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DIFFERENTIAL SUPERORDINATION THEOREM AND SOME SANDWICH-TYPE RESULTS INVOLVING CONVOLUTION

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Abstract. In the present paper, we study the operator

$$\Gamma\left(w,\ zw';\ z\right) = w^{\gamma}\left\{a + \frac{b}{w} + cw + d\frac{zw'}{w}\right\}^{\beta};\ w \in \mathbb{D} = \mathbb{C}\backslash\{0\},\ z \in \mathbb{E},$$

where $\mathbb{E} = \{z\ ;\ |z| < 1\}$ and $a,\ b,\ c,\ d,\ \beta,\ \gamma$ be complex numbers such that $d \neq 0,\ \beta \neq 0$, to obtain superordination theorems which generalise and unify various well known results. In what follows, all the powers taken are the principle ones.

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1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in the unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of the functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$$

Let $\mathcal{A}$ be the class of functions f, analytic in the unit disk $\mathbb{E}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.
Let $\mathcal{S}$ denote the class of all analytic univalent functions f defined in the open unit disk $\mathbb{E}$ which are normalized by the conditions $f(0) = f'(0) - 1 = 0$. The Taylor series expansion of any function $f \in \mathcal{S}$ is

$$f(z) = z + a_2 z^2 + a_3 z^3 + ...$$

Let the functions f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g written as $f \prec g$ in $\mathbb{E}$, if there exists a Schwarz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1$, $\phi(0) = 0$ and $|\phi(z)| \leq |z| < 1$) such that

$$f(z) = g(\phi(z)), \ |z| < 1.$$

Let $\Gamma : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Gamma(p(z), zp'(z); z) \prec h(z), \ \Gamma(p(0), 0; 0) = h(0). \tag{1}$$

A univalent function q is called dominant of the differential subordination 1 if $p(0) = q(0)$ and $p \prec q$ for all p satisfying 1. A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of 1, is said to be the best dominant of 1. The best dominant is unique up to the rotation of $\mathbb{E}$.
Let $\Psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic and univalent function in domain $\mathbb{C}^2 \times \mathbb{E}$, h be analytic function in $\mathbb{E}$, p be analytic and univalent in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$. Then p is called the solution of the first order differential superordination if

$$h(z) \prec \Psi(p(z), zp'(z); z), h(0) = \Psi(p(0), 0; 0). \tag{2}$$

An analytic function q is called a subordinant of the differential superordination2 if $q \prec p$ for all p satisfying 2. A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of 2, is said to be the best subordinant of 2. The best subordinant is unique up to the rotation of $\mathbb{E}$.
Let $f(z) = \sum_{k=1}^{\infty} a_k z^k$ and $g(z) = \sum_{k=1}^{\infty} b_k z^k$ be two analytic functions, then convolution of f and g, written as $f * g$ is defined by

$$(f * g)(z) = \sum_{k=1}^{\infty} a_k b_k z^k.$$

A function $f \in \mathcal{A}$ is said to be starlike in the open unit disk $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \ z \in \mathbb{E}.$$

Let $\mathcal{S}^*$ denote the subclass of $\mathcal{S}$ consisting of all univalent starlike functions with respect to the origin.
A function $f \in \mathcal{A}$ is said to be convex in the open unit disk $\mathbb{E}$, if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ z \in \mathbb{E}.$$

Let $\mathcal{K}$ denote the class of convex functions.
A function $f \in \mathcal{A}$ is said to be close-to-convex in $\mathbb{E}$, if there exists a starlike function g (not necessarily normalized) such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \ z \in \mathbb{E}.$$

In addition, if g is normalized by the conditions $g(0) = 0 = g'(0) - 1$, then the class of close-to-convex functions is denoted by $\mathcal{C}$.
A function $f \in \mathcal{A}$ is called parabolic starlike in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, \ z \in \mathbb{E}, \tag{3}$$

and the class of such functions is denoted by S_P.
A function $f \in \mathcal{A}$ is said to be uniformly convex in $\mathbb{E}$, if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, \ z \in \mathbb{E}, \tag{4}$$

and let UCV denote the class of all such functions.
A function $f \in \mathcal{A}$ is said to be uniformly close-to-convex in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \left|\frac{zf'(z)}{g(z)} - 1\right|, \ z \in \mathbb{E}, \tag{5}$$

for some $g \in \mathcal{S}_P$. Let UCC denote the class of all such functions. Note that the function $g(z) \equiv z \in \mathcal{S}_P$. Therefore, for $g(z) \equiv z$, condition 5 becomes:

$$\Re\left(f'(z)\right) > \left|f'(z) - 1\right|, \ z \in \mathbb{E}. \tag{6}$$

Ronning [8] and Ma and Minda [5] studied the domain Ω and the function $q(z)$ defined below:

$$\Omega = \left\{u + iv : u > \sqrt{(u-1)^2 + v^2}\right\}.$$

Clearly the function

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$

maps the unit disk $\mathbb{E}$ onto the domain Ω. Hence the conditions 3 and 6 are, respectively, equivalent to

$$\frac{zf'(z)}{f(z)} \prec q(z), \ z \in \mathbb{E},$$

and

$$f'(z) \prec q(z).$$

Let ϕ be analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0$ and $\Re(\phi'(0)) > 0$. Then, the function $f \in \mathcal{A}$ is said to be $\phi-$ like in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > 0, \ z \in \mathbb{E}.$$

This concept was introduced by Brickman [1]. He proved that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is $\phi-$ like for some analytic function ϕ. Later, Ruscheweyh [9] investigated the following general class of $\phi-$like functions:
Let ϕ be analytic in a domain containing $f(\mathbb{E})$, where $\phi(0) = 0$, $\phi'(0) = 1$ and $\phi(w) \neq 0$ for some $w \in f(\mathbb{E})\backslash\{0\}$, then the function $f \in \mathcal{A}$ is called $\phi-$like with respect to a univalent function q, $q(0) = 1$, if

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \ z \in \mathbb{E}.$$

A function $f \in \mathcal{A}$ is said to be parabolic $\phi-$ like in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > \left|\frac{zf'(z)}{\phi(f(z))} - 1\right|, \ z \in \mathbb{E}. \tag{7}$$

Equivalently, condition 7 can be written as:

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2.$$

Our work is inspired by various differential operators in literature (see ref. [7], [10], [11], [12]).
In this study, we derive a differential superordination theorem which generalises various known results. Further, combining with the corresponding differential subordination theorem (see Kaur et al. [3]), we obtain some sandwich-type theorems. As special cases of our main results, we find sufficient conditions for normalised analytic functions to be parabolic $\phi-$like, parabolic starlike, uniform convex, uniform close-to-convex, $\phi-$like, starlike, convex and close-to-convex.

2. Preliminaries

We shall need the following definition and lemma to prove our main result.

Definition 1. *([6], Definition 2, p.817)Denote by $\mathbb{Q}$, the set of all functions $f(z)$ that are analytic and injective on $\bar{\mathbb{E}} \setminus \mathbb{E}(f)$, where*

$$\mathbb{E}(f) = \left\{ \zeta \in \partial\mathbb{E} \ : \ \lim_{z\to\zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{E} \setminus \mathbb{E}(f)$.

Lemma 1. *([2]). Let q be univalent in $\mathbb{E}$ and let θ and φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\varphi[q(z)],\ h(z) = \theta[q(z)] + Q_1(z)$ and suppose that either*
(i) Q_1 is starlike and
(ii) $\Re\left(\dfrac{\theta' q(z)}{\varphi(q(z)})\right) > 0$ for all $z \in \mathbb{E}$.
If $p \in \mathcal{H}[1,\ 1] \cap \mathbb{Q}$ with $p(\mathbb{E}) \subset \mathbb{D}$ and $\theta[p(z)] + zp'(z)\varphi[p(z)]$ is univalent in $\mathbb{E}$ and

$$\theta[q(z)] + zq'(z)\varphi[q(z)] \prec \theta[p(z)] + zp'(z)\varphi[p(z)],\ z \in \mathbb{E},$$

then $q(z) \prec p(z)$ and q is the best subordinant.

3. A Superordination Theorem

Theorem 2. *Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*
(i) $\Re\left[1 + \dfrac{zq''(z)}{q'(z)} + \left(\dfrac{\gamma}{\beta} - 1\right)\dfrac{zq'(z)}{q(z)}\right] > 0$ and
(ii) $\Re\left[\dfrac{b}{d}\left(\dfrac{\gamma}{\beta} - 1\right)\dfrac{1}{q(z)} + \dfrac{c}{d}\left(\dfrac{\gamma}{\beta} + 1\right)q(z) + \dfrac{a\gamma}{d\beta}\right] > 0.$
If the analytic function $p \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$, with $p(z) \neq 0,\ z \in \mathbb{E}$, satisfies the following differential superordination

$$\Gamma\left(q(z),\ zq'(z);\ z\right) \prec\ \Gamma\left(p(z),\ zp'(z);\ z\right) \tag{8}$$

where

$$\Gamma\left(w,\ zw';\ z\right) = w^{\gamma}\left\{a + \frac{b}{w} + cw + d\frac{zw'}{w}\right\}^{\beta};\ w \in \mathbb{D} = \mathbb{C}\backslash\{0\},\ z \in \mathbb{E}, \tag{9}$$

and $a,\ b,\ c,\ d,\ \beta,\ \gamma$ be complex numbers such that $d \neq 0,\ \beta \neq 0$ then

$$q(z) \prec p(z),\ z \in \mathbb{E},$$

and $q(z)$ is the best subordinant.

Proof. Let the functions Θ and Υ be defined as:

$$\Theta(w) = aw^{\frac{\gamma}{\beta}} + bw^{\frac{\gamma}{\beta}-1} + cw^{\frac{\gamma}{\beta}+1}$$

and

$$\Upsilon(w) = dw^{\frac{\gamma}{\beta}-1}$$

Obviously, the functions Θ and Υ are analytic in domain $\mathbb{D} = \mathbb{C}\backslash\{0\}$ and $\Upsilon(w) \neq 0$ in $\mathbb{D}$.
Therefore,

$$Q_1(z) = \Upsilon(q(z))zq'(z) = d(q(z))^{\frac{\gamma}{\beta}-1}zq'(z)$$

and
$h(z) = \Theta(q(z)) + Q_1(z)$

$$= a(q(z))^{\frac{\gamma}{\beta}} + b(q(z))^{\frac{\gamma}{\beta}-1} + c(q(z))^{\frac{\gamma}{\beta}+1} + d(q(z))^{\frac{\gamma}{\beta}-1}zq'(z)$$

On differentiating, we get

$$\frac{zQ_1'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)}$$

and

$$\frac{\Theta'(q(z))}{\Upsilon(q(z))} = \frac{b}{d}\left(\frac{\gamma}{\beta} - 1\right)\frac{1}{q(z)} + \frac{c}{d}\left(\frac{\gamma}{\beta} + 1\right)q(z) + \frac{a\gamma}{d\beta}.$$

In view of the given conditions (i) and (ii) we see that Q_1 is starlike and $\Re\left(\frac{\Theta'(q(z))}{\Upsilon(q(z))}\right) > 0$.
Therefore, the proof, now follows from the Lemma 1.

Remark 1. *By replacing $\gamma = \delta,\ \beta = 1,\ a = \gamma,\ b = 0,\ c = \beta,$ and $d = \alpha$, in Theorem 2, we obtain Theorem 2 of [11].*

4. Corollaries

Remark 2. *On writing $p(z) = \left(\frac{z^{1-\lambda}(f * \Phi)'(z)}{((f * \Psi)(z))^{1-\lambda}}\right)^{\alpha}; z \in \mathbb{E},\ 0 \leq \lambda \leq 1,\ \alpha(\neq 0) \in \mathbb{C}$ in Theorem 2, we get*

Corollary 3. *Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ which satisfy conditions (i) and (ii) of Theorem 2. If $f \in \mathcal{A}$ and analytic functions $\Phi,\ \Psi$ such that $\left(\dfrac{z^{1-\lambda}(f*\Phi)'(z)}{((f*\Psi)(z))^{1-\lambda}}\right)^{\alpha} \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ with $\left(\dfrac{z^{1-\lambda}(f*\Phi)'(z)}{((f*\Psi)(z))^{1-\lambda}}\right)^{\alpha} \neq 0,\ z \in \mathbb{E},\ \alpha(\neq 0) \in \mathbb{C},\ 0 \leq \lambda \leq 1$, satisfy the differential superordination*

$$\Gamma\big(q(z),\ zq'(z);\ z\big) \prec \Gamma\left\{\left(\frac{z^{1-\lambda}(f*\Phi)'(z)}{((f*\Psi)(z))^{1-\lambda}}\right)^{\alpha},\ z\left[\left(\frac{z^{1-\lambda}(f*\Phi)'(z)}{((f*\Psi)(z))^{1-\lambda}}\right)^{\alpha}\right]';\ z\right\}$$

where $a,\ b,\ c,\ d,\ \beta,\ \gamma,\ \alpha$ be complex numbers such that $d \neq 0,\ \beta \neq 0,\ \alpha \neq 0$ and Γ is given by 9, then

$$q(z) \prec \left(\frac{z^{1-\lambda}(f*\Phi)'(z)}{((f*\Psi)(z))^{1-\lambda}}\right)^{\alpha},\ z \in \mathbb{E},$$

and $q(z)$ is the best subordinant.

Remark 3. *By selecting $p(z) = \left(1 + \dfrac{z^{2-\lambda}(f*\Phi)''(z)}{(z(f*\Psi)'(z))^{1-\lambda}}\right)^{\alpha};\ z \in \mathbb{E},\ 0 \leq \lambda \leq 1,\ \alpha(\neq 0) \in \mathbb{C}$ in Theorem 2, we obtain*

Corollary 4. *Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ which satisfy conditions (i) and (ii) of Theorem 2. If $f \in \mathcal{A}$ and analytic functions $\Phi,\ \Psi$ such that $\left(1 + \dfrac{z^{2-\lambda}(f*\Phi)''(z)}{(z(f*\Psi)'(z))^{1-\lambda}}\right)^{\alpha} \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ with $\left(1 + \dfrac{z^{2-\lambda}(f*\Phi)''(z)}{(z(f*\Psi)'(z))^{1-\lambda}}\right)^{\alpha} \neq 0,\ z \in \mathbb{E},\ \alpha(\neq 0) \in \mathbb{C},\ 0 \leq \lambda \leq 1$, satisfy the differential superordination*

$$\Gamma\big(q(z),\ zq'(z);\ z\big) \prec \Gamma\left\{\left(1 + \frac{z^{2-\lambda}(f*\Phi)''(z)}{(z(f*\Psi)'(z))^{1-\lambda}}\right)^{\alpha},\ z\left[\left(1 + \frac{z^{2-\lambda}(f*\Phi)''(z)}{(z(f*\Psi)'(z))^{1-\lambda}}\right)^{\alpha}\right]';\ z\right\}$$

where $a,\ b,\ c,\ d,\ \beta,\ \gamma,\ \alpha$ be complex numbers such that $d \neq 0,\ \beta \neq 0,\ \alpha \neq 0$ and Γ is given by 9, then

$$q(z) \prec \left(1 + \frac{z^{2-\lambda}(f*\Phi)''(z)}{(z(f*\Psi)'(z))^{1-\lambda}}\right)^{\alpha},\ z \in \mathbb{E},$$

and $q(z)$ is the best subordinant.

Remark 4. *By taking $p(z) = \dfrac{(f*\Phi)(z)}{(f*\Psi)(z)}$, in Theorem 2, we have the following result.*

Corollary 5. *Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ which satisfy conditions (i) and (ii) of Theorem 2. If $f \in \mathcal{A}$ and analytic functions $\Phi,\ \Psi$ such that $\dfrac{(f*\Phi)(z)}{(f*\Psi)(z)} \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ with $\dfrac{(f*\Phi)(z)}{(f*\Psi)(z)} \neq 0,\ z \in \mathbb{E}$, satisfy the differential superordination*

$$\Gamma\left(q(z),\ zq'(z);\ z\right) \prec\ \Gamma\left[\frac{(f*\Phi)(z)}{(f*\Psi)(z)},\ z\left(\frac{(f*\Phi)(z)}{(f*\Psi)(z)}\right)';\ z\right]$$

where $a,\ b,\ c,\ d,\ \beta,\ \gamma,\ \alpha$ be complex numbers such that $d \neq 0,\ \beta \neq 0,\ \alpha \neq 0$ and Γ is given by 9, then

$$q(z) \prec \frac{(f*\Phi)(z)}{(f*\Psi)(z)},\ z \in \mathbb{E},$$

and $q(z)$ is the best subordinant.

Remark 5. *On selecting $p(z) = \dfrac{z(f*g)'(z)}{\phi((f*g)(z))}$; $f,\ g \in \mathcal{A}$, in Theorem 2, we get the following result.*

Corollary 6. *Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ which satisfy conditions (i) and (ii) of Theorem 2. Let ϕ be analytic function in the domain containing $(f*g)(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in (f*g)(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$ such that $\dfrac{z(f*g)'(z)}{\phi((f*g)(z))} \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ with $\dfrac{z(f*g)'(z)}{\phi((f*g)(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$\Gamma\left(q(z),\ zq'(z);\ z\right) \prec\ \Gamma\left[\frac{z(f*g)'(z)}{\phi((f*g)(z))},\ z\left(\frac{z(f*g)'(z)}{\phi((f*g)(z))}\right)';\ z\right]$$

where $a,\ b,\ c,\ d,\ \beta,\ \gamma,\ \alpha$ be complex numbers such that $d \neq 0,\ \beta \neq 0,\ \alpha \neq 0$ and Γ is given by 9, then

$$q(z) \prec \frac{z(f*g)'(z)}{\phi((f*g)(z))},\ z \in \mathbb{E},$$

and $q(z)$ is the best subordinant.

By taking $g(z) = \dfrac{z}{1-z}$, in Corollary 6, we get

Corollary 7. *Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ which satisfy conditions (i) and (ii) of Theorem 2. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, such that $\dfrac{zf'(z)}{\phi(f(z))} \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ with $\dfrac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$\Gamma\left(q(z),\ zq'(z);\ z\right) \prec\ \Gamma\left[\frac{zf'(z)}{\phi(f(z))},\ z\left(\frac{zf'(z)}{\phi(f(z))}\right)';\ z\right]$$

where $a,\ b,\ c,\ d,\ \beta,\ \gamma,\ \alpha$ be complex numbers such that $d \neq 0,\ \beta \neq 0,\ \alpha \neq 0$ and Γ is given by 9, then

$$q(z) \prec \frac{zf'(z)}{\phi(f(z))},\ z \in \mathbb{E},$$

and $q(z)$ is the best subordinant.

5. Deductions

By selecting the particular values of $\gamma,\ \beta,\ a,\ b,\ c$ and d in previous corollaries, we get various known results and some of them are given below.

1. By taking $\gamma = 0,\ \beta = 1,\ a = \rho,\ b = \eta,\ c = \delta$ and $d = \mu$ in Corollary 3, we get Theorem 3.1 of [12].
2. By setting $\gamma = 0,\ \beta = 1,\ a = \rho,\ b = \eta,\ c = \delta$ and $d = \mu$ in Corollary 4, we obtain Theorem 3.2 of [12].
3. By replacing $\gamma = \delta,\ \beta = 1,\ a = \gamma,\ b = 0,\ c = \beta$ and $d = \alpha$ in Corollary 5, we get Theorem 4 of [11].
4. By taking $\gamma = 0,\ \beta = 1,\ a = b = 0,\ c = \alpha$ and $d = \gamma$ in Corollary 6, we obtain Theorem 2.5 of [10].
5. By choosing $\gamma = \delta,\ \beta = 1,\ a = \gamma,\ b = 0,\ c = \beta$ and $d = \alpha$ in Corollary 6, we get Theorem 8 of [11].
6. On replacing $\gamma = 0,\ \beta = 1,\ a = 0 = b,\ c = a,\ d = b$ in Corollary 7, we obtain Theorem 33 of [4].

Remark 6. *By combining Theorem 2 and the corresponding result for differential subordination (see Kaur et al. [3]), we have the following "sandwich-type result".*

6. Sandwich-Type Theorems

Theorem 8. *Let $q_1,\ q_2$ ($q_1(z) \neq 0,\ q_2(z) \neq 0,\ z \in \mathbb{E}$), be univalent functions in $\mathbb{E}$, such that*
(i) $\Re\left[1 + \frac{zq_i''(z)}{q_i'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq_i'(z)}{q_i(z)}\right] > 0$ *and*
(ii) $\Re\left[\frac{b}{d}\left(\frac{\gamma}{\beta} - 1\right)\frac{1}{q_i(z)} + \frac{c}{d}\left(\frac{\gamma}{\beta} + 1\right)q_i(z) + \frac{a\gamma}{d\beta}\right] > 0;\ i = 1,\ 2.$
If the analytic function $p \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$, with $p(z) \neq 0,\ z \in \mathbb{E}$, satisfy

$$\Gamma\left(q_1(z),\ zq_1'(z);\ z\right) \prec\ \Gamma\left(p(z),\ zp'(z);\ z\right) \prec \Gamma\left(q_2(z),\ zq_2'(z);\ z\right)$$

where $a,\ b,\ c,\ d,\ \beta,\ \gamma$ *be complex numbers such that* $d \neq 0,\ \beta \neq 0$ *and* $\Gamma\left(p(z),\ zp'(z);\ z\right)$ *is univalent and is given by 9, then*

$$q_1(z) \prec p(z) \prec q_2(z),\ z \in \mathbb{E},$$

where $q_1(z)$ *and* $q_2(z)$ *are the best subordinant and the best dominant respectively.*

Remark 7. *By taking* $p(z) = \dfrac{z(f*g)'(z)}{\phi((f*g)(z))};\ f,\ g \in \mathcal{A}$, *in Theorem 8, we have*

Theorem 9. *Let* $q_1,\ q_2\ (q_1(z) \neq 0,\ q_2(z) \neq 0,\ z \in \mathbb{E})$*, be univalent functions in* $\mathbb{E}$ *which satisfy conditions (i) and (ii) of Theorem 8. Let* ϕ *be analytic function in the domain containing* $(f*g)(\mathbb{E})$ *such that* $\phi(0) = 0 = \phi'(0) - 1$ *and* $\phi(w) \neq 0$ *for* $w \in (f*g)(\mathbb{E})\backslash\{0\}$*. If* $f,\ g \in \mathcal{A}$ *such that* $\dfrac{z(f*g)'(z)}{\phi((f*g)(z))} \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ *with* $\dfrac{z(f*g)'(z)}{\phi((f*g)(z))} \neq 0,\ z \in \mathbb{E}$*, satisfy*

$$\Gamma\left(q_1(z),\ zq_1'(z);\ z\right) \prec\ \Gamma\left[\frac{z(f*g)'(z)}{\phi((f*g)(z))},\ z\left(\frac{z(f*g)'(z)}{\phi((f*g)(z))}\right)';\ z\right]$$
$$\prec \Gamma\left(q_2(z),\ zq_2'(z);\ z\right),$$

where $a,\ b,\ c,\ d,\ \beta,\ \gamma,\ \alpha$ *be complex numbers such that* $d \neq 0,\ \beta \neq 0,\ \alpha \neq 0$ *and* Γ *is given by 9, then*

$$q_1(z) \prec \frac{z(f*g)'(z)}{\phi((f*g)(z))} \prec q_2(z),\ z \in \mathbb{E},$$

where $q_1(z)$ *and* $q_2(z)$ *are the best subordinant and the best dominant respectively.*

By selecting $g(z) = \dfrac{z}{1-z}$, in Theorem 9, we get

Theorem 10. *Let* $q_1,\ q_2\ (q_1(z) \neq 0,\ q_2(z) \neq 0,\ z \in \mathbb{E})$*, be univalent functions in* $\mathbb{E}$ *which satisfy conditions (i) and (ii) of Theorem 8. Let* ϕ *be analytic function in the domain containing* $f(\mathbb{E})$ *such that* $\phi(0) = 0 = \phi'(0) - 1$ *and* $\phi(w) \neq 0$ *for* $w \in f(\mathbb{E})\backslash\{0\}$*. If* $f \in \mathcal{A}$*, such that* $\dfrac{zf'(z)}{\phi(f(z))} \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ *with* $\dfrac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$*, satisfy*

$$\Gamma\left(q_1(z),\ zq_1'(z);\ z\right) \prec\ \Gamma\left[\frac{zf'(z)}{\phi(f(z))},\ z\left(\frac{zf'(z)}{\phi(f(z))}\right)';\ z\right]$$
$$\prec \Gamma\left(q_2(z),\ zq_2'(z);\ z\right),$$

where $a,\ b,\ c,\ d,\ \beta,\ \gamma,\ \alpha$ *be complex numbers such that* $d \neq 0,\ \beta \neq 0,\ \alpha \neq 0$ *and* Γ *is given by 9, then*

$$q_1(z) \prec \frac{zf'(z)}{\phi(f(z))} \prec q_2(z),\ z \in \mathbb{E},$$

where $q_1(z)$ *and* $q_2(z)$ *are the best subordinant and the best dominant respectively.*

When we choose $\phi(z) = z,\ g(z) = \dfrac{z}{1-z}$, in Theorem 9, we have

Theorem 11. *Let* $q_1,\ q_2\ (q_1(z) \neq 0,\ q_2(z) \neq 0,\ z \in \mathbb{E})$*, be univalent functions in* $\mathbb{E}$ *which satisfy conditions (i) and (ii) of Theorem 8. If* $f \in \mathcal{A}$ *such that* $\dfrac{zf'(z)}{f(z)} \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ *with* $\dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$*, satisfy*

$$\Gamma\left(q_1(z),\ zq_1'(z);\ z\right) \prec\ \Gamma\left[\frac{zf'(z)}{f(z)},\ z\left(\frac{zf'(z)}{f(z)}\right)';\ z\right]$$
$$\Gamma\left(q_2(z),\ zq_2'(z);\ z\right)$$

where $a,\ b,\ c,\ d,\ \beta,\ \gamma,\ \alpha$ *be complex numbers such that* $d \neq 0,\ \beta \neq 0,\ \alpha \neq 0$ *and* Γ *is given by 9, then*

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),\ z \in \mathbb{E},$$

where $q_1(z)$ *and* $q_2(z)$ *are the best subordinant and the best dominant respectively.*

When we select $\phi(z) = z,\ g(z) = \dfrac{z}{(1-z)^2}$, in Theorem 9, we have

Theorem 12. *Let* $q_1,\ q_2\ (q_1(z) \neq 0,\ q_2(z) \neq 0,\ z \in \mathbb{E})$*, be univalent functions in* $\mathbb{E}$ *which satisfy conditions (i) and (ii) of Theorem 8. If* $f \in \mathcal{A}$ *such that* $\left(\dfrac{zf''(z)}{f'(z)} + 1\right) \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ *with* $\dfrac{zf''(z)}{f'(z)} \neq -1,\ z \in \mathbb{E}$*, satisfy*

$$\Gamma\left(q_1(z),\ zq_1'(z);\ z\right) \prec \Gamma\left[1 + \frac{zf''(z)}{f'(z)},\ z\left(1 + \frac{zf''(z)}{f'(z)}\right)';\ z\right]$$
$$\Gamma\left(q_2(z),\ zq_2'(z);\ z\right)$$

where $a,\ b,\ c,\ d,\ \beta,\ \gamma,\ \alpha$ *be complex numbers such that* $d \neq 0,\ \beta \neq 0,\ \alpha \neq 0$ *and* Γ *is given by 9, then*

$$q_1(z) \prec 1 + \frac{zf''(z)}{f'(z)} \prec q_2(z),\ z \in \mathbb{E},$$

where $q_1(z)$ *and* $q_2(z)$ *are the best subordinant and the best dominant respectively.*

Remark 8. *On putting* $p(z) = \left(\dfrac{z^{1-\lambda}(f*\Phi)'(z)}{((f*\Psi)(z))^{1-\lambda}}\right)^{\alpha}$; $z \in \mathbb{E},\ 0 \le \lambda \le 1,\ \alpha(\neq 0) \in \mathbb{C}$ *in Theorem 8, we obtain*

Theorem 13. *Let* $q_1,\ q_2$ $(q_1(z) \neq 0,\ q_2(z) \neq 0,\ z \in \mathbb{E})$*, be univalent functions in* $\mathbb{E}$ *which satisfy conditions* (i) *and* (ii) *of Theorem 8. If* $f \in \mathcal{A}$ *and analytic functions* $\Phi,\ \Psi$ *such that* $\left(\dfrac{z^{1-\lambda}(f*\Phi)'(z)}{((f*\Psi)(z))^{1-\lambda}}\right)^{\alpha} \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ *with* $\left(\dfrac{z^{1-\lambda}(f*\Phi)'(z)}{((f*\Psi)(z))^{1-\lambda}}\right)^{\alpha} \neq 0,\ z \in \mathbb{E},\ \alpha(\neq 0) \in \mathbb{C},\ 0 \le \lambda \le 1,$ *satisfy*

$$\begin{aligned}&\Gamma\left(q_1(z),\ zq_1'(z);\ z\right)\\ &\quad \prec \Gamma\left\{\left(\frac{z^{1-\lambda}(f*\Phi)'(z)}{((f*\Psi)(z))^{1-\lambda}}\right)^{\alpha},\ z\left[\left(\frac{z^{1-\lambda}(f*\Phi)'(z)}{((f*\Psi)(z))^{1-\lambda}}\right)^{\alpha}\right]';\ z\right\}\\ &\qquad \prec \Gamma\left(q_2(z),\ zq_2'(z);\ z\right)\end{aligned}$$

where $a,\ b,\ c,\ d,\ \beta,\ \gamma,\ \alpha$ *be complex numbers such that* $d \neq 0,\ \beta \neq 0,\ \alpha \neq 0$ *and* Γ *is given by 9, then*

$$q_1(z) \prec \left(\frac{z^{1-\lambda}(f*\Phi)'(z)}{((f*\Psi)(z))^{1-\lambda}}\right)^{\alpha} \prec q_2(z),\ z \in \mathbb{E},$$

where $q_1(z)$ *and* $q_2(z)$ *are the best subordinant and the best dominant respectively.*

When we select $\Phi(z) = \dfrac{z}{1-z}$, $\lambda = 1$, in above theorem, we get

Theorem 14. *Let* $q_1,\ q_2$ $(q_1(z) \neq 0,\ q_2(z) \neq 0,\ z \in \mathbb{E})$*, be univalent functions in* $\mathbb{E}$ *which satisfy conditions* (i) *and* (ii) *of Theorem 8. If* $f \in \mathcal{A}$ *such that* $f'(z) \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ *with* $f'(z) \neq 0,\ z \in \mathbb{E},$ *satisfy*

$$\begin{aligned}\Gamma\left(q_1(z),\ zq_1'(z);\ z\right) \prec\ &\Gamma\left\{(f'(z))^{\alpha},\ z\left[(f'(z))^{\alpha}\right]';\ z\right\}\\ &\prec \Gamma\left(q_2(z),\ zq_2'(z);\ z\right)\end{aligned}$$

where $a,\ b,\ c,\ d,\ \beta,\ \gamma,\ \alpha$ *be complex numbers such that* $d \neq 0,\ \beta \neq 0,\ \alpha \neq 0$ *and* Γ *is given by 9, then*

$$q_1(z) \prec \left(f'(z)\right)^{\alpha} \prec q_2(z),\ z \in \mathbb{E},$$

where $q_1(z)$ *and* $q_2(z)$ *are the best subordinant and the best dominant respectively.*

By taking $\alpha = 1$ in above theorem, we have

Theorem 15. *Let $q_1,\ q_2\ (q_1(z) \neq 0,\ q_2(z) \neq 0,\ z \in \mathbb{E})$, be univalent functions in $\mathbb{E}$ which satisfy conditions (i) and (ii) of Theorem 8. If $f \in \mathcal{A}$ such that $f'(z) \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ with $f'(z) \neq 0,\ z \in \mathbb{E}$, satisfy*

$$\Gamma\left(q_1(z),\ zq_1'(z);\ z\right) \prec \Gamma\left\{f'(z),\ zf''(z);\ z\right\} \prec \Gamma\left(q_2(z),\ zq_2'(z);\ z\right)$$

where $a,\ b,\ c,\ d,\ \beta,\ \gamma,\ \alpha$ be complex numbers such that $d \neq 0,\ \beta \neq 0,\ \alpha \neq 0$ and Γ is given by 9, then

$$q_1(z) \prec f'(z) \prec q_2(z),\ z \in \mathbb{E},$$

where $q_1(z)$ and $q_2(z)$ are the best subordinant and the best dominant respectively.

7. Applications

Remark 9. *By taking $\beta = 1,\ \gamma = 1,\ q_1(z) = 1 + r_1 z$ and $q_2(z) = 1 + r_2 z;\ 0 < r_1 < r_2 \leq 1$ in Theorems 10, 11, 12 and 15 then after some calculations, we have checked that for $\frac{c}{d} \geq 0$ and $\frac{a}{d} > 0$, $q_1(z)$ and $q_2(z)$ satisfy conditions (i) and (ii) of Theorem 8. Hence we get the following corollaries from Theorems 10, 11, 12 and 15 respectively.*

Corollary 16. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, such that $\frac{zf'(z)}{\phi(f(z))} \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ with $\frac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$\begin{aligned} &b + r_1 dz + a(1 + r_1 z) + c(1 + r_1 z)^2 \\ &\prec b + \frac{azf'(z)}{\phi(f(z))} + c\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + \frac{dzf'(z)}{\phi(f(z))}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z[\phi(f(z))]'}{\phi(f(z))}\right) \\ &\prec b + r_2 dz + a(1 + r_2 z) + c(1 + r_2 z)^2, \end{aligned}$$

where $a,\ b,\ c,\ d,\ \beta,\ \gamma,\ \alpha$ be complex numbers such that $d \neq 0,\ \beta \neq 0,\ \alpha \neq 0$, $\frac{c}{d} \geq 0$ and $\frac{a}{d} > 0$ then

$$1 + r_1 z \prec \frac{zf'(z)}{\phi(f(z))} \prec 1 + r_2 z,\ 0 < r_1 < r_2 \leq 1,\ z \in \mathbb{E}.$$

i.e. f is ϕ–like.

Corollary 17. *If $f \in \mathcal{A}$, such that $\frac{zf'(z)}{f(z)} \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ with $\frac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$b + r_1dz + a(1 + r_1z) + c(1 + r_1z)^2$$
$$\prec b + \frac{azf'(z)}{f(z)} + c\left(\frac{zf'(z)}{f(z)}\right)^2 + \frac{dzf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)$$
$$\prec b + r_2dz + a(1 + r_2z) + c(1 + r_2z)^2,$$

where $a,\ b,\ c,\ d,\ \beta,\ \gamma,\ \alpha$ be complex numbers such that $d \neq 0,\ \beta \neq 0,\ \alpha \neq 0$, $\frac{c}{d} \geq 0$, and $\frac{a}{d} > 0$, then

$$1 + r_1z \prec \frac{zf'(z)}{f(z)} \prec 1 + r_2z,\ 0 < r_1 < r_2 \leq 1,\ z \in \mathbb{E}.$$

i.e. $f \in \mathcal{S}^*$.

Corollary 18. *If $f \in \mathcal{A}$ such that $\left(\frac{zf''(z)}{f'(z)} + 1\right) \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ with $\frac{zf''(z)}{f'(z)} \neq -1,\ z \in \mathbb{E}$, satisfy*

$$b + r_1dz + a(1 + r_1z) + c(1 + r_1z)^2 \prec b + a\left(1 + \frac{zf''(z)}{f'(z)}\right) + c\left(1 + \frac{zf''(z)}{f'(z)}\right)^2$$
$$+dz\left(\frac{zf'(z)f'''(z) + f'(z)f''(z) - z[f''(z)]^2))}{[f'(z)]^2}\right)$$
$$\prec b + r_2dz + a(1 + r_2z) + c(1 + r_2z)^2,$$

where $a,\ b,\ c,\ d,\ \beta,\ \gamma,\ \alpha$ be complex numbers such that $d \neq 0,\ \beta \neq 0,\ \alpha \neq 0$, $\frac{c}{d} \geq 0$, and $\frac{a}{d} > 0$, then

$$1 + r_1z \prec 1 + \frac{zf''(z)}{f'(z)} \prec 1 + r_2z,\ 0 < r_1 < r_2 \leq 1,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}$.

Corollary 19. *If $f \in \mathcal{A}$ such that $f'(z) \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ with $f'(z) \neq 0,\ z \in \mathbb{E}$, satisfy*

$$b + r_1dz + a(1 + r_1z) + c(1 + r_1z)^2 \prec b + af'(z) + c[f'(z)]^2 + dzf''(z)$$
$$\prec b + r_2dz + a(1 + r_2z) + c(1 + r_2z)^2,$$

where $a,\ b,\ c,\ d,\ \beta,\ \gamma,\ \alpha$ *be complex numbers such that* $d \neq 0,\ \beta \neq 0,\ \alpha \neq 0,$ $\dfrac{c}{d} \geq 0,$ *and* $\dfrac{a}{d} > 0$*, then*

$$1 + r_1 z \prec f'(z) \prec 1 + r_2 z,\ 0 < r_1 < r_2 \leq 1,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{C}$*.*

By selecting $r_1 = \dfrac{1}{2},\ r_2 = 1,\ a = 2,\ b = 1,\ c = 3,\ d = 2$ in corollaries 16, 17, 18 and 19, we obtain the following examples:

Example 1. *Let* ϕ *be analytic function in the domain containing* $f(\mathbb{E})$ *such that* $\phi(0) = 0 = \phi'(0) - 1$ *and* $\phi(w) \neq 0$ *for* $w \in f(\mathbb{E})\backslash\{0\}$*. If* $f \in \mathcal{A}$*, such that* $\dfrac{zf'(z)}{\phi(f(z))} \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ *with* $\dfrac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$*, satisfy*

$$\begin{aligned} &\frac{3}{4}z^2 + 5z + 6 \\ &\prec 1 + \frac{2zf'(z)}{\phi(f(z))} + 3\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + \frac{2zf'(z)}{\phi(f(z))}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z[\phi(f(z))]'}{\phi(f(z))}\right) \\ &\qquad \prec 3z^2 + 10z + 6, \end{aligned}$$

then

$$1 + \frac{z}{2} \prec \frac{zf'(z)}{\phi(f(z))} \prec 1 + z,\ z \in \mathbb{E}.$$

i.e. f *is* $\phi-$*like.*

Example 2. *If* $f \in \mathcal{A}$*, such that* $\dfrac{zf'(z)}{f(z)} \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ *with* $\dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$*, satisfy*

$$\begin{aligned} &\frac{3}{4}z^2 + 5z + 6 \\ &\prec 1 + \frac{2zf'(z)}{f(z)} + 3\left(\frac{zf'(z)}{f(z)}\right)^2 + \frac{2zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \\ &\qquad \prec 3z^2 + 10z + 6, \end{aligned}$$

then

$$1 + \frac{z}{2} \prec \frac{zf'(z)}{f(z)} \prec 1 + z,\ z \in \mathbb{E}.$$

i.e. $f \in \mathcal{S}^*$*.*

Example 3. *If $f \in \mathcal{A}$ such that* $\left(\frac{zf''(z)}{f'(z)}+1\right) \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ *with* $\frac{zf''(z)}{f'(z)} \neq -1,\ z \in \mathbb{E}$, *satisfy*

$$\frac{3}{4}z^2+5z+6 \prec 1+2\left(1+\frac{zf''(z)}{f'(z)}\right)+3\left(1+\frac{zf''(z)}{f'(z)}\right)^2$$

$$+2z\left(\frac{zf'(z)f'''(z)+f'(z)f''(z)-z[f''(z)]^2))}{[f'(z)]^2}\right) \prec 3z^2+10z+6,$$

then

$$1+\frac{z}{2} \prec 1+\frac{zf''(z)}{f'(z)} \prec 1+z,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}$.

Example 4. *If $f \in \mathcal{A}$ such that* $f'(z) \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ *with* $f'(z) \neq 0,\ z \in \mathbb{E}$, *satisfy*

$$\frac{3}{4}z^2+5z+6 \prec 1+2f'(z)+3[f'(z)]^2+2zf''(z) \prec 3z^2+10z+6,$$

then

$$1+\frac{z}{2} \prec f'(z) \prec 1+z,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{C}$.

Using Mathematica 7.0, we plot the images of unit disk $\mathbb{E}$ under the functions $w_1(z) = 6+5z+\frac{3}{4}z^2$ and $w_2(z) = 6+10z+3z^2$, which are given by Figure 3.1 and the images of unit disk $\mathbb{E}$ under the functions $q_1(z) = 1+\frac{z}{2}$ and $q_2(z) = 1+z$, which are given by Figure 3.2. Therefore, from Examples 1 and 2, we see that the differential operators $\frac{zf'(z)}{\phi(f(z))}$ and $\frac{zf'(z)}{f(z)}$ take values in the light shaded region of Figure 3.2 when the differential operators

$$1+\frac{2zf'(z)}{\phi(f(z))}+3\left(\frac{zf'(z)}{\phi(f(z))}\right)^2+\frac{2zf'(z)}{\phi(f(z))}\left(1+\frac{zf''(z)}{f'(z)}-\frac{z[\phi(f(z))]'}{\phi(f(z))}\right)$$

and

$$1+\frac{2zf'(z)}{f(z)}+3\left(\frac{zf'(z)}{f(z)}\right)^2+\frac{2zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)$$

take values in the light shaded region of Figure 3.1. Thus the function $f(z)$ is $\phi-$like and starlike in $\mathbb{E}$ respectively. Similarly, in Examples 3 and 4, we notice that the differential operators $1 + \frac{zf''(z)}{f'(z)}$ and $f'(z)$ take values in the light shaded region of Figure 3.2 when the operators

$$1 + 2\left(1 + \frac{zf''(z)}{f'(z)}\right) + 3\left(1 + \frac{zf''(z)}{f'(z)}\right)^2$$
$$+2z\left(\frac{zf'(z)f'''(z) + f'(z)f''(z) - z[f''(z)]^2}{[f'(z)]^2}\right)$$

and $1 + 2f'(z) + 3[f'(z)]^2 + 2zf''(z)$ take values in the light shaded region of Figure 3.1 and hence the function $f(z)$ is convex and close-to-convex in $\mathbb{E}$ respectively.

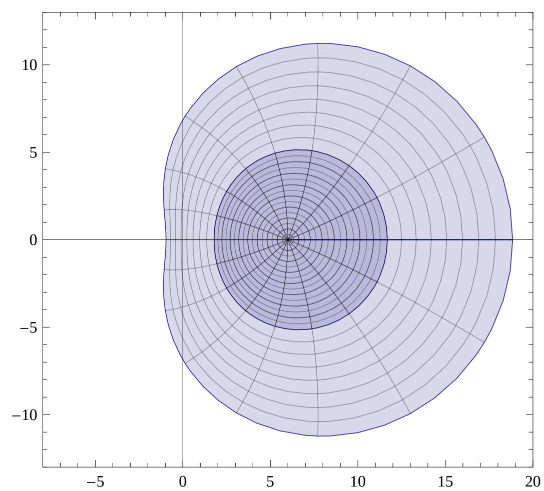

Figure 3.1

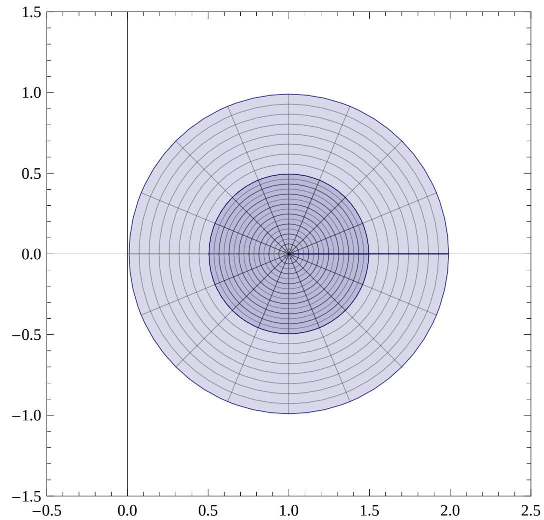

Figure 3.2

Remark 10. *Again, when we select* $\beta = 1,\ \gamma = 1,\ q_1(z) = e^{z/4}$ *and* $q_2(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$ *in Theorems 10, 11, 12 and 15, then after having some calculations, we found that for* $\frac{c}{d} \geq 0$ *and* $\frac{a}{d} > 0$, $q_1(z)$ *and* $q_2(z)$ *satisfy conditions* (i) *and* (ii) *of Theorem 8. Therefore, we get the following corollaries from Theorems 10, 11, 12 and 15 respectively.*

Corollary 20. *Let* ϕ *be analytic function in the domain containing* $f(\mathbb{E})$ *such that* $\phi(0) = 0 = \phi'(0) - 1$ *and* $\phi(w) \neq 0$ *for* $w \in f(\mathbb{E})\backslash\{0\}$. *If* $f \in \mathcal{A}$, *such that* $\frac{zf'(z)}{\phi(f(z))} \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ *with* $\frac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, *satisfy*

$$b + ce^{z/2} + \left(a + \frac{d}{4}z\right)e^{z/4} \prec b + \frac{azf'(z)}{\phi(f(z))} + c\left(\frac{zf'(z)}{\phi(f(z))}\right)^2$$

$$+\frac{dzf'(z)}{\phi(f(z))}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z[\phi(f(z))]'}{\phi(f(z))}\right)$$

$$\prec b + a\left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\} + c\left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^2$$

$$+\frac{4d\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

where $a,\ b,\ c,\ d$ *be complex numbers such that* $d\neq 0,\ \dfrac{c}{d}\geq 0$ *and* $\dfrac{a}{d}>0$ *then*

$$e^{z/4}\prec\frac{zf'(z)}{\phi(f(z))}\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E}.$$

i.e. f *is parabolic* $\phi-$*like.*

Corollary 21. *If* $f\in\mathcal{A}$, *such that* $\dfrac{zf'(z)}{f(z)}\in\mathcal{H}[q(0),\ 1]\cap\mathbb{Q}$ *with* $\dfrac{zf'(z)}{f(z)}\neq 0,\ z\in\mathbb{E}$, *satisfy*

$$b+ce^{z/2}+\left(a+\frac{d}{4}z\right)e^{z/4}\prec b+\frac{azf'(z)}{f(z)}+c\left(\frac{zf'(z)}{f(z)}\right)^2$$

$$+\frac{dzf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\prec b+a\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}$$

$$+c\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^2+\frac{4d\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

where $a,\ b,\ c,\ d$ *be complex numbers such that* $d\neq 0,\ \dfrac{c}{d}\geq 0$ *and* $\dfrac{a}{d}>0$ *then*

$$e^{z/4}\prec\frac{zf'(z)}{f(z)}\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E}.$$

i.e. $f\in S_P$.

Corollary 22. *If* $f\in\mathcal{A}$ *such that* $\left(\dfrac{zf''(z)}{f'(z)}+1\right)\in\mathcal{H}[q(0),\ 1]\cap\mathbb{Q}$ *with* $\dfrac{zf''(z)}{f'(z)}\neq -1,\ z\in\mathbb{E}$, *satisfy*

$$b+ce^{z/2}+\left(a+\frac{d}{4}z\right)e^{z/4}\prec b+a\left(1+\frac{zf''(z)}{f'(z)}\right)+c\left(1+\frac{zf''(z)}{f'(z)}\right)^2$$

$$+dz\left(\frac{zf'(z)f'''(z)+f'(z)f''(z)-z[f''(z)]^2))}{[f'(z)]^2}\right)$$

$$\prec b+a\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}+c\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^2$$

$$+\frac{4d\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

where $a,\ b,\ c,\ d$ *be complex numbers such that* $d \neq 0,\ \frac{c}{d} \geq 0,$ *and* $\frac{a}{d} > 0,$ *then*

$$e^{z/4} \prec 1+\frac{zf''(z)}{f'(z)} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z \in \mathbb{E},$$

i.e. $f \in UCV$.

Corollary 23. *If* $f \in \mathcal{A}$ *such that* $f'(z) \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ *with* $f'(z) \neq 0,\ z \in \mathbb{E}$, *satisfy*

$$b+ce^{z/2}+\left(a+\frac{d}{4}z\right)e^{z/4} \prec b+af'(z)+c[f'(z)]^2+dzf''(z)$$

$$\prec b+a\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}+c\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^2$$

$$+\frac{4d\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

where $a,\ b,\ c,\ d$ *be complex numbers such that* $d \neq 0,\ \frac{c}{d} \geq 0,$ *and* $\frac{a}{d} > 0,$ *then*

$$e^{\frac{z}{4}} \prec f'(z) \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z \in \mathbb{E},$$

i.e. $f \in UCC$.

By taking $a = 2,\ b = 1,\ c = 3,\ d = 2$ in corollaries 20, 21, 22 and 23, we have

Example 5. *Let* ϕ *be analytic function in the domain containing* $f(\mathbb{E})$ *such that* $\phi(0) = 0 = \phi'(0) - 1$ *and* $\phi(w) \neq 0$ *for* $w \in f(\mathbb{E})\backslash\{0\}$. *If* $f \in \mathcal{A}$, *such that* $\frac{zf'(z)}{\phi(f(z))} \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ *with* $\frac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, *satisfy*

$$1+3e^{z/2}+\left(2+\frac{z}{2}\right)e^{z/4}$$

$$\prec 1+\frac{2zf'(z)}{\phi(f(z))}+3\left(\frac{zf'(z)}{\phi(f(z))}\right)^2+\frac{2zf'(z)}{\phi(f(z))}\left(1+\frac{zf''(z)}{f'(z)}-\frac{z[\phi(f(z))]'}{\phi(f(z))}\right)$$

$$\prec 1+2\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}+3\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^2$$

$$+\frac{8\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

then

$$e^{z/4} \prec \frac{zf'(z)}{\phi(f(z))} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E}.$$

i.e. f *is parabolic* $\phi-$*like.*

Example 6. *If* $f\in\mathcal{A}$, *such that* $\dfrac{zf'(z)}{f(z)}\in\mathcal{H}[q(0),\ 1]\cap\mathbb{Q}$ *with* $\dfrac{zf'(z)}{f(z)}\neq 0,\ z\in\mathbb{E}$, *satisfy*

$$1+3e^{z/2}+\left(2+\frac{z}{2}\right)e^{z/4} \prec 1+\frac{2zf'(z)}{f(z)}+3\left(\frac{zf'(z)}{f(z)}\right)^2$$

$$+\frac{2zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec 1+2\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}$$

$$+3\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^2+\frac{8\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

then

$$e^{z/4} \prec \frac{zf'(z)}{f(z)} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E}.$$

i.e. $f\in S_P$.

Example 7. *If* $f\in\mathcal{A}$ *such that* $\left(\dfrac{zf''(z)}{f'(z)}+1\right)\in\mathcal{H}[q(0),\ 1]\cap\mathbb{Q}$ *with* $\dfrac{zf''(z)}{f'(z)}\neq -1,\ z\in\mathbb{E}$, *satisfy*

$$1+3e^{z/2}+\left(2+\frac{z}{2}\right)e^{z/4} \prec 1+2\left(1+\frac{zf''(z)}{f'(z)}\right)+3\left(1+\frac{zf''(z)}{f'(z)}\right)^2$$

$$+2z\left(\frac{zf'(z)f'''(z)+f'(z)f''(z)-z[f''(z)]^2))}{[f'(z)]^2}\right)$$

$$\prec 1+2\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}+3\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^2$$

$$+\frac{8\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

then

$$e^{\frac{z}{4}} \prec 1+\frac{zf''(z)}{f'(z)} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E},$$

i.e. $f\in UCV$.

Example 8. *If* $f \in \mathcal{A}$ *such that* $f'(z) \in \mathcal{H}[q(0),\ 1] \cap \mathbb{Q}$ *with* $f'(z) \neq 0,\ z \in \mathbb{E}$, *satisfy*

$$1 + 3e^{z/2} + \left(2 + \frac{z}{2}\right) e^{z/4} \prec 1 + 2f'(z) + 3[f'(z)]^2 + 2zf''(z)$$

$$\prec 1 + 2\left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\} + 3\left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^2$$

$$+\frac{8\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

then

$$e^{\frac{z}{4}} \prec f'(z) \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z \in \mathbb{E},$$

i.e. $f \in UCC$.

Again by using Mathematica 7.0, the images of unit disk $\mathbb{E}$ under the functions $w_3(z) = 1 + 3e^{z/2} + \left(2 + \frac{z}{2}\right) e^{z/4}$ and

$$w_4(z) = 1 + 2\left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}$$

$$+3\left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^2 + \frac{8\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)$$

are given in Figure 3.3. In Figure 3.4, the images of unit disk $\mathbb{E}$ under the functions $q_3(z) = e^{\frac{z}{4}}$ and $q_4(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$ are given. From Examples 5 and 6, we can conclude that the the differential operators $\frac{zf'(z)}{\phi(f(z))}$ and $\frac{zf'(z)}{f(z)}$ take values in the light shaded region of Figure 3.4 when the differential operators

$$1 + \frac{2zf'(z)}{\phi(f(z))} + 3\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + \frac{2zf'(z)}{\phi(f(z))}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z[\phi(f(z))]'}{\phi(f(z))}\right)$$

and

$$1 + \frac{2zf'(z)}{f(z)} + 3\left(\frac{zf'(z)}{f(z)}\right)^2 + \frac{2zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)$$

take values in the light shaded region of Figure 3.3. Therefore, the function $f(z)$ is parabolic $\phi-$like and parabolic starlike in $\mathbb{E}$ respectively. Similarly, from Examples

7 and 8, we can say that the differential operators $1+\dfrac{zf''(z)}{f'(z)}$ and $f'(z)$ take values in the light shaded region of Figure 3.4 when the operators

$$1+2\left(1+\frac{zf''(z)}{f'(z)}\right)+3\left(1+\frac{zf''(z)}{f'(z)}\right)^2 +2z\left(\frac{zf'(z)f'''(z)+f'(z)f''(z)-z[f''(z)]^2}{[f'(z)]^2}\right)$$

and $1+2f'(z)+3[f'(z)]^2+2zf''(z)$ take values in the light shaded region of Figure 3.3. Thus the function $f(z)$ is uniformly convex and uniformly close-to-convex in $\mathbb{E}$ respectively.

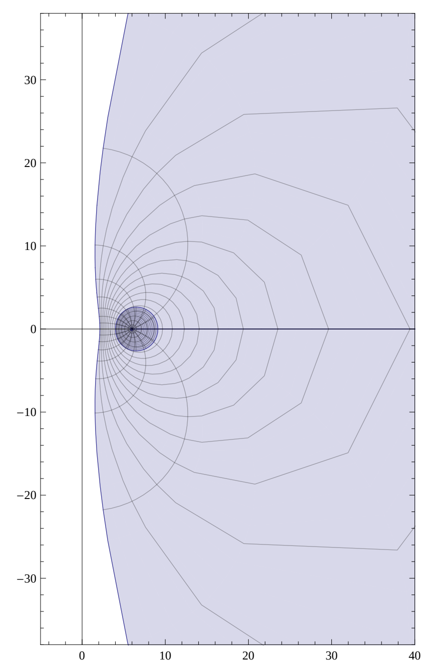

Figure 3.3

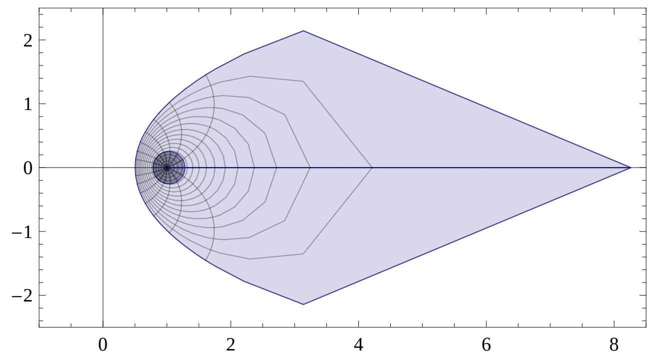

Figure 3.4

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