PROPAGATION OF STRESS WAVES IN ELASTIC SOLIDS

Ion Al. Crăciun

Abstract. The propagation of waves in elastic media under dynamic loads (*stress waves*) are investigated. The nature of deformation, stress, stress–strain relations, and equation of motion are some objectives of investigation. General decompositions of elastic waves are studied. Two planar waves in an infinite isotropic elastic medium, and then time–harmonic solutions of the wave equations are analyzed. Spherically symmetric waves in three–dimensional space from a point source, radially symmetric waves in a solid infinite cylinder of radius a, and waves propagated over the surface of an elastic body are studied. Finally, particular solutions of the Navier equations are given.

2000 Mathematics Subject Classification: 74B05, 74J05, 74J10, 74J15.
Keywords: elastic waves, time–harmonic solutions, spherically and radially symmetric waves.

1. Introduction

The field equations of physics for a given continuous medium arise from the three conservation equations: conservation of mass, momentum, and energy. For an elastic medium these equations of motion are called the *Navier equations*. From them, a rich variety of stress waves is obtained. The first two conservation laws are common to all media, while the energy equation defines a particular medium through the equation of state. When we deal with an elastic solid we are concerned with the dynamic variables: stress and strain. The energy equation for an elastic solid yields a relation between the stress and strain. For small–amplitude deformations there is a linear relation between the stress and strain, which is given by Hooke's law.

A knowledge of wave propagation in elastic solids, which we call *stress wave propagation*, is very important to engineers in many fields (mechanical, civil, aeronautical, etc), where an investigation of the dynamic effects of various loads on engineering materials is considered.

2. Notation and Mathematical Preliminaries

Physical quantities are mathematically represented by tensors of various orders. The equations describing physical laws are tensor equations. Quantities that are not associated with any special direction and are measured by a single number are represented by scalars, or tensors of order zero. Tensors of order one are vectors, which represent quantities that are characterized by a direction as well as a magnitude. More complicated physical quantities are represented by tensors of order greater than one.

Throughout this paper light–faced Roman or Greek letters stand for scalars, Roman letters in boldface denote vectors, while lower case Greek letters in boldface denote second–order tensors.

2.1 Indicial notation

A sistem of fixed rectangular Cartesian coordinates is sufficient for the presentation of the theory. In indicial notation, the coordinates axes may be denoted by x_j and the unit base vectors by $\mathbf{e}_j$, where $j = 1, 2, 3$. In the sequel, subscripts assume the values $1, 2, 3$ unless explicitly otherwise specified. If the components of a vector $\mathbf{u}$ are denoted by u_j, we have

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3. \tag{1}$$

Since summation of the type (1) frequently occur in the mathematical description of the mechanics of a continuum medium, we introduce the *summation convention*, whereby a repeated subscript implies a summation. Equation (1) may be rewritten as

$$\mathbf{u} = u_j\mathbf{e}_j. \tag{2}$$

As another example of the use of the summation convention, the scalar product of the two vectors is expressed as

$$\mathbf{u} \cdot \mathbf{v} = u_j v_j = u_1v_1 + u_2v_2 + u_3v_3. \tag{3}$$

As opposed to the free index in u_j, which may assume any one of the values $1, 2, 3$, the index j in (2) and (3) is a bound index or a dummy index, which must assume all three values $1, 2$ and 3

Quantities assume two free indices as subscripts, such as τ_{ij}, denote components of a tensor of second rank $\boldsymbol{\tau}$, and similarly three free indices define a tensor of rank three. A well-known special tensor of rank two is the *Kronecker delta*, whose components are defined as

$$\delta_{ij} \quad = \quad \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

A frequently–used special tensor of rank three is the *alternating tensor*, whose components are defined as follows:

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if} \quad ijk \quad \text{represents an even permutation of} \quad 123 \\ \ \ 0 & \text{if} \quad \text{any} \quad \text{two of the } ijk \text{ indices are equal} \\ -1 & \text{if} \quad ijk \quad \text{represents an odd permutation of} \quad 123. \end{cases}$$

By the use of the alternating tensor and the summation convention, the components of the cross product $\mathbf{h} = \mathbf{u} \times \mathbf{v}$ may be expressed as

$$h_i = \varepsilon_{ijk} u_j v_k.$$

In extended notation the components of $\mathbf{h}$ are

$$h_1 = u_2 v_3 - u_3 v_2, \quad h_2 = u_3 v_1 - u_1 v_3, \quad h_3 = u_1 v_2 - u_2 v_1.$$

2.2 Vector operators

Particularly significant in vector calculus is the Hamilton's vector operator (or nabna) denoted by $\boldsymbol{\nabla}$, which is given by

$$\boldsymbol{\nabla} = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3}.$$

When applied to the scalar field $\varphi(x_1, x_2, x_3)$, the vector operator $\boldsymbol{\nabla}$ yields a vector field which is known as the *gradient* of the scalar field,

$$\operatorname{grad} \varphi = \boldsymbol{\nabla} \varphi = \frac{\partial \varphi}{\partial x_1} \mathbf{e}_1 + \frac{\partial \varphi}{\partial x_2} \mathbf{e}_2 + \frac{\partial \varphi}{\partial x_3} \mathbf{e}_3.$$

In indicial notation, partial differentiation is commonly denoted by a comma, and thus

$$\operatorname{grad} \varphi = \boldsymbol{\nabla} \varphi = \varphi_{,k} \mathbf{e}_k.$$

The appearence of a single subscript in $\varphi_{,k}$ indicates that $\varphi_{,k}$ are the components of a tensor of rank one, i. e.., a vector.

In a vector field, denoted by $\mathbf{u}(\mathbf{x})$, the components of the vector are functions af spatial coordinates. The components are denoted by $u_i(x_1, x_2, x_3)$. Assuming that functions $u_i(x_1, x_2, x_3)$ are differentiable, the nine partial derivatives $\dfrac{\partial u_i}{\partial x_j}(x_1, x_2, x_3)$ can be written in indicial notation as $u_{i,j}$. It can be shown that $u_{i,j}$ are the components of a second–rank tensor.

When the vector operator $\boldsymbol{\nabla}$ operates on a vector in a manner analogous to scalar multiplication, the result is a scalar field, termed the *divergence* of the vector field $\mathbf{u}(\mathbf{x})$

$$\operatorname{div}\mathbf{u} = \boldsymbol{\nabla}\cdot\mathbf{u} = u_{i,i}.$$

By taking the cross product of $\boldsymbol{\nabla}$ and $\mathbf{u}$, we obtain a vector termed the *curl* of $\mathbf{u}$, denoted by $\operatorname{curl}\mathbf{u} = \boldsymbol{\nabla}\times\mathbf{u}$. If $\mathbf{q} = \boldsymbol{\nabla}\times\mathbf{u}$, the components of $\mathbf{q}$ are

$$q_i = \varepsilon_{ijk}u_{k,j}.$$

The *Laplace operator* ∇^2 is obtained by taking the divergence of a gradient. The Laplacian of a twice differentiable scalar field is another scalar field,

$$\operatorname{div}\operatorname{grad}\varphi = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\varphi = \nabla^2\varphi = f_{,ii}.$$

The Laplacian of a vector field is another vector field denoted by $\nabla^2\mathbf{u}$

$$\nabla^2\mathbf{u} = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\mathbf{u} = u_{k,jj}\mathbf{e}_k.$$

2.3 Gauss' Theorem

One of the most important integral theorem of tensor analysis, known as *Gauss' theorem*, relates a volume integral to a surface integral over the bounding surface of the volume. Consider a convex region B of volume V, bounded by a surface S which possesses a piecewise continuously turning tangent plane. Such a region is said to be regular. Now let us consider a tensor field $\tau_{jkl\cdots p}$, and let every component of $\tau_{jkl\cdots p}$ be continuously differentiable in B. Then Gauss' theorem states

$$\int_V \tau_{jkl\cdots p,i}dV = \int_S n_i\tau_{jkl\cdots p}dA, \tag{4}$$

where n_i are the components of the unit vector along the outer normal to the surface S. If equation (4) is written with the three components of a vector $\mathbf{u}$

successively substituted for $\tau_{jkl\cdots p}$, and if the three resulting expressions are added, the result is

$$\int_V u_{i,i} dV = \int_S n_i u_i dA. \tag{5}$$

Equation (5) is the well–known *divergence theorem* of a vector calculus which states that the integral of the outer normal component of a vector over a closed surface is equal to the integral of the divergence of the vector over volume bounded by the closed surface.

2.4 Notation

The equations governing the linearized theory of elasticity are presented in the following commonly used notation:

position (radius) vector :	$\mathbf{x}$(coordinates x_i)
displacement vector :	$\mathbf{u}$(components u_i)
small strain tensor :	$\boldsymbol{\varepsilon}$(components ε_{ij})
stress tensor :	$\boldsymbol{\tau}$(components τ_{ij}).

3. Fundamental Concepts in Elasticity

In this section we shall investigate the nature of deformation, strain, stress, stress–strain relations, equations of motion for the stress components, and equations of motion for the displacement.

3.1 Deformations

An elastic material is a deformable, continuous medium which suffers no energy loss when its deformed state returns to the equilibrium state. The deformation of any medium is a purely geometric concept. Since strain is derived from deformation, strain is also a geometrical concept. A continuum is a medium that has a continuous distribution of matter in the sense that its molecular and crystalline structure is neglected. This means that we can define mathematically a differential volume element dV that has the same continuous properties as the material in the large. This concept of a continuum is based on an averaging process, where we take advantage of the large number of molecules in a differential volume element to smear out the effects of individual molecules.

To describe a deformable continuum we consider two states of the medium or body:

1. The undeformed or equilibrium configuration or state;
2. The deformed configuration.

Any two neighboring points P_1, P_2 in the body in its undeformed state under a deformation suffer the transformation $P_1 \mapsto P_1'$, $P_2 \mapsto P_2'$. The distance between the two undeformed points changes in the deformed state. If $P_1'P_2' < P_1P_2$ that part of the body undergoes a compression; if $P_1'P_2' > P_1P_2$ we have tension. Clearly, if there is no change we have the equilibrium state. We use the Lagrange representation. A point in the undeformed state is given by the coordinates $\mathbf{X} = (X_1, X_2, X_3)$, where $\mathbf{X}$ is the radius vector. In the deformed state, that point goes into the Eulerian coordinates given by the radius vector $\mathbf{x}$, where $\mathbf{x} = (x_1, x_2, x_3)$. The displacement vector $\mathbf{u}$ is defined by $\mathbf{u} = \mathbf{x} - \mathbf{X}$. Since we use the Lagrange representation, we have $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, $\mathbf{u} = \mathbf{u}(\mathbf{X}, t)$, and all the dynamic and thermodynamic variables are functions of $(\mathbf{X}, t)$). We have thus defined deformation (compression or tension) in terms of the displacement vector.

3.2 Strain Tensor

For the equilibrium state, let $\mathbf{X} = (X_1, X_2, X_3)$ and for the deformed state let $\mathbf{x} = (x_1, x_2, x_3)$ be the radius vectors of an arbitrary point of the body. In the equilibrium state a particle occupies the volume element $dV_X = dX_1 dX_2 dX_3$. In the deformed state that particle occupies the volume element

$$dV_x = dx_1 dx_2 dx_3.$$

A deformation is given by the regular transformation $\mathbf{X} \mapsto \mathbf{x}$, which has a unique inverse transformation. The relation between these two volume elements is

$$dV_x = \det(J_{\mathbf{X}}(\mathbf{x}))\, \mathbf{dV_a},$$

where $\det(J_{\mathbf{X}}(\mathbf{x}))$ is the determinant of $J_{\mathbf{X}}(\mathbf{x})$, the Jacobian of the transformation $\mathbf{X} \mapsto \mathbf{x}$. $J_{\mathbf{X}}(\cdot)$ is also called the *mapping function* and is given by the matrix

$$J_{\mathbf{X}}(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_3} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{pmatrix}.$$

The principle of conservation mass is given by

$$\rho_x \, dV_x \,=\, \rho_X \, dV_X.$$

The *compression ratio* R is defined by

$$R = \frac{\rho_x}{\rho_X} = \Big[\det(J_{\mathbf{X}}(\mathbf{x}))\Big]^{-1} = \det(J_{\mathbf{x}}(\mathbf{X})).$$

We develop the strain tensor in the form of a matrix by considering a curve C_X in the body in the deformed state. Under the mapping $\mathbf{X} \mapsto \mathbf{x}$, this curve maps into the curve $C_{\mathbf{x}}$ in the deformed state. The two curves are composed by the same particles. The column matrix $d\mathbf{X}$ has the components dX_1, dX_2, dX_3, while the row matrix is the transpose $d\mathbf{X}^*$. We have $d\mathbf{X} = (dX_1, dX_2, dX_3)^*$, and similarly for $d\mathbf{x}$. Let ds_X be an element of arc lenght of C_X and let ds_x be an element of arc length of C_x under the transformation $d\mathbf{X} \mapsto d\mathbf{x}$ which is given by

$$d\mathbf{x} \,=\, J_{\mathbf{X}}(\mathbf{x}) \, d\mathbf{X}. \tag{6}$$

The magnitude of ds_X is the square root of the expression

$$(ds_X)^2 \,=\, (d\mathbf{X})^*(d\mathbf{X}).$$

Similarly, we have

$$(ds_x)^2 \,=\, (d\mathbf{x})^*(d\mathbf{x}).$$

The transpose of equation (6) is $(d\mathbf{x})^* = (d\mathbf{X})^*(J_{\mathbf{X}}(\mathbf{x}))^*$. Using this expression and equation (6), we get

$$(ds_x)^2 \,=\, (d\mathbf{x})^*(d\mathbf{x}) \,=\, (d\mathbf{X})^*(J_{\mathbf{X}}(\mathbf{x}))^* J_{\mathbf{X}}(\mathbf{x})(d\mathbf{X}).$$

Suppose the transformation $\mathbf{X} \mapsto \mathbf{x}$ has the property that for every curve C_X all arc lengths are unchanged in being transformed to the corresponding curve C_x. It follows that $(d\mathbf{x})^*(d\mathbf{x}) = (d\mathbf{X})^*(d\mathbf{X})$, so that $(J_{\mathbf{X}}(\mathbf{x}))^* J_{\mathbf{X}}((\mathbf{X})) = \mathbf{E}$, the 3×3 identity matrix. This means that $J_{\mathbf{X}}(\mathbf{x})$ is the *rotation matrix* yielding a rigid–body rotation (no deformation), where $\det(J_{\mathbf{X}}(\mathbf{x})) > 0$. Since the measure of strain can be considered as a deviation from a pure rotation, we may take the expression $(J_{\mathbf{X}}(\mathbf{x}))^* J_{\mathbf{X}}((\mathbf{X})) \,-\, \mathbf{E}$ as twice the three–dimensional strain tensor. Thus, we have

$$\varepsilon \,=\, \frac{1}{2}((J_{\mathbf{X}}(\mathbf{x}))^* J_{\mathbf{X}}((\mathbf{X})) \,-\, \mathbf{E}), \tag{7}$$

where ε is the strain tensor. The reason for the factor of 1/2 in equation (7) is seen when we derive the linear approximation of ε. For this approximation we have $|(ds_x - ds_X)/ds_X| \ll 1$, so that

$$\frac{(ds_x)^2 - (ds_X)^2}{(ds_X)^2} = \frac{(ds_x - ds_X)(ds_x + ds_X)}{(ds_X)^2} \approx$$
$$\frac{2\, ds_X(ds_x - ds_X)}{(ds_X)^2} = 2\,\frac{ds_x - ds_X}{ds_X}.$$

Let ε_L be the linear strain tensor. We have

$$\frac{ds_x - ds_X}{ds_X} = \Big(\frac{d\mathbf{X}}{ds_X}\Big)^* \varepsilon_L \Big(\frac{d\mathbf{X}}{ds_X}\Big),$$

where equation (7) is used for the linear strain strain tensor.

3.3 Strain as a Function of Displacement

We now obtain the strain tensor matrix ε as a function of the displacement vector $\mathbf{u}$, where $\mathbf{u} = (u_1, u_2, u_3)$. We have $\mathbf{u} = \mathbf{x} - \mathbf{X}$, the components being $u_i = x_i - X_i$. Instead of calculating the right–hand side of (7) and setting $\mathbf{x} = \mathbf{u} + \mathbf{X}$ to obtain ε, we introduce $\mathbf{K} = J_{\mathbf{X}}(\mathbf{u})$, the Jacobian of $\mathbf{u}$, so that

$$K = \begin{pmatrix} \dfrac{\partial u_1}{\partial X_1} & \dfrac{\partial u_1}{\partial X_2} & \dfrac{\partial u_1}{\partial X_3} \\ \dfrac{\partial u_2}{\partial X_1} & \dfrac{\partial u_2}{\partial X_2} & \dfrac{\partial u_2}{\partial X_3} \\ \dfrac{\partial u_3}{\partial X_1} & \dfrac{\partial u_3}{\partial X_2} & \dfrac{\partial u_3}{\partial X_3} \end{pmatrix}.$$

It follows that $\mathbf{J} = \mathbf{K} + \mathbf{E}$, so that the strain tensor in matrix form becomes

$$\varepsilon = \frac{1}{2}(K + K^*) + K\,K^* = \varepsilon_L + \varepsilon_N. \tag{8}$$

The linear part of relation (8) is given by the matrix

$$\varepsilon_L = \begin{pmatrix} \dfrac{\partial u_1}{\partial X_1} & \dfrac{1}{2}\Big(\dfrac{\partial u_1}{\partial X_2} + \dfrac{\partial u_2}{\partial X_1}\Big) & \dfrac{1}{2}\Big(\dfrac{\partial u_1}{\partial X_3} + \dfrac{\partial u_3}{\partial X_1}\Big) \\ \dfrac{1}{2}\Big(\dfrac{\partial u_2}{\partial X_1} + \dfrac{\partial u_1}{\partial X_2}\Big) & \dfrac{\partial u_2}{\partial X_2} & \dfrac{1}{2}\Big(\dfrac{\partial u_2}{\partial X_3} + \dfrac{\partial u_3}{\partial X_2}\Big) \\ \dfrac{1}{2}\Big(\dfrac{\partial u_3}{\partial X_1} + \dfrac{\partial u_1}{\partial X_3}\Big) & \dfrac{1}{2}\Big(\dfrac{\partial u_3}{\partial X_2} + \dfrac{\partial u_2}{\partial X_3}\Big) & \dfrac{\partial u_3}{\partial X_3} \end{pmatrix}. \tag{9}$$

The diagonal elements of ε represent the pure or normal components of the strain, while the off–diagonal elements are the components of the shear strain. The strain tensor ε is represented by a symmetric matrix. This is true for both the linear and nonlinear parts, that is for both ε_L and ε_N.

3.4 Linear Momentum and the Stress Tensor

A basic postulate in the theory of continuous media is that the mechanical action of the material points which are situated on one side of an arbitrary material surface within a body upon those on the other side can be completely accounted for the prescribing a suitable surface traction on this surface. Thus if a surface element has a unit outward normal $\mathbf{n}$ we introduce the surface traction $\mathbf{t}$, defining a force per unit area. The surface tractions generally depend on the orientation of $\mathbf{n}$ as well as on the location $\mathbf{x}$ of the surface element.

Suppose we remove from a body a closed region $V + S$, where S is the boundary. The surface S is subjected to a distribution of surface tractions $\mathbf{t}(\mathbf{x}, t)$. Each mass element of the body may be subjected to a body force per unit mass, $\mathbf{F}(\mathbf{x}, t)$. According to the principle of balance of linear momentum, the instantaneous rate of change of the linear momentum of a body is equal to the resultant external force acting on the body at the particular instant of time. In the linearized theory this leads to the equation

$$\int_S \mathbf{t} dA + \int_V \rho \mathbf{F} dV = \int_V \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{10}$$

By means of the *tetrahedron argument*, equation (10) subsequently leads to the stress tensor $\boldsymbol{\tau}$ with components τ_{kl}, where

$$t_l = \tau_{kl} n_k \tag{11}$$

Equation (11) is the Cauchy stress formula. Physically τ_{kl} is the component in the x_i-direction of the traction on the surface with the unit normal $\mathbf{i}_k$.

By substitution of $t_l = \tau_{kl} n_k$, equation (10) is rewritten in an indicial notation as

$$\int_S \tau_{kl} n_k dA + \int_V \rho F_l dV = \int_V \rho \frac{\partial^2 u_l}{\partial t^2}.$$

The surface integral can be transformed into a volume integral by Gauss' theorem, and we obtain

$$\int_V \Big(\frac{\partial \tau_{kl}}{\partial x_k} + \rho F_l - \rho\frac{\partial^2 u_l}{\partial t^2}\Big) dV = 0.$$

Since V may be an arbitrary part of the body it follows that wherever the integral is continuous, we have

$$\frac{\partial \tau_{kl}}{\partial x_k} + \rho F_l = \rho\frac{\partial^2 u_l}{\partial t^2}. \tag{12}$$

This is Cauchy's first law of motion and it represents the conservation of linear momentum. These equations of motion are therefore Newton's equations of motion for a continuum. They are the linearized equations of motion, since the nonlinear terms for the particle acceleration, which represent the convective terms, are neglected. Since they are linear, the Lagrangian and Eulerian representations are the same; the difference between these two representations appears only in the nonlinear terms. The equations of motion are valid for any continuous medium (solid, liquid, gas) since they do not invoke the energy equation that defines that material. Clearly, the equations of motion give an incomplete description of the physical situation, since we have three equations and six components of the stress tensor $\boldsymbol{\tau}$ and three components of the displacement vector $\mathbf{u}$. The additional requisite equations are given from the conservation of energy, which supplies the *constitutivwe equations*; these we take as Hooke's law for an isotropic elastic medium.

3.5 Balance of Moment of Momentum

For the linearized theory the principle of moment of momentum states

$$\int_S (\mathbf{x}\times\mathbf{t})dA + \int_V (\mathbf{x}\times\mathbf{F})dV = \int_V \rho\,\frac{\partial}{\partial t}\Big(\mathbf{x}\times\frac{\partial \mathbf{u}}{\partial t}\Big)dV.$$

Simplifying the right–hand side and introducing indicial notation, this equation can be written as

$$\int_S \varepsilon_{klm}x_l t_m dA + \int_V \varepsilon_{klm}x_l F_m dV = \int_V \rho\,\varepsilon_{klm}x_l\frac{\partial^2 u_l}{\partial t^2}. \tag{13}$$

Elimination of t_m from the surface integral and the use of Gauss' theorem result in

$$\int_S \varepsilon_{klm} x_l \tau_{km} n_k dA = \int_V \varepsilon_{klm}(\delta_{lk}\tau_{km} + x_l \tau_{km,k}) dV.$$

By virtue of the first law of motion, equation (13) reduces to

$$\int_V \varepsilon_{klm}\delta_{lk}\tau_{km} dV$$

or

$$\varepsilon_{klm}\tau_{lm} = 0.$$

This result implies that

$$\tau_{lm} = \tau_{ml},$$

i. e., the stress tensor is symmetric.

4. The Navier Equations of Motion for the Displacement

In this section we derive the linear equations of motion for an elastic medium in terms of the components of the displacement vector by using Hooke's law. These equations are called the Navier equations. In many problems in elasticity it is more convenient to obtain the equations of motion for the displacement vector and then derive the stress field from the definition of strain and Hooke's law.

4.1 Stress–Strain Relations

In general terms, the linear relation between the components of the stress tensor and the components of the strain tensor is

$$\tau_{ij} = C_{ijkl}\varepsilon_{kl},$$

where

$$C_{ijkl} = C_{jikl} = C_{klij} = C_{ijlk}.$$

Thus, 21 of the 81 components of the tensor C_{ijkl} are independent. The medium is *elastically homogeneous* if the coefficient C_{ijkl} are constants. The material is *elastically isotropic* when there are no preferred directions in the material, and the elastic constants must be the same whatever the orientation of the Cartesian coordinate system in which the components of τ_{ij} and ε_{ij} are evaluated. It can be shown that the constants C_{ijkl} may be expressed as

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

Hooke's law then assumes the well–known form

$$\tau_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}. \tag{14}$$

Equation (14) contains two elastic constants λ and μ, which are known as *Lamé's elastic constants.*

The trace (sum of the diagonal elements) of the strain matrix Θ is called *dilatation.* It is a measure of the relative change in volume of the body due to a compression or dilatation. It is given by

$$\Theta = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}. \tag{15}$$

Hooke's law can be also expressed by the set of equations

$$\tau_{ij} = \lambda\delta_{ij}\Theta + 2\mu\varepsilon_{ij}. \tag{16}$$

Let Φ be the trace of stress matrix so that

$$\Phi = \tau_{11} + \tau_{22} + \tau_{33}. \tag{17}$$

We have

$$\Phi = (3\lambda + 2\mu)\Theta. \tag{18}$$

The inverse of equation (16) is

$$\varepsilon_{ij} = \frac{\lambda}{2\mu(3\lambda + 2\mu)}\Phi\delta_{ij} + \frac{1}{2\mu}\tau_{ij}. \tag{19}$$

The Lamé constants λ, μ can be expressed in terms of the two elastic constants *Young's modulus* E and *Poisson's ratio* σ. It can be shown that

$$\begin{aligned} \lambda &= \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \qquad \mu = \frac{E}{2(1+\sigma)} \\ \sigma &= \frac{\lambda}{2(\lambda+\mu)}, \qquad E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}. \end{aligned} \tag{20}$$

4.2 Equations of Motion for the Displacements

We obtain the linear strain in tensor form from equation (9) by using the x coordinate system instead of the Lagrangian variables $\mathbf{X}$ and the dispalacement vector $\mathbf{u}$. The ijth component of the linear strain tensor is $\varepsilon_{ij} = u_{i,j}$. Since the strain tensor is symmetric, we have

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{21}$$

$$\tau_{kl,k} = \mu\nabla^2 u_l + (\lambda + \mu)(\boldsymbol{\nabla} \cdot \mathbf{u})_{,l}. \tag{22}$$

In this way, the lth equation of motion (12) becomes

$$\mu\nabla^2 u_i + (\lambda + \mu)(\boldsymbol{\nabla} \cdot \mathbf{u})_{,i} = \rho(-F_i + u_{i,tt}). \tag{23}$$

The vector form of the Navier equations of motion for the displacement can be obtained from (23) by multiplying with the unit vector $\mathbf{e}_i$ of the x_i axis followed by addition over i. Thus, we have

$$\mu\nabla^2\mathbf{u} + (\lambda + \mu)\boldsymbol{\nabla}\Theta = \rho(-\mathbf{F} + \mathbf{u}_{tt}). \tag{24}$$

Using the well known identity

$$\nabla^2\mathbf{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \operatorname{curl}\boldsymbol{\Psi} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \boldsymbol{\nabla} \times \boldsymbol{\Psi},$$

where $\boldsymbol{\Psi} = \operatorname{curl}\mathbf{u} = \boldsymbol{\nabla} \times \mathbf{u}$ is the curl of displacement vector $\mathbf{u}$, equation (24) becomes

$$-\mu\boldsymbol{\nabla} \times \boldsymbol{\Psi} + (\lambda + 2\mu)\boldsymbol{\nabla}\Theta = \rho(-\mathbf{F} + \mathbf{u}_{,tt}). \tag{25}$$

$\boldsymbol{\Psi}$ is called the rotation vector. Note that the vector form of the Navier equations are independent of the coordinate system. From the form given by (25) we now derive two types of waves: (1) longitudinal waves involving the wave equation for the dilatation, and (2) transverse waves, involving the rotation vector.

(1) We take the divergence of each term of equation (25) and use the fact that $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{\Psi} = 0$. We obtain

$$\frac{\lambda + 2\mu}{\rho}\nabla^2\Theta = \Theta_{,tt} - \boldsymbol{\nabla} \cdot \mathbf{F}. \tag{26}$$

Equation (26) is the vector form of the three–dimensional nonhomogeneous wave equation for Θ, the nonhomogeneous term being the one involving $\mathbf{F}$. We set

$$\frac{\lambda + 2\mu}{\rho} = c_L^2.$$

It will be seen below that c_L is the longitudinal wave velocity.

(2) Next we get the equation of motion for $\mathbf{\Psi}$ by taking the curl of each term in equation (25). We use the fact that $\mathbf{\nabla} \times \Theta = \mathbf{0}$ and obtain

$$\frac{\mu}{\rho} \nabla^2 \mathbf{\Psi} = \mathbf{\Psi}_{,tt} - \mathbf{\nabla} \times \mathbf{F}.$$

We set

$$\frac{\mu}{\rho} = c_T^2,$$

where c_T is the velocity of the rotational vector, which, we shall shall show, is the velocity of a transverse wave.

5. Propagation of Plane Elastic Waves

In this section we study the plane waves in an infinite isotropic elastic medium. A *plane wave* is defined as one whose wave front is a planar surface normal to the direction of the propagating wave. If the wave front is normal to x_1 axis, then the displacement vector is a function of coordinate x_1 and time t, and all derivatives with respect to x_2 and x_3 in equation (24) are zero. For simplicity we set $\mathbf{F} = \mathbf{0}$. Since the dilatation becomes $\Theta = u_{1,1}$, we see that the three scalar equations in (24) become

$$\begin{cases} c_L^2 u_{1,11} & = & u_{1,tt} \\ c_T^2 u_{2,11} & = & u_{2,tt} \\ c_T^2 u_{3,11} & = & u_{3,tt}. \end{cases} \tag{27}$$

The first equation of (27) is the one–dimensional wave equation for the displacement component u_1 in the direction of wave propagation x_1. This gives a longitudinal wave, so that c_L is indeed the longitudinal wave velocity. The second equation of (27) is the wave equation for u_2. The direction of wave propagation is still x_1, so that $u_2 = u_2(x_1, t)$ gives particle vibrations in the plane of the wave front (the x_2 direction), which is normal to x_1. This means that we hve a transverse wave, and c_T is indeed the transverse wave velocity. Similarly for the third equation, where $u_3(x_1, t)$ shows transverse wave propagation with the same velocity c_T. From their definitions it is seen that the velocity of longitudinal waves is always greater than that of transverse waves.

6. General decomposition of Elastic Waves

By using the definitions of c_L and c_T we can write the vector Navier equation (24) in the form

$$c_T^2\nabla^2\mathbf{u} + (c_L^2 - c_T^2)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) = \mathbf{u}_{,tt}, \tag{28}$$

where, again, $\mathbf{F} = \mathbf{0}$. We now split equation (28) into two vector equations by decomposing the dispalacement vector as follows:

$$\mathbf{u} = \mathbf{u}_L + \mathbf{u}_T. \tag{29}$$

We shal show that $\mathbf{u}_L$ represents a longitudinal wave and $\mathbf{u}_T$ a transverse wave. A longitudinal wave is rotationless, which means that

$$\boldsymbol{\nabla}\times\mathbf{u}_L = \mathbf{0}.$$

It follows from vector analysis that a scalar function of space and time exists such that

$$\mathbf{u}_L = \operatorname{grad}\phi = \boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3, \tag{30}$$

where ϕ is called the scalar potential. On the other hand, $\mathbf{u}_T$ satisfies the equation

$$\operatorname{div}\mathbf{u}_T = 0. \tag{31}$$

This clearly means that a transverse wave suffers no change in volume (an *equivoluminal wave*) but is rotational, so that

$$\mathbf{u} = \boldsymbol{\nabla}\times\boldsymbol{\psi}, \tag{32}$$

where $\boldsymbol{\psi}$ is the vector potential.

Using equations (30) and (32), equation (29) becomes

$$\mathbf{u} = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\times\boldsymbol{\psi}. \tag{33}$$

Equation (32) tells us that the displacement vector can be decomposed into an irotational vector and an equivoluminal vector.

We now insert equation (29) into (28), take the divergence of each term, and use equation (31). We obtain

$$\nabla \cdot (c_L^2 \nabla^2 \mathbf{u}_L - \mathbf{u}_{L,tt}) = 0.$$

Since $\boldsymbol{\nabla} \times \mathbf{u}_L = \mathbf{0}$ and $\nabla \cdot (\quad) = 0$, it follows that the terms in parantheses are also zero, yielding

$$c_L^2 \nabla^2 \mathbf{u}_L - \mathbf{u}_{L,tt} = \mathbf{0}. \tag{34}$$

Equation (34) is the vector wave equation for the displacement representing longitudinal wave (irrotational waves), since the wave velocity is c_L. Since $\mathbf{u}_L = \boldsymbol{\nabla}\phi$, it is clear that the scalar potential ϕ also satisfies the wave equation with the same velocity.

Similarly, inserting equation (29) into (28), taking the curl of each term, and using the fact that $\boldsymbol{\nabla} \times \mathbf{u_L} = \mathbf{0}$, we obtain

$$\boldsymbol{\nabla} \times (c_T^2 \nabla^2 \mathbf{u}_T - \mathbf{u}_{T,tt}) = 0.$$

Since $\nabla \cdot (\quad) = 0$, it follows that

$$c_T^2 \nabla^2 \mathbf{u}_T - \mathbf{u}_{T,tt} = \mathbf{0}. \tag{35}$$

Equation (35) is the vector wave equation for $\mathbf{u}_T$, whose solutions yield transverse, equivoluminal, rotational waves. It follows that the vector potential $\boldsymbol{\psi}$ also staisfies this wave equation.

Sometimes it is easier to solve the wave equations in terms of the scalar and vector potentials. Then the displacement can be obtained from equation (33).

7. Characteristic Surfaces for Planar Waves

The wave front of a planar wave is a plane normal to the direction of wave propagation. Let the normal to the wave front be $\boldsymbol{\nu} = (l, m, n)$, where l, m, n are the direction cosines of $\boldsymbol{\nu}$. Let the radius vector $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ be directed from the origin to aby point on the wave front. The scalar product $\boldsymbol{\nu} \cdot \mathbf{x}$ is the projection of $\mathbf{x}$ on the wave front.

All solutions of the wave equations (34) and (35) are of the form

$$\begin{aligned} \mathbf{u}_L &= \mathbf{f}(\boldsymbol{\nu} \cdot \mathbf{x} + ct) + \mathbf{g}(\boldsymbol{\nu} \cdot \mathbf{x} - ct), \\ \mathbf{u}_T &= \mathbf{F}(\boldsymbol{\nu} \cdot \mathbf{x} + ct) + \mathbf{G}(\boldsymbol{\nu} \cdot \mathbf{x} - ct), \end{aligned} \tag{36}$$

where $\mathbf{f}$, $\mathbf{g}$, $\mathbf{F}$ and $\mathbf{G}$ are arbitrary functions of the indicated arguments which are called the phases of the waves, $(\boldsymbol{\nu}\cdot\mathbf{x}+ct)$, and $(\boldsymbol{\nu}\cdot\mathbf{x}-ct)$ are called the phases of the waves. The functions $\mathbf{f}(\boldsymbol{\nu}\cdot\mathbf{x}+ct)$, $\mathbf{F}(\boldsymbol{\nu}\cdot\mathbf{x}+ct)$ represent progressing waves, while $\mathbf{g}(\boldsymbol{\nu}\cdot\mathbf{x}-ct)$ and $\mathbf{G}(\boldsymbol{\nu}\cdot\mathbf{x}-ct)$ are the regressing waves. It is also true that there are no solutions that are not of the form given by equations (36). If we set $\boldsymbol{\nu}\cdot\mathbf{x}-ct=$ const., we obtain a characteristic surface; this is a planar wave front that progress into the medium in the direction normal to the wave front with a wave velocity equal to c. Setting $\boldsymbol{\nu}\cdot\mathbf{x}+ct=$ const. yields a characteristic surface that regresses in the direction oposite to that of the normal to the wave front. Inserting $\mathbf{u}=\mathbf{u}_L+\mathbf{u}_T$, where $\mathbf{u}_L$ and $\mathbf{u}_T$ are given in (36), into equation (24), and setting then $x=y=z=0$, yields the following quadratic equation for c^2 :

$$(\rho\, c^2-\mu)(\rho\, c^2-\lambda-2\mu)=0. \tag{37}$$

The roots of equation (37) are $c^2=c_L^2,\ c_T^2$. This substantiates the fact that, in any direction $\boldsymbol{\nu}$, there are two planar waves, one longitudinal and the other transverse.

8. Time–Harmonic Solutions of the Reduced Wave Equations

We now investigate *time–harmonic* solutions to the wave equations for longitudinal and transverse waves. These are waves whose time–dependent parts are of the form $e^{\pm i\omega t}$.

We write the displacement vector $\mathbf{u}$ in the form

$$\mathbf{u}=\mathrm{Re}\,[\mathbf{U}(x_1,x_2,x_3)\,e^{\pm i\omega t}], \tag{38}$$

where Re[ ] is the real part of the bracket. In equation (38), $\mathbf{u}$ stands for either a longitudinal or a transverse wave. Substituting equation (38) into equations (34) and (35) and factoring out the exponentials yields

$$\begin{aligned}\nabla^2\mathbf{U}_L+k_L^2\mathbf{U}_L &= \mathbf{0},\\ \nabla^2\mathbf{U}_T+k_T^2\mathbf{U}_T &= \mathbf{0},\end{aligned} \tag{39}$$

where k_L and k_T are the wave numbers for the longitudinal and transverse waves, respectively, and are given by

$$k_L=\frac{\omega}{c_L},\quad k_T=\frac{\omega}{c_T}.$$

Equations (39) are called the *reduced wave equations* for $\mathbf{U}_L$ and $\mathbf{U}_T$, respectively. The reduced wave equations are of eliptic type. Let $\mathbf{U}$ stands for $\mathbf{U}_L$ and $\mathbf{U}_T$. Solutions of the reduced wave equation $\nabla^2\mathbf{U} + k^2\mathbf{U} = \mathbf{0}$ are of the form

$$\mathbf{U} = \bar{\mathbf{U}}\, e^{ik(\boldsymbol{\nu}\cdot\mathbf{x})},$$

where $\bar{\mathbf{U}}$ is a constant vector. Combining this solution of the reduced wave equation with the time–dependent solution given by equation (38) yields the time–harmonic solutions

$$\mathbf{u} = \mathrm{Re}\,[\bar{\mathbf{U}}\, e^{ik(\boldsymbol{\nu}\cdot\mathbf{x}\pm ct)}], \tag{40}$$

where $c = \omega/k$.

The minus sign in the phase gives a progressing travelling wave and the plus sign a regressing wave, ω is the same for both a longitudinal and transverse waveform, while c and k depend on the waveform.

The time–harmonic solutions expressed by equations (40) are special case of the general solution given by (36). Note that we have the same phase of the progressing and regressing waves, given by $(\boldsymbol{\nu}\cdot\mathbf{x} \pm ct)$, so that planes of constant phase yield the traveling wave fronts.

9. Spherically Symmetric Waves

Spherically symmetric waves are produced in three–dimensional space from a point source. For simplicity we consider the wave equation for the generic scalar $f(r,t)$, which may stand for the components of $\mathbf{u}$, the scalar potential ϕ, or the components of the vector potential $\boldsymbol{\psi}$. Setting $r^2 = x_1^2 + x_2^2 + x_3^2$ and neglecting angular dependence, we get the wave equation for f in spherical coordinates:

$$c^2\left[f_{,rr} + \left(\frac{2}{r}\right) f_{,r}\right] = f_{,tt}. \tag{41}$$

We get time–harmonic solutions of equation (41) by setting

$$f(r,t) = g(r)e^{-i\omega t}. \tag{42}$$

Inserting equation (42) into (41) yields

$$\frac{d^2g}{dr^2} + \left(\frac{2}{r}\right)\frac{dg}{dr} + k^2 g = 0, \qquad k = \frac{\omega}{c}.$$

This ordinary differential equations for g may be rewritten as

$$\frac{d^2(rg)}{dr} + k^2(rg) = 0. \tag{43}$$

Note that equation (43) is a second order ordinary differential equation for rg where the first–derivative term is absent. Setting $w = rg$ gives $w'' + k^2 w = 0$, so that the general solution of equation (43) is

$$g(r) = \frac{1}{r})e^{\pm ikr}.$$

Multiplying this solution by the time–dependent part given by equation (42) yelds time–harmonic solutions of the spherical wave equation as a linear combination of terms of the form

$$\frac{1}{r}\,e^{i(kr-ct)}, \quad \frac{1}{r}\,e^{i(kr+ct)} \tag{44}$$

The first expression of equation (44) represents an outgoing attenuated spherical wave (emanating from the point source), while the second expression represents an incoming wave (from infiniti where the amplitude is zero) going toward the source. Note that there is a singularity at the source. The phase of the wave is $(kr \mp ct)$. Setting the phase $(k + ct)$ equal to a constant and varying time generates an outgoing spherical wave front, and setting $(k - ct)$ equal to a constant generates an incoming wave; these are the characteristic surfasces.

We may obtain a more general solution to the spherical wave equation (41) by writing it as

$$c^2(rf)_{,rr} = (rf)_{,tt}.$$

Therefore the general solution for the spherical wave equation is

$$f(r,t) = \frac{1}{r}F(r - ct) + \frac{1}{r}G(r + ct), \tag{45}$$

where F and G are arbitrary functions of their arguments to be specifically determined by the boundary and initial conditions. The argument of F is the phase $(r-ct)$ and thus represent an outgoing wave, while the argument of G is $(r+ct)$ and represents an incoming wave. All solutions of the spherical wave equation are of the form given by equation (45), and there are no solutions that are not of those forms.

10. Curvilinear Orthogonal Coordinates

In this section we digress by investigating the transformations from Cartesian to curvilinear orthogonal coordinates and then specialize to the transformation to cylindrical coordinates.

Let (q_1, q_2, q_3) be coordinates of a point in any system. The Cartesian coordinates x_1, x_2, x_3 will be functions of these coordinates, so that the set of functions

$$x_1 = x_1(q_1, q_2, q_3), \quad x_2 = x_2(q_1, q_2, q_3), \quad x_3 = x_3(q_1, q_2, q_3), \tag{46}$$

is a regular transformation, which can be written as $\mathbf{x} = \mathbf{x}(\mathbf{q})$. Consequently, there is the inverse transformation

$$q_1 = q_1(x_1, x_2, x_3), \quad q_2 = q_2(x_1, x_2, x_3), \quad q_3 = q_3(x_1, x_2, x_3).$$

The three equations $q_i = c_i$, where c_i are constants, represent three families of surfaces whose lines of intersection form three families of curved lines. These lines of intersection will be used as the coordinate lines in our curvilinear coordinate system. Thus, the position of a point in space can be defined by the values of three coordinates q_1, q_2 and q_3. The local coordinate directions at a point are tangent to the three coordinate lines intersecting at the point.

The Jacobian matrix of transformation (46) is a 3×3 matrix whose kth column has the elements $x_{k,i}$. This matrix will be denoted by $J_{\mathbf{q}}(\mathbf{x})$.

The differentials dx_i are now expanded in terms of (dq_1, dq_2, dq_3), so that

$$dx_i = \frac{\partial x_i}{\partial q_1} dq_1 + \frac{\partial x_i}{\partial q_2} dq_2 + \frac{\partial x_i}{\partial q_3} dq_3, \quad i = 1, 2, 3. \tag{47}$$

In the matrix form, equations (47) can be written as

$$d\mathbf{x}^* = d\mathbf{q}^* \Big(J_{\mathbf{q}}(\mathbf{x}) \Big)^*,$$

where $d\mathbf{x}^*$ is the transpose matrix of the one column matrix $d\mathbf{x}$ whose components are dx_1, dx_2, dx_3. Similarly, for $d\mathbf{q}^*$. Let ds be an element of length. It follows that

$$ds^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij} dq_i dq_j, \tag{48}$$

where

$$g_{ij} = g_{ji} = \sum_{k=1}^{3} \frac{\partial x_k}{\partial q_i} \cdot \frac{\partial x_k}{\partial q_j} \tag{49}$$

are called the *metric coefficients*.

The quadratic form (48) can be written as

$$ds^2 = d\mathbf{q}^* \Big(J_{\mathbf{q}}(\mathbf{x})\Big)^* J_{\mathbf{q}}(\mathbf{x}) d\mathbf{q}.$$

If we consider the element of length ds_i, where $i = 1, 2, 3$, that corresponds to a change from q_i to $q_i + dq_i$, we have

$$ds_1 = \sqrt{g_{11}}\, dq_1, \quad ds_2 = \sqrt{g_{22}}\, dq_2, \quad ds_3 = \sqrt{g_{33}}\, dq_3.$$

To shorten the notation we introduce the *scale factors*

$$h_1 = \sqrt{g_{11}}, \quad h_2 = \sqrt{g_{22}}, \quad h_3 = \sqrt{g_{33}},$$

which are in general functions of the coordinates q_j.

For an orthogonal system of coordinates the surfaces $q_1 =$ const, $q_2 =$ const, $q_3 =$ const intersect each other at right angles, and all g_{ij}, with $i \neq j$, are equal to zero. We choose an orthonormal right–handed basis whose unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are respectively directed in the sense of increase of the coordinates q_1, q_2 and q_3. The following well-known relations hold

$$\begin{aligned} \mathbf{e}_i \cdot \mathbf{e}_j &= \delta_{ij}, \\ \mathbf{e}_i \times \mathbf{e}_j &= \mathbf{e}_k, \end{aligned} \tag{50}$$

where in (50) the indices i, j and k are in cyclic order. A major difference between curvilinear coordinates and Cartesian coordinates is that the coordinates q_1, q_2 and q_3 are not necessarily measured in lengths. This difference manifests itself in the appearence of scale factors in the relation between the infinitesimal displacement vector $d\mathbf{x}$ and the infinitesimal variations dq_1, dq_2 and dq_3, namely,

$$d\mathbf{x} = \mathbf{e}_1 h_1 dq_1 + \mathbf{e}_2 h_2 dq_2 + \mathbf{e}_3 h_3 dq_3. \tag{51}$$

The scale factors h_i are in general functions of the coordinates q_j.

All equations in Cartesian coordinates which do not involve space derivatives and which pertain to properties at a point carry over unchanged into curvilinear coordinates. If space derivatives are involved, however, equations do not directly carry over, since the differential operators such as gradient, divergence, curl, and the Laplacian assume different forms.

We consider first the gradient operator $\boldsymbol{\nabla}$. When applied to a scalar φ it gives a vector $\boldsymbol{\nabla}\varphi$, with components which we call f_1, f_2 and f_3. Thus

$$\boldsymbol{\nabla}\varphi = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3. \tag{52}$$

The increment of φ due to a change of position $d\mathbf{x}$ is

$$d\varphi = \boldsymbol{\nabla}\varphi \cdot d\mathbf{x} = h_1 f_1 dq_1 + h_2 f_2 dq_2 + h_3 f_3 dq_3, \tag{53}$$

where (51) has been used. The increment $d\varphi$ can also be written as

$$d\varphi = \frac{\partial\varphi}{\partial q_1}dq_1 + \frac{\partial\varphi}{\partial q_2}dq_2 + \frac{\partial\varphi}{\partial q_3}dq_3, \tag{54}$$

whence it can be concluded that

$$\boldsymbol{\nabla} = \frac{\mathbf{e}_1}{h_1}\frac{\partial}{\partial q_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial}{\partial q_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial}{\partial q_3}. \tag{55}$$

If the operation (55) is applied to the scalar function φ, then from (53) and (54) it results that f_i in (52) is equal to $\dfrac{1}{h_i}\dfrac{\partial\varphi}{\partial q_i}$, so that the gradient of the scalar function φ gives

$$\operatorname{grad}\varphi = \boldsymbol{\nabla}\varphi = \frac{1}{h_1}\frac{\partial\varphi}{\partial q_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial\varphi}{\partial q_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial\varphi}{\partial q_3}\mathbf{e}_3. \tag{56}$$

Let us consider an other vector function $\mathbf{F}$ which in the $\mathbf{q}$ orthogonal system is expressed by

$$\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3. \tag{57}$$

where (F_1, F_2, F_3) are the components of $\mathbf{F}$ along the axes of the $\mathbf{q}$ system depending of q_1, q_2 and q_3.

The *divergence* of $\mathbf{F}$ in this orthogonal system of coordinates is

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3F_1)}{\partial q_1} + \frac{\partial(h_3h_1F_2)}{\partial q_2} + \frac{\partial(h_1h_2F_3)}{\partial q_3}\right], \tag{58}$$

while the *curl* of the same vector is given by

$$\begin{aligned} \boldsymbol{\nabla}\times\mathbf{F} &= \frac{1}{h_2h_3}\left[\frac{\partial(h_3F_3)}{\partial q_2}-\frac{\partial(h_2F_2)}{\partial q_3}\right]\mathbf{e}_1+ \\ &+ \frac{1}{h_3h_1}\left[\frac{\partial(h_1F_1)}{\partial q_3}-\frac{\partial(h_3F_3)}{\partial q_1}\right]\mathbf{e}_2+ \\ &+ \frac{1}{h_1h_2}\left[\frac{\partial(h_2F_2)}{\partial q_1}-\frac{\partial(h_1F_1)}{\partial q_2}\right]\mathbf{e}_3. \end{aligned} \tag{59}$$

Note that the expression (59) of the vector $\boldsymbol{\nabla}\times\mathbf{F}$ can be write in the form of a determinant, namely,

$$\boldsymbol{\nabla}\times\mathbf{F}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1u_1 & h_2u_2 & h_3u_3 \end{vmatrix}. \tag{60}$$

The *Laplacian operator* of a scalar function can easily be derived by using (58) and (55). Thus

$$\nabla^2\varphi=\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\varphi=\frac{1}{h_1h_2h_3}\sum_{n=1}^{3}\frac{\partial}{\partial q_n}\left[\frac{h_1h_2h_3}{h_n^2}\frac{\partial\varphi}{\partial q_n}\right]. \tag{61}$$

In cylindrical coordinate we choose

$$q_1=r,\quad q_2=\theta,\quad q_3=z.$$

The corresponding scale factors and unit base vectors are

$$\begin{gathered} h_1=1,\quad h_2=r,\quad h_3=1 \\ \mathbf{e}_1=\mathbf{e}_r,\quad \mathbf{e}_2=\mathbf{e}_\theta,\quad \mathbf{e}_3=\mathbf{e}_z. \end{gathered} \tag{62}$$

The relations between the cylindrical coordinates (r,θ,z) and the Cartesian coordinates (x_1,x_2,x_3) are

$$\left\{\begin{aligned} x_1 &= r\cos\theta, \\ x_2 &= r\sin\theta, \\ x_3 &= z. \end{aligned}\right. \tag{63}$$

In this case, r is the length of the projection on the plane (x_1, x_2) of the radius vector, θ is the angle between this projected radius vector and x_1 axis measured in the trigonometric sense, and z is the axis of the cylinder which is the same as x_3 axis. Cartesian coordinate x_1 and x_2 are in a plane normal to the z axis. Here, $q_1 = r$, $q_2 = \theta$, and $q_3 = z$.

The Jacobian matrix of transformation (63) is given by

$$J_{\mathbf{q}}(\mathbf{x}) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{64}$$

Starting with equations (49), (63), and (64) we find that the cylindrical coordinates system is orthogonal and

$$ds_1 = dr, \quad ds_2 = r\, d\theta, \quad ds_3 = dz. \tag{65}$$

Let the components of a vector $\mathbf{F}$ be F_r, F_θ, F_z in the r, θ, and z directions, respectively. Following (58), (59), and (62), we have:

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{1}{r}\left[\frac{\partial(r\,F_r)}{\partial r} + \frac{\partial F_\theta}{\partial \theta} + r\frac{\partial F_z}{\partial z}\right]; \tag{66}$$

$$\begin{aligned} \boldsymbol{\nabla}\times\mathbf{F} \;=\; & \left[\frac{1}{r}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}\right]\mathbf{e}_r + \\ & + \left[\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right]\mathbf{e}_\theta + \frac{1}{r}\left[\frac{\partial (rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right]\mathbf{e}_z, \end{aligned} \tag{67}$$

where $\mathbf{e}_r$ is the unit vector of the projected radius vector on the plane (x_1, x_2), $\mathbf{e}_\theta$ is a unit vector tangent in the point (r, θ, z) to the circle of r radius and the centre in the point $(0, 0, z)$, belonging to the plane parallel with (x_1, x_2) passing throught this point, and $\mathbf{e}_z$ is the unit vector of x_3 axis. These three unit vectors stand for $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ in equation (57), and the set $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ forms a right–handed orthogonal basis in the three–dimensional space.

If $\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_z\mathbf{e}_z$ is the displacement vector refered to cylindrical coordinates system, then the dilatation $\nabla\mathbf{u} = \Theta$, the gradient of Θ, and the rotation vector $\boldsymbol{\Psi} = \boldsymbol{\nabla}\times\mathbf{u}$ are given by

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \Theta = \frac{1}{r}\left[\frac{\partial(r\,u_r)}{\partial r} + \frac{\partial u_\theta}{\partial\theta} + r\frac{\partial u_z}{\partial z}\right], \tag{68}$$

$$\boldsymbol{\nabla}\Theta = \frac{\partial\Theta}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Theta}{\partial\theta}\mathbf{e}_\theta + \frac{\partial\Theta}{\partial z}\mathbf{e}_z, \tag{69}$$

$$\begin{aligned}\boldsymbol{\Psi} = \boldsymbol{\nabla}\times\mathbf{u} \;\; &= \;\; \left[\frac{1}{r}\frac{\partial u_z}{\partial\theta} - \frac{\partial u_\theta}{\partial z}\right]\mathbf{e}_r + \\ &\quad + \left[\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right]\mathbf{e}_\theta + \frac{1}{r}\left[\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right]\mathbf{e}_z.\end{aligned} \tag{70}$$

From (70) we derive that the components of the vector $\boldsymbol{\Psi}$ are

$$\begin{cases} \Psi_r &= \dfrac{1}{r}\dfrac{\partial u_z}{\partial\theta} - \dfrac{\partial u_\theta}{\partial z} \\ \Psi_\theta &= \dfrac{\partial u_r}{\partial z} - \dfrac{\partial u_z}{\partial r}, \\ \Psi_\theta &= \dfrac{1}{r}\left[\dfrac{\partial(ru_\theta)}{\partial r} - \dfrac{\partial u_r}{\partial\theta}\right]. \end{cases} \tag{71}$$

Our aim is to use the metric coefficients for the cylindrical coordinate system along with the above expressions for the vector and scalar functions to derive the Navier equations in the form given by equation (25) into cylindrical coordinates.

11. The Navier Equations in Cylindrical Coordinates

Using equations (68), (69), (70), and (59), where $\mathbf{F} = \boldsymbol{\Psi}$, the Navier equations (25), divided by ρ, become in extended form

$$\begin{cases} c_L^2\dfrac{\partial\Theta}{\partial r} & - \quad c_T^2\left[\dfrac{1}{r}\dfrac{\partial\Psi_z}{\partial\theta} - \dfrac{\partial\Psi_\theta}{\partial z}\right] & = \dfrac{\partial^2 u_r}{\partial t^2}, \\ c_L^2\dfrac{1}{r}\dfrac{\partial\Theta}{\partial\theta} & - \quad c_T^2\left[\dfrac{\partial\Psi_r}{\partial z} - \dfrac{\partial\Psi_z}{\partial r}\right] & = \dfrac{\partial^2 u_\theta}{\partial t^2}, \\ c_L^2\dfrac{\partial\Theta}{\partial z} & - \quad c_T^2\dfrac{1}{r}\left[\dfrac{\partial(r\Psi_\theta)}{\partial r} - \dfrac{\partial\Psi_r}{\partial\theta}\right] & = \dfrac{\partial^2 u_z}{\partial t^2}. \end{cases} \tag{72}$$

For longitudinal waves without rotation, the rotation vector $\boldsymbol{\Psi}$ vanishes and equation (72) reduces to the wave equation for the dilatation Θ. For transverse waves, Θ vanishes and equation (72) becomes the wave equation for $\boldsymbol{\Psi}$.

12. Radially Symmetric Waves

We now investigate the propagation of radially symmetric waves in a solid infinite cylinder of radius a. Some authors ([**?**], [**?**] etc) take the following approach. They solve the Navier equations (72) for the dilatation and rotation components. From them they obtain the displacement and rotation. Then they apply the results to an infinite cylinder with free–surface boundary conditions.

The approach to be used here is to take advantage of the fact that an elastic medium has no internal friction, so that scalar and vector potentials exist. We shall make use of this fact by solving the wave equations for these potentials in cylindrical coordinates, and from these solutions we obtain the displacement vector, the stress components, etc.

Radially symmetric waves in a solid cylinder are symmetric about the cylinder axis z, so that the angular displacement component u_θ vanishes and the other components do not depend on the angle θ. Each particle of the cylinder oscillates in the (r, z) plane. It turns out that the rotation vector $\mathbf{\Psi} = \mathbf{\nabla} \times \mathbf{u}$ has a nonzero angular component that is independent of θ. We consider an infinite train of time–harmonic waves of a single frequency along a solid infinite cylinder such that the displacement is a simple harmonic function of z. The cylindrical coordinates are (r, θ, z), where where θ is the angle that the radius vector r makes with the x axis. The x, y coordinates are in a plane normal to the cylinder axis z. The displacement vector is $\mathbf{u} = (u_r, 0, u_z)$, where the components are functions of (r, z, t). We see that the rotation vector has the form $\mathbf{\nabla} \times \mathbf{u} = (0, \dfrac{\partial u_r}{\partial z} - \dfrac{\partial u_z}{\partial r}, 0) = (0, \Psi_\theta, 0)$, where Ψ_θ is also a function of (r, z, t).

In our axisymmetric case, the relation between the components of the displacement vector to the scalar and vector potentials (33) become

$$\begin{aligned} u_r &= \frac{\partial \phi}{\partial r} - \frac{\partial \psi}{\partial z}, \\ u_z &= \frac{\partial \phi}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) = \frac{\partial \phi}{\partial z} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r}. \end{aligned} \tag{73}$$

The wave equations for the potentials ϕ and ψ in axisymmetric cylindrical coordinates become

$$
\begin{aligned}
c_L^2\left[\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}\right] &= \frac{\partial^2\phi}{\partial t^2}\\
c_T^2\left[\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2}\right] &= \frac{\partial^2\psi}{\partial t^2}.
\end{aligned}
\tag{74}
$$

The only nonzero components of the stress tensor are

$$
\begin{aligned}
\tau_{rr} &= \lambda\left(\frac{u_r}{r}+\frac{\partial u_r}{\partial r}+\frac{\partial u_z}{\partial z}\right)+2\mu\frac{\partial u_r}{\partial r},\\
\tau_{rz} &= \mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right),
\end{aligned}
\tag{75}
$$

since the cylinder is axisymmetric so that the angular components of the stress tensor vanish.

The boundary conditions are on a free surface, which means that on that on the surface $r=a$ we have

$$
\tau_{rr}=0,\quad \tau_{zz}=0,\quad \text{at } r=a. \tag{76}
$$

To solve for the potentials we take time harmonic solutions. Furthermore, since the wave front propagates in the z direction, we assume the potentials are harmonic in z. Therefore we can separate the $r-$dependent parts of the potentials and write them in the form

$$
\phi=AF(r)e^{i(kz\pm\omega t)},\quad \psi=BG(r)e^{i(kz\pm\omega t)}, \tag{77}
$$

where A and B are constants and F and G are functions of r to be determined. Note that the same phase $kz\pm\omega t$ is used for the scalar and vector potentials. At any instant t, potentials ϕ and ψ are periodic functions of z with wavelength Λ, where $\Lambda=2\pi/k$. The quantity $k=2\pi/\Lambda$, which counts the number of wavelength over 2π is termed the wavenumber. At any position the potentials ϕ and ψ are time–harmonic with time period T, where $T=2\pi/\omega$. The circular frequency ω is given by $\omega=k\cdot c$.

Inserting equations (77) into the appropriate wave equations (74) yields

$$
\begin{aligned}
\frac{d^2F}{dr^2}+\frac{1}{r}\frac{dF}{dr}+\beta^2F &= 0,\\
\frac{d^2G}{dr^2}+\frac{1}{r}\frac{dG}{dr}+\gamma^2G &= 0,
\end{aligned}
\tag{78}
$$

where

$$\beta^2 = \frac{\omega^2}{c_L^2} - k^2, \quad \gamma^2 = \frac{\omega^2}{c_T^2} - k^2. \tag{79}$$

The solutions of equation (78) that has no singularity at $r = 0$ is Bessel's functions of order zero, so that the potentials become, for progressing wave fronts,

$$\phi = AJ_0(\beta r)e^{i(kz-\omega t)}, \quad \psi = BJ_0(\gamma r)e^{i(kz-\omega t)} \tag{80}$$

Using (80), equation (73) for the displacement components become

$$\begin{aligned} u_r &= \left[A\frac{d}{dr}J_0(\beta r) + ikB\frac{d}{dr}J_0(\gamma r)\right]e^{i(kz-\omega t)} \\ u_z &= \left[ikAJ_0(\beta r) + \gamma^2 BJ_0(\gamma r)\right]e^{i(kz-\omega t)}. \end{aligned} \tag{81}$$

Starting with (69) and (81) we derive that the divergence of the displacement vector in the case of radially symmetric waves is given by

$$\Theta = \boldsymbol{\nabla}\cdot\mathbf{u} = -\frac{\omega^2}{c_L^2}AJ_0(\beta r)\,e^{i(kz-\omega t)}. \tag{82}$$

Using now the constitutive equations (75), the expressions of u_r and u_θ given in (81), and equation (82) we find that the components τ_{rr}, τ_{rz} of the stress tensor are

$$\begin{aligned} \tau_{rr} &= \left\{\left[2\mu\frac{d^2}{dr^2}J_0(\beta r) - \lambda\frac{\omega^2}{c_L^2}J_0(\beta r)\right]A + 2\mu ikB\frac{d^2}{dr^2}J_0(\gamma r)\right\}e^{i(kz-\omega t)}, \\ \tau_{rz} &= \mu\left[2ikA\frac{d}{dr}J_0(\beta r) + \left(\frac{\omega^2}{c_L^2} - 2k^2\right)B\frac{d}{dr}J_0(\gamma r)\right]e^{i(kz-\omega t)}. \end{aligned} \tag{83}$$

To satisfy the boundary conditions (76) we must to have

$$\begin{aligned} \left[2\mu\frac{d^2}{dr^2}J_0(\beta r)\Big|_{r=a} - \lambda\frac{\omega^2}{c_L^2}J_0(\beta a)\right]A + \quad 2\mu ik\frac{d^2}{dr^2}J_0(\gamma r)\Big|_{r=a}B \quad &= 0, \\ 2ik\frac{d}{dr}J_0(\beta r)\Big|_{r=a}A \quad + \left(\frac{\omega^2}{c_T^2} - 2k^2\right)\frac{d}{dr}J_0(\gamma r)\Big|_{r=a}B &= 0. \end{aligned} \tag{84}$$

The system (84) is a set of two algebraic homogeneous equations for the constants A and B. Therefore, for nontrivial solutions thwe determinant must equal zero, yielding

$$\begin{vmatrix} 2\mu\dfrac{d^2}{dr^2}J_0(\beta r)\Big|_{r=a} - \lambda\dfrac{\omega^2}{c_L^2}J_0(\beta a) & 2\mu ik\dfrac{d^2}{dr^2}J_0(\gamma r)\Big|_{r=a} \\ 2ik\dfrac{d}{dr}J_0(\beta r)\Big|_{r=a} & \left(\dfrac{\omega^2}{c_T^2}-2k^2\right)\dfrac{d}{dr}J_0(\gamma r)\Big|_{r=a} \end{vmatrix}. \tag{85}$$

This is the *period equation*. It is difficult to discuss this equation in its general form, other than to perform numerical analyses for special cases. But if we have a thin cylinder, than the radius a is small, therefore βa and γa will be small enough to neglect fourth–order terms. This is seen by expanding the Bessel function

$$J_0(x) = 1 - \frac{1}{2}x^2 + \frac{1}{64}x^4 - \cdots, \tag{86}$$

where x stands for βa or γa. Using equation (86) in the expansion of equation (85) and keeping only the quadratic terms, we get the required approximation for the period equation. From this we obtain an approximation for the longitudinal wave speed:

$$c_L = \frac{\omega}{k} = \sqrt{\frac{E}{\rho}\Big(1 - \frac{1}{4}\sigma^2k^2a^2\Big)}, \tag{87}$$

where E is the Young's modulus and σ is the Poisson's ratio of the cylinder.

13. Waves Propagated over the Surface of an Elastic Body

Lord Rayleigh investigated a type of surface wave runing along the planar interface between air and an isotropic elastic solid in which the amplitude of the wave damps off exponentially as it penetrates the solid. It was anticipated by Lord Rayleigh that solutions of this type might approximate the behaviour of seismic waves observed during earthquakes [*Proceedings of the London Mathematical Society*, vol. 17 (1887), or *Scientific Papers*, vol. 2, p. 441].

When such surface waves are studied by the Rayleigh's treatment, it is considered that the (x, y) plane bounded by the free surface $y = 0$ has air in its upper plane $y > 0$ and an isotropic elastic solid in the lower plane $y < 0$. We assume that monochromatic progressing waves (single frequency) are propagated in the positive x direction as a result of forces applied in the solid at some distance from the surface (for example, the forces that produce earthquakes). Since the nature of the disturbing force is not specified, there are infinitely many solutions to these wave equations. However, using Rayleigh's approach we obtain solutions that are exponentially damped.

We assume that all the dependent variables are functions of (x, y, t) and that the displacement vector $\mathbf{u} = (u, v, 0)$. The rotation vector $\mathbf{\Psi}$ has the form $\mathbf{\Psi} = (0, 0, \Psi)$, where $\Psi = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$. The two–dimensional wave equations for the scalat potential ϕ and the component Ψ of the vector potential $\mathbf{\Psi}$ are

$$\begin{aligned} c_L^2\Big(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\Big) &= \frac{\partial^2\phi}{\partial t^2}, \\ c_T^2\Big(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2}\Big) &= \frac{\partial^2\Psi}{\partial t^2}. \end{aligned} \tag{88}$$

The boundary conditions at $y = 0$ are of type of a free surface, so that the shear stress τ_{xy} and the normal stress τ_{yy} vanish on the x axis. In this case, Hooke's law gives us

$$\tau_{xy} = 2\mu\varepsilon_{xy}, \quad \tau_{yy} = \lambda\Theta + 2\mu\varepsilon_{yy}, \quad \Theta = \varepsilon_{xx} + \varepsilon_{yy}, \tag{89}$$

for two–dimensional stress and strain. The components u, v, and w of displacement vector $\mathbf{u}$ become

$$u = \frac{\partial\phi}{\partial x} + \frac{\partial\Psi}{\partial y}, \quad v = \frac{\partial\phi}{\partial y} - \frac{\partial\Psi}{\partial x}, \quad w = 0. \tag{90}$$

The strain tensor has the nonzero components

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\Psi}{\partial x\partial y}, \\ \varepsilon_{xy} &= \frac{1}{2}\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) = \frac{\partial^2\phi}{\partial x\partial y} + \frac{1}{2}\Big(-\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2}\Big), \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} = \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\Psi}{\partial y\partial x}. \end{aligned} \tag{91}$$

Inserting equations (91) into (89) yields the boundary conditions for the components of the stress tensor at $y = 0$:

$$\begin{aligned} 2\frac{\partial^2\phi}{\partial x\partial y}(x,0) - \frac{\partial^2\Psi}{\partial x^2}(x,0) + \frac{\partial^2\Psi}{\partial y^2}(x,0) &= 0, \\ \lambda\nabla^2\phi(x,0) + 2\mu\Big(\frac{\partial^2\phi}{\partial y^2}(x,0) - \frac{\partial^2\Psi}{\partial y\partial x}(x,0)\Big) &= 0. \end{aligned} \tag{92}$$

The problem is to find time–harmonic solutions to (88) that exponentially decay with y, are progressing waves in the x direction, and satisfy the boundary conditions (92). To this end we take the potentials in the form

$$\phi(x,y) = Ae^{-ay}e^{ik(x-ct)}, \quad \Psi(x,y) = Be^{-by}e^{ik(x-ct)}, \quad a > 0,\, b > 0, \tag{93}$$

where A is the amplitude of the scalar potential, B is the amplitude of the vector potential, and a and b are the *decay constants*, which are determined from the wave equations. Note that the wave number k and the frequency ω are the same for each potential. Inserting equations (93) into (88) yields

$$\begin{aligned} a^2 - k^2\left[1 - \left(\frac{c}{c_L}\right)^2\right] &= 0, \\ b^2 - k^2\left[1 - \left(\frac{c}{c_T}\right)^2\right] &= 0. \end{aligned} \tag{94}$$

The wave velocity c is not equal to c_L or c_T. However, if $a = 0$ then $c = c_L$, or if $b = 0$ then $c = c_T$. But neither a nor b can vanish.

To satisfy the boundary conditions we insert equation (93) into equation (92) and obtain

$$\begin{aligned} -2iakA + (b^2 + k^2)B &= 0, \\ \left[2\mu a^2 + \lambda(a^2 - k^2)\right]A + 2i\mu bkB &= 0. \end{aligned} \tag{95}$$

Equations (95) are a pair of homogeneous algebraic equations for the complex constants A and B. As usual, we set the determinant equal to zero in order to have nontrivial solutions for A and B. We get

$$4\mu abk^2 - (b^2 + k^2)\left[2\mu a^2 + \lambda(a^2 - k^2)\right] = 0. \tag{96}$$

Eliminating a and b from equation (96) by appealing to (94) we derive that equation (96) becomes a cubic in $(c/c_T)^2$, which can be put in the form

$$s^3 - 8s^2 + 8(3 - 2r)s - 16(1 - r) = 0, \tag{97}$$

where

$$s = \left(\frac{c}{c_T}\right)^2, \quad r = \left(\frac{c_T}{c_L}\right)^2.$$

Using the approximation $\lambda = \mu$ (Poisson's condition), we get $r = 1/3$ and equation (97) becomes

$$(s - 4)(3s^2 - 12s + 8) = 0. \tag{98}$$

The roots of equation (98) are

$$s_1 = 4, \quad s_2 = 2 + \frac{2\sqrt{3}}{3}, \quad s_3 = 2 - \frac{2\sqrt{3}}{3}.$$

It is easily seen that the only root that yields positive values for a and b is $s = 2 - 2\sqrt{3}/3$. We thereby obtain the following relationship between c and c_T :

$$c = \sqrt{s}c_T = 0.9194\, c_T.$$

For the case of an incompressible body we have $\Theta = 0$. This gives $r = 0$, so that the velocity of a Rayleigh wave becomes

$$c = \sqrt{s}c_T = 0.9553\, c_T.$$

We have seen that in either case c is slightly less than the velocity of an equivoluminal wave.

We note that c is independent of frequency. This means that there is no dispersion, that is the wave shape is maintained. Having determined c in terms of c_T and c_L, we can then calculate the decay constants a and b, which determine the rate at which the potentials attenuate with depth (note that y is positive downward). From the definition of the wave number k, both a and b are proportional to ω and $a > b$. This means that for a given frequency, irrotational waves attenuate faster than equivoluminal waves. We also see that waves of higher frequency are attenuated more rapidly than those of lower frequency.

Seismographic signatures often depict waves similar in structure to Rayleigh waves. However, seismograph records of distant eartquakes indicate dispersion, which means dependence of c on ω. This arises mainly because of the inhomogeneity of the earth, and also because of the viscoelastic properties of the earth.

14. Special cases

As we know, the vector Navier equation (24) is satisfied with a vector of the form

$$\mathbf{u} = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \boldsymbol{\Psi}, \tag{99}$$

where the potentials ϕ and $\boldsymbol{\Psi}$ have to satisfy the wave equations (suppose that $\mathbf{F} = \mathbf{0}$)

$$c_L^2 \nabla^2 \phi = \frac{\partial^2 \phi}{\partial t^2}, \qquad c_T^2 \nabla^2 \boldsymbol{\Psi} = \frac{\partial^2 \boldsymbol{\Psi}}{\partial t^2}. \tag{100}$$

A class of particular solutions of the Navier equation (24) can be generated by seeking solutions of (100) of the form

$$\phi = Ae^{-ay+i(x-\omega t)}, \qquad \boldsymbol{\Psi} = \mathbf{e}_2 e^{-by+i(x-\omega t)}, \tag{101}$$

where A, $\mathbf{e}_2$, $a > 0$, and $b > 0$ are constants. It is seen that in this case the displacement vector (99) has the form $\mathbf{u} = (u, v, 0)$, and u and v are independent of z. The functions ϕ and $\boldsymbol{\Psi}$ in (101) are solutions of (100) if

$$a = \sqrt{1 + \frac{\omega^2}{c_L^2}}, \qquad b = \sqrt{1 + \frac{\omega^2}{c_T^2}}.$$

Since we have a two–dimensional problem it results that only the third component of the vector $\mathbf{B}$ is nonzero. Putting $B_3 = B$, we follow to determine the constants A and B from adequate boundary conditions.

When the gradient and curl are expressed in cylindrical coordinates (r, θ, z), then for axially symmetric problems,

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial \Psi}{\partial z}, \quad u_\theta = 0, \quad u_z = \frac{\partial \phi}{\partial z} + \frac{\partial \Psi}{\partial r} + \frac{1}{r}\Psi, \tag{102}$$

and the functions ϕ and Ψ must to satisfy the following differential equations

$$c_L^2 \nabla^2 \phi = \frac{\partial^2 \phi}{\partial t^2}, \qquad c_T^2 \nabla^2 \Psi = \frac{\partial^2 \Psi}{\partial t^2}, \tag{103}$$

where ∇^2 is the Laplace's operator in cylindrical coordinates which in axially symmetric case has the form

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

Particular solutions of equations (103) may be seeking of the form

$$\phi = F(r)e^{i(z-\omega t)}, \quad \Psi = G(r)e^{i(z-\omega t)},$$

where F and G are solutions of the following ordinary differential equations

$$\begin{aligned} F''(r) + \frac{1}{r}F(r) + \left(\frac{\omega^2}{c_L^2} - 1\right)F(r) &= 0, \\ G''(r) + \frac{1}{r}G(r) + \left(\frac{\omega^2}{c_T^2} - 1\right)G(r) &= 0. \end{aligned} \tag{104}$$

The solutions of equations (104) may be expressed with Bessel's function of order zero

$$F(r) = J_0(\beta_1 r), \quad G(r) = J_0(\gamma_1 r), \tag{105}$$

where

$$\beta_1 = \sqrt{\frac{\omega^2}{c_L^2} - 1}, \quad \gamma_1 = \sqrt{\frac{\omega^2}{c_T^2} - 1}.$$

If we have to solve a boundary value problem in axially symmetric case, then the functions ϕ and Ψ must to be of the form

$$\phi = AF(r)e^{i(z-\omega t)}, \quad \Psi = BG(r)e^{i(z-\omega t)}, \tag{106}$$

where $F(r)$ and $G(r)$ are given in equation (105), and A and B are constants which will be determined from boundary conditions.

References

[1] Love, A. E. H., A Treatise on the Mathematical Theory of Elasticity. Dover, 1944.

[2] Davis, J. L., Wave Propagation in Solids and Fluids. Springer–Verlag, 1988.

[3] Kolsky, H., Stress Waves in Solids. Dover, 1963.

[4] Achenbach, J. D., Wave Propagation in Elastic Solids. North–Holland Publishing Company, 1973.

[5] Magnus, W., Oberhettinger, F., Soni, R. P., Formulas and Theoremas for the Special Functions of Mathematical Physics. Springer–Verlag, 1966.

Author:

Ion Crăciun
Department of Mathematics
Technical University of Iasi
Address: Iaşi, Bulevardul Carol I, nr. 11
e-mail: *ialcraciun@yahoo.com*