SOME MECHANICAL SYSTEMS AND THEIR EQUILIBRIUM STATES

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Abstract. In the first part of the paper some theoretical results (including the Lyapunov-Malkin theorem) are presented, followed in the second part by some of its applications in geometrical mechanics.

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1. Theoretical aspects

To explain the stability concept, we need some basic notions and results from the theory of dynamical systems.

The lows of Dynamics are usually presented as equations of motion, which we will write as differential equations:

$$\dot{x} = f(x) \tag{1.1}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

is a variable describing the state of the system. The function

$$f : \mathbb{R}^n \to \mathbb{R}^n$$

is smooth of x, and

$$\dot{x} = \frac{dx}{dt}.$$

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The set of all allowed x forms the state space of (1.1). When the time advances, the system's state is changed.

Definition 1.1 *An point $x_e \in \mathbb{R}^n$ is called equilibrium point for the system (1.1) if:*

$$f(x_e) = 0.$$

Remark. It is clear that the constant function

$$x(t) = x_e$$

is a solution for (1.1) and from the existence and uniqueness theorem it results that does not exists other solution containing x_e. So the unique trajectory starting at x_e is x_e itself, i.e. x_e does not changes in time.

Definition 1.2. *Let x_e be an equilibrium state for (1.1). We will say that x_e is nonlinear stable (or Lyapunov stable) if for any neighborhood U of x_e there exists a neighborhood V of x_e, $V \subset U$ such that any solution $x(t)$, initially in V (i.e. $x(0) \in V$), never leaves U.*

Definition 1.3. *If V in Definition 2 can be chosen such that*

$$\lim_{t\to\infty} x(t) = x_e,$$

then x_e is called asymptotically stable.

Definition 1.4. An equilibrium state x_e that is not stable is called unstable.

Let us consider the following system of differential equations of order one:

$$\begin{cases} \dot{x} = Ax + X(x,y) \\ \dot{y} = By + Y(x,y) \end{cases} \tag{1.2}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, A and B are constant matrices such that all eigenvalues of A are of nonzero real parts and all eigenvalues of B are of zero real parts, and the functions X, Y satisfy the following conditions:

i) $X(0,0) = 0$,

ii) $dX(0,0) = 0$,

iii) $Y(0,0) = 0$,
iv) $dY(0,0) = 0$.

We will take now the particular case of (1.2) in which the matrix B is O_n. The equations (1.2) become:

$$\begin{cases} \dot{x} = Ax + X(x,y) \\ \dot{y} = Y(x,y) \end{cases} \tag{2.3}$$

Theorem 1.1. (Lyapunov-Malkin) *Under the above conditions, if all eigenvalues of A have negative real parts and $X(x,y)$ and $Y(x,y)$ vanish when $x = 0$, then the equilibrium state*

$$x = 0, \ y = 0$$

*of the system (**??**) is nonlinear stable with respect to (x,y) and asymptotically stable with respect to X.*

For the proof of this basic result see Zenkov, Bloch and Marsden [6].

2. Applications of Lyapunov-Malkin Theorem

Example 2.1. (Volterra model with two controls on axes Ox_1 and Ox_2) The Volterra model equations with two controls on axes Ox_1 and Ox_2 are writing in the following form:

$$\begin{cases} \dot{x}_1 = x_1 x_2 \\ \dot{x}_2 = x_2 x_3 - x_1 x_2 + u_1 \\ \dot{x}_3 = -x_1 x_3 + u_2 \end{cases} \tag{2.1}$$

The controls u_1 and u_2 will be written as:

$$u_1(x_1, x_2, x_3) = \alpha x_2$$
$$u_2(x_1, x_2, x_3) = \beta x_3$$

where $\alpha, \beta \in$, $\alpha, \beta < 0$. Then the dynamics described by (2.1) takes the following form:

$$\begin{cases} \dot{x}_1 = x_1 x_2 \\ \dot{x}_2 = x_2 x_3 - x_1 x_2 + \alpha x_2 \\ \dot{x}_3 = -x_1 x_3 + \beta x_3 \end{cases} \tag{2.2}$$

If we consider:

$$x = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}, \quad y = x_1,$$
$$X(x,y) = \begin{bmatrix} x_2x_3 - x_1x_2 \\ -x_1x_3 \end{bmatrix},$$
$$Y(x,y) = x_1x_2$$

then the system (2.2) can be written in the following equivalent form:

$$\begin{cases} \dot{x} = Ax + X(x,y) \\ \dot{y} = Y(x,y) \end{cases} \tag{2.3}$$

where:

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

Now we must verify the conditions from the Lyapunov-Malkin theorem for system (2.3). We have successively:

i) $X(0,0) = \begin{bmatrix} x_2x_3 - x_1x_2 \\ -x_1x_3 \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$

$Y(0,0) = [x_1x_2]_{(0,0,0)} = 0.$

ii) $X(0,y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (\forall)\ y \in R,$

$Y(0,y) = 0, \quad (\forall)\ y \in R.$

iii) $$\frac{DX}{Dx}\Big|_{(0,0)} = \begin{bmatrix} \frac{\partial(x_2x_3 - x_1x_2)}{\partial x_2} & \frac{\partial(x_2x_3 - x_1x_2)}{\partial x_3} \\ \frac{\partial(-x_1x_3)}{\partial x_2} & \frac{\partial(-x_1x_3)}{\partial x_3} \end{bmatrix}_{(0,0,0)}$$
$$= \begin{bmatrix} x_3 - x_1 & x_2 \\ 0 & -x_1 \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\frac{DX}{Dy}\Big|_{(0,0)} = \begin{bmatrix} \frac{\partial(x_2x_3 - x_1x_2)}{\partial x_1} \\ \frac{\partial(-x_1x_3)}{\partial x_1} \end{bmatrix}_{(0,0,0)}$$
$$= \begin{bmatrix} -x_2 \\ -x_3 \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\frac{DY}{Dx}\Big|_{(0,0)} = \begin{bmatrix} \frac{\partial(x_1x_2)}{\partial x_2} & \frac{\partial(x_1x_2)}{\partial x_3} \end{bmatrix}_{(0,0,0)}$$
$$= \begin{bmatrix} x_1 & 0 \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

$\frac{DY}{Dy}\Big|_{(0,0)} = [x_2]_{(0,0,0)} = [0].$

iv) The characteristic polynomial of A is

$$P_A(x) = \det \begin{bmatrix} \alpha - x & 1 \\ 0 & \beta - x \end{bmatrix} = (\alpha - x)(\beta - x),$$

and it has negative roots. So we have the eigenvalues of the matrix A, $x_1 = \alpha < 0,\ x_2 = \beta < 0$.

We can conclude, by Lyapunov-Malkin theorem, that:

Proposition 2.1. The equilibrium state $(0, 0, 0)$ for the system (2.2) is nonlinear stable.

Example 2.2. (The ball-plate poblem) The equations for the ball-plate problem with two controls on axes Op_2 şi Op_3 are given by:

$$\begin{cases} \dot{p}_1 = -p_2 p_3 \\ \dot{p}_2 = p_1 p_3 + u_2 \\ \dot{p}_3 = u_3 \end{cases} \tag{2.4}$$

The controls u_2 and u_3 will be written as:

$\begin{cases} u_2(p_1, p_2, p_3) = mp_3 \\ u_3(p_1, p_2, p_3) = -mp_2 + 3mp_3 \end{cases}$

where $e < 0, e \in R$. Then the dynamics described by (2.4) takes the following form:

$$\begin{cases} \dot{p}_1 = -p_2 p_3 \\ \dot{p}_2 = p_1 p_3 + mp_3 \\ \dot{p}_3 = -ep_2 + 3mp_3 \end{cases} \tag{2.5}$$

where $m < 0$, $m \in R$. Now we consider:

$$x = \begin{bmatrix} p_2 \\ p_3 \end{bmatrix}, \quad y = p_1,$$
$$X(x, y) = \begin{bmatrix} p_1 p_3 \\ 0 \end{bmatrix},$$
$$Y(x, y) = -p_2 p_3.$$

and the system (2.5) can be written in the following equivalent form:

$$\begin{cases} \dot{x} = Ax + X(x, y) \\ \dot{y} = Y(x, y) \end{cases} \tag{2.6}$$

where:

$$A = \begin{bmatrix} 0 & m \\ -m & 3m \end{bmatrix}.$$

Now we must verify the conditions from the Lyapunov-Malkin theorem for system (2.6). We have successively:

i) $X(0,0) = \begin{bmatrix} p_1 p_3 \\ 0 \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$Y(0,0) = [-p_2 p_3]_{(0,0,0)} = [0]$.

ii) $X(0,y) = \begin{bmatrix} p_1 p_3 \\ 0 \end{bmatrix}_{(p_1, p_2=0, p_3=0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (\forall)\ y \in R^2,$

$Y(0,y) = [-p_2 p_3]_{(p_1, p_2=0, p_3=0)} = [0], \quad (\forall)\ y \in R^2.$

iii) $$\left.\frac{DX}{Dx}\right|_{(0,0)} = \begin{bmatrix} \frac{\partial p_1 p_3}{\partial p_2} & \frac{\partial p_1 p_3}{\partial p_3} \\ \frac{\partial 0}{\partial p_2} & \frac{\partial 0}{\partial p_2} \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 & p_1 \\ 0 & 0 \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\left.\frac{DX}{Dy}\right|_{(0,0)} = \begin{bmatrix} \frac{\partial p_1 p_3}{\partial p_1} \\ \frac{\partial 0}{\partial p_1} \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} p_3 \\ 0 \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\left.\frac{DY}{Dx}\right|_{(0,0)} = \begin{bmatrix} \frac{\partial(-p_2 p_3)}{\partial p_2} & \frac{\partial(-p_2 p_3)}{\partial p_3} \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} -p_3 & -p_2 \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

$$\left.\frac{DY}{Dy}\right|_{(0,0)} = \begin{bmatrix} \frac{\partial(-p_2 p_3)}{\partial p_1} \end{bmatrix}_{(0,0,0)} = [0]$$

iv) Again the characteristic polynomial of A is

$$P_A(x) = \det \begin{bmatrix} -x & m \\ -m & 3e - x \end{bmatrix} = x^2 - 3mx + m^2,$$

and it has negative roots. So we have the eigenvalues of the matrix A, $x_{1,2} = \dfrac{3m \pm m\sqrt{5}}{2} < 0$, for $m < 0$.

We can conclude, by Lyapunov-Malkin theorem, that:

Proposition 2.2 The equilibrium state $(0,0,0)$ for the system (2.6) is nonlinear stable.

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