda)x\right)\right]d\lambda$$

*for* $x\in[a,b]$.

*Proof.* In the proof of (16), we apply integration by parts, then we have

$$H_1 = \int_0^{\frac{1}{2}}\left(\lambda-\frac{1}{6}\right)\left[f'\left(\lambda x+(1-\lambda)a\right)-f'\left(\lambda a+(1-\lambda)x\right)\right]d\lambda$$

$$= \frac{2}{3(x-a)}f\left(\frac{a+x}{2}\right)+\frac{1}{6(x-a)}\left[f(a)+f(x)\right]-\frac{1}{(x-a)^2}\int_a^x f(t)dt$$

$$H_2 = \int_{\frac{1}{2}}^1\left(\frac{5}{6}-\lambda\right)\left[f'\left(\lambda x+(1-\lambda)b\right)-f'\left(\lambda b+(1-\lambda)x\right)\right]d\lambda$$

$$= \frac{2}{3(b-x)}f\left(\frac{b+x}{2}\right)+\frac{1}{6(x-a)}\left[f(a)+f(x)\right]-\frac{1}{(b-x)^2}\int_x^b f(t)dt.$$

If we add $H_1$ from $H_1$ and multiply by $(x-a)$ and $(b-x)$, respectively we obtain proof of the identity (16).

**Remark 8.** *If in Lemma 19, we take* $x=a$ *or* $x=b$ *then, the identity (16) becomes the identity (4) of Lemma 1 by Alomari et. al. in [1] for* $s=1$.

**Corollary 20.** *Under assumption of Lemma 19, if we choose* $x=\frac{a+b}{2}$ *the identity (16) reduces to*

$$\frac{1}{6}\left[4f\left(\frac{3a+b}{4}\right)+4f\left(\frac{3b+a}{4}\right)+f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)dt$$

$$= \frac{b-a}{4}\left\{\int_0^{\frac{1}{2}}\left(\lambda-\frac{1}{6}\right)\left[f'\left(\lambda\frac{a+b}{2}+(1-\lambda)a\right)-f'\left(\lambda a+(1-\lambda)\frac{a+b}{2}\right)\right]d\lambda\right.$$

$$\left.+\int_{\frac{1}{2}}^{1}\left(\frac{5}{6}-\lambda\right)\left[f'\left(\lambda\frac{a+b}{2}+(1-\lambda)b\right)-f'\left(\lambda b+(1-\lambda)\frac{a+b}{2}\right)\right]d\lambda\right\}.$$

Finally, we extend some estimates of the Simpson type inequalities for functions whose first derivatives absolute values are convex as follows:

**Theorem 21.** *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]\right.\\&\left.-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|\\\leq\;&\frac{5(b-a)}{72}\left|f'(x)\right|+\frac{5(x-a)\left|f'(a)\right|+5(b-x)\left|f'(b)\right|}{72}\end{aligned}\tag{17}$$

*for $x\in[a,b]$.*

*Proof.* From Lemma 19 and using the convexity of $|f'|$, then we have

$$\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|$$

$$\leq (x-a)\int_0^{\frac{1}{2}}\left|\lambda-\frac{1}{6}\right|\left[\left|f'(\lambda x+(1-\lambda)a)\right|+\left|f'(\lambda a+(1-\lambda)x)\right|\right]d\lambda$$

$$+(b-x)\int_{\frac{1}{2}}^{1}\left|\frac{5}{6}-\lambda\right|\left[\left|f'(\lambda x+(1-\lambda)b)\right|+\left|f'(\lambda b+(1-\lambda)x)\right|\right]d\lambda$$

$$\leq (x-a)\left[\left|f'(x)\right|+\left|f'(a)\right|\right]\int_0^{\frac{1}{2}}\left|\lambda-\frac{1}{6}\right|d\lambda+(b-x)\left[\left|f'(x)\right|+\left|f'(a)\right|\right]\int_{\frac{1}{2}}^{1}\left|\frac{5}{6}-\lambda\right|d\lambda$$

$$= \frac{5(b-a)}{72}\left|f'(x)\right| + \frac{5(x-a)\left|f'(a)\right| + 5(b-x)\left|f'(b)\right|}{72}.$$

This completes the proof.

**Remark 9.** *If in Theorem 21, we take $x=a$ or $x=b$ then, the inequality (17) reduces to the inequality (2.2) of Theorem 2.1 by Sarikaya et. al. in [20].*

**Corollary 22.** *Under assumption of Theorem 21, if we choose $x=\frac{a+b}{2}$ the identity (17) reduces to*

$$\begin{aligned}
&\left|\frac{1}{6}\left[4f\left(\frac{3a+b}{4}\right)+4f\left(\frac{3b+a}{4}\right)+f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
\leq\ & \frac{5(b-a)}{144}\left[\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}\right] \\
\leq\ & \frac{5(b-a)}{72}\left(\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}\right).
\end{aligned}$$

**Theorem 23.** *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for some $q>1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]\right. \\
&\left.-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \qquad (18)\\
\leq\ & \frac{1}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}} \\
&\times\left\{(x-a)\left[\left(\frac{\left|f'(x)\right|^q+\left|f'(a)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+\left|f'(x)\right|^q}{2}\right)^{\frac{1}{q}}\right]\right. \\
&\left.+(b-x)\left[\left(\frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(b)\right|^q+\left|f'(x)\right|^q}{2}\right)^{\frac{1}{q}}\right]\right\}
\end{aligned}$$

*where $x\in[a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.*

*Proof.* From Lemma 19, Hölder's inequality and the convexity of $|f'|^q$, we find that

$$\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|$$

$$
\begin{aligned}
\leq\ & (x-a)\left(\int_0^{\frac{1}{2}}\left|\lambda-\frac{1}{6}\right|^p d\lambda\right)^{\frac{1}{p}} \\
& \times\left[\left(\int_0^{\frac{1}{2}}\left|f'(\lambda x+(1-\lambda)a)\right|^q d\lambda\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}\left|f'(\lambda a+(1-\lambda)x)\right|^q d\lambda\right)^{\frac{1}{q}}\right] \\
& +(b-x)\left(\int_{\frac{1}{2}}^1\left|\frac{5}{6}-\lambda\right|^p d\lambda\right)^{\frac{1}{p}} \\
& \times\left[\left(\int_{\frac{1}{2}}^1\left|f'(\lambda x+(1-\lambda)b)\right|^q d\lambda\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\left|f'(\lambda b+(1-\lambda)x)\right|^q d\lambda\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{1}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}}\left\{(x-a)\left[\left(\int_0^{\frac{1}{2}}\left[\lambda\left|f'(x)\right|^q+(1-\lambda)\left|f'(a)\right|^q\right]d\lambda\right)^{\frac{1}{q}}\right.\right. \\
& \left.\left.+\left(\int_0^{\frac{1}{2}}\left[\lambda\left|f'(a)\right|^q+(1-\lambda)\left|f'(x)\right|^q\right]d\lambda\right)^{\frac{1}{q}}\right]\right\} \\
& +(b-x)\left[\left(\int_0^{\frac{1}{2}}\left[\lambda\left|f'(x)\right|^q+(1-\lambda)\left|f'(b)\right|^q\right]d\lambda\right)^{\frac{1}{q}}\right. \\
& \left.\left.+\left(\int_0^{\frac{1}{2}}\left[\lambda\left|f'(b)\right|^q+(1-\lambda)\left|f'(x)\right|^q\right]d\lambda\right)^{\frac{1}{q}}\right]\right\} \\
\leq\ & \frac{1}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}} \\
& \times\left\{(x-a)\left[\left(\frac{\left|f'(x)\right|^q+\left|f'(a)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+\left|f'(x)\right|^q}{2}\right)^{\frac{1}{q}}\right]\right. \\
& \left.+(b-x)\left[\left(\frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(b)\right|^q+\left|f'(x)\right|^q}{2}\right)^{\frac{1}{q}}\right]\right\}
\end{aligned}
$$

This completes the proof.

**Remark 10.** *If in Theorem 23, we take $x = a$ or $x = b$ then, the inequality (18) reduces to the inequality (2.3) of Theorem 2.3 by Sarikaya et. al. in [20].*

**Corollary 24.** *Under assumption of Theorem 23, if we choose $x = \frac{a+b}{2}$ the inequality (18) reduces to*

$$\left|\frac{1}{6}\left[4f\left(\frac{3a+b}{4}\right)+4f\left(\frac{3b+a}{4}\right)+f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{(b-a)}{2(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}}$$

$$\times\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(b-a)}{2(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}}$$

$$\times\left[\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right].$$

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