A C-CONTINUUM X IS METRIZABLE IF AND ONLY IF IT ADMITS A WHITNEY MAP FOR C(X)

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Abstract. The purpose of this paper is to prove that a C-continuum X is metrizable if and only if it admits a Whitney map for $C(X)$ (Theorem 2.5).

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1. Preliminaries

In [1] J.J. Charatonik and W.J. Charatonik showed that the non-metric indecomposable continuum example given by Gutek and Hagopian [3] will support a Whitney map on $C(X)$ the hyperspace of subcontinua.

In this paper we shall show that such examples are not in the class of C-continua, since a C-continuum X admits a Whitney map for $C(X)$ if and only if it is metrizable (Theorem 2.5).

All spaces in this paper are compact Hausdorff and all mappings are continuous. Let X be a space. We define its hyperspaces as the following sets:

$$2^X = \{F \subseteq X : F \text{ is closed and nonempty } \},$$
$$C(X) = \left\{F \in 2^X : F \text{ is connected } \right\},$$
$$C^2(X) = C(C(X)),$$
$$X(n) = \left\{F \in 2^X : F \text{ has at most } n \text{ points } \right\},\ n \in \mathbb{N}.$$

For any finitely many subsets S_1, ..., S_n, let

$$\langle S_1, ..., S_n\rangle = \left\{F \in 2^X : F \subset \bigcup_{i=1}^{n} S_i,\ and\ F \cap S_i \neq \emptyset,\ for\ each\ i\right\}.$$

The topology on 2^X is the Vietoris topology, i.e., the topology with a base $\{< U_1, ..., U_n >: U_i$ is an open subset of X for each i and each $n < \infty$ $\}$, and $C(X)$ is a subspace of 2^X.

Let Λ be a subspace of 2^X. By a *Whitney map* for Λ [7, p. 24, (0.50)] we will mean any mapping $W : \Lambda \to [0, +\infty)$ satisfying

a) if $A, B \in \Lambda$ such that $A \subset B$ and $A \neq B$, then $W(A) < W(B)$ and

b) $W(\{x\}) = 0$ for each $x \in X$ such that $\{x\} \in \Lambda$.

If X is a metric continuum, then there exists a Whitney map for 2^X and $C(X)$ ([7, pp. 24-26], [4, p. 106]). On the other hand, if X is non-metrizable, then it admits no Whitney map for 2^X [1]. It is known that there exist non-metrizable continua which admit and ones which do not admit a Whitney map for $C(X)$ [1].

The notion of an irreducible mapping was introduced by Whyburn [9, p. 162]. If X is a continuum, a surjection $f : X \rightarrow Y$ is *irreducible* provided no proper subcontinuum of X maps onto all of Y under f. Some theorems for the case when X is semi-locally-connected are given in [9, p. 163].

A mapping $f : X \rightarrow Y$ is said to be *hereditarily irreducible* [7, p. 204, (1.212.3)] provided that for any given subcontinuum Z of X, no proper subcontinuum of Z maps onto $f(Z)$.

A mapping $f : X \rightarrow Y$ is *light* (*zero-dimensional*) if all fibers $f^{-1}(y)$ are hereditarily disconnected (zero-dimensional or empty) [2, p. 450], i.e., if $f^{-1}(y)$ does not contain any connected subsets of cardinality larger that one ($\dim f^{-1}(y) \leq 0$). Every zero-dimensional mapping is light, and in the realm of mappings with compact fibers the two classes of mappings coincide.

Lemma 1.1. *Every hereditarily irreducible mapping is light.*

If X is a metric continuum, then there exists a Whitney map for 2^X and $C(X)$ ([7, pp. 24-26], [4, p. 106]). On the other hand, if X is non-metrizable, then it admits no Whitney map for 2^X [1]. It is known that there exist non-metrizable continua which admit and ones which do not admit a Whitney map for $C(X)$ [1].

We shall use the notion of inverse system as in [2, pp. 135-142]. An inverse system is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$. We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *σ-directed* if for each sequence $a_1, a_2, ..., a_k, ...$ of the members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

Theorem 1.2. [6, Theorem 1.8, p. 397]. *Let X be a compact Hausdorff space such that $w(X) > \aleph_0$. There exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric compacta X_a such that X is homeomorphic to* $\lim \mathbf{X}$.

The following result is an external characterization of non-metric continua which admit a Whitney map.

Theorem 1.3. [6, Theorem 2.3, p. 398]. *Let X be a non-metric continuum. Then X admits a Whitney map for $C(X)$ if and only if for each σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of continua which admit a Whitney map and $X = \lim \mathbf{X}$ there exists a cofinal subset $B \subset A$ such that for every $b \in B$ the projection p_b : $\lim X \rightarrow X_b$ is hereditarily irreducible.*

Now we give another form of Theorem 1.3.

Theorem 1.4. *Let X be a non-metric continuum. Then X admits a Whitney map for $C(X)$ if and only if for each σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of continua which admit a Whitney map and $X = \lim \mathbf{X}$ there exists a cofinal subset $B \subset A$ such that for every $b \in B$ the projection $C(p_b) : \lim C(X) \to C(X_b)$ is light.*

Proof. This follows from Theorem 1.3 and the fact that $C(f) : C(X) \to C(Y)$ is light if and only if $f : X \to Y$ is hereditarily irreducible [7, p. 204, (1.212.3)].

2. C-continua and Whitney maps

A continuum X is said to be a *C-continuum* provided for each triple x, y, z of points of X, there exists a subcontinuum C of X which contains x and exactly one of the points y and z [10, p. 326].

A *generalized arc* is a Hausdorff continuum with exactly two non-separating points (end points) x, y. Each separable arc is homeomorphic to the closed interval $I = [0, 1]$.

We say that a space X is *arcwise connected* if for every pair x, y of points of X there exists a generalized arc L with end points x, y.

Lemma 2.1. *Each arcwise connected continuum is a C-continuum.*

Proof. Let x, y, z be a triple of points of an arcwise connected continuum X. There exists an arc $[x, y]$ with endpoints x and y. If $z \notin [x, y]$, then the proof is completed. If $z \in [x, y]$, then subarc $[x, z]$ contains x and z, but not y. The proof is completed.

Lemma 2.2. *The cartesian product $X \times Y$ of two non-degenerate continua X and Y is a C-continuum.*

Proof. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be a triple of points of the product $X \times Y$. Now we have $x_2 \neq x_3$ or $y_2 \neq y_3$. We will give the proof in the case $x_2 \neq x_3$ since the proof in the case $y_2 \neq y_3$ is similar. Now we have two disjoint continua $Y_2 = \{(x_2, y) : y \in Y\}$ and $Y_3 = \{(x_3, y) : y \in Y\}$. If $(x_1, y_1) \in Y_2$ or $(x_1, y_1) \in Y_3$, the proof is completed. Let $(x_1, y_1) \notin Y_2$ and $(x_1, y_1) \notin Y_3$. Consider the continua $X_2 = \{(x, y_2) : x \in X\}$ and $X_3 = \{(x, y_3) : x \in X\}$. The continuum $Y_1 = \{(x_1, y) : y \in Y\}$ contains a point (x_1, p) such that $(x_1, p) \notin X_2 \cup X_3$. Let $X_p = \{(x, p) : x \in X\}$. It is clear that a continuum $Y_1 \cup X_p \cup Y_2$ contains the points (x_1, y_1) and (x_2, y_2) but not (x_3, y_3). Similarly, a continuum $Y_1 \cup X_p \cup Y_3$ contains the points (x_1, y_1) and (x_3, y_3) but not (x_2, y_2). The proof is completed.

The concept of aposyndesis was introduced by Jones in [5]. A continuum X is *aposyndetic* provided it is true that if x and y are any two points of X, then some closed connected neighborhood of x misses y.

A continuum is said to be *semi-aposyndetic* [4, p. 238, Definition 29.1], if for every $p \neq q$ in X, there exists a subcontinuum M of X such that $\mathrm{Int}_X(M)$ contains one of the points p, q and $X \setminus M$ contains the other one. Each locally connected continuum is semi-aposyndetic.

Proposition 1.[10, Theorem 1, p. 326]. *If the continuum X is aposyndetic, the X is the C-continuum.*

Remark 1. There exists a C-continuum continuum which is not aposyndetic [10, p. 327]wilder. (See countable harmonic fan).

Remark 2. There exists a C-continuum continuum which is not arcwise connected [10 p. 328].

A continuum X is said to be *colocally connected* provided that for each point $x \in X$ and each open se $U \ni x$ there exists an open set V containing x such that $V \subset U$ and $X \setminus U$ is connected.

Lemma 2.3.*Each colocally connected continuum X is a C-continuum.*

*Proof.*Let x, y, z be a triple of points of X. Now, $U = X \setminus \{x, y\}$ is an open set U such that $z \in U$. From the colocal connectedness of X it follows that there exists an open set V such that $z \in V \subset U$ and $X \setminus V$ is connected.Hence, X is a C-continuum since the continuum $X \setminus V$ contains the points x and y.

Lemma 2.4.*The cartesian product $X \times Y$ of two non-degenerate continua is a colocally connected continuum and, consequently, a C-continuum.*

*Proof.*Let (x, y) be a point of $X \times Y$. We have to prove that there exists a neighbourhood $U = U_x \times U_y$ of (x, y) such that $E = X \times Y \setminus U$ is connected. We may assume that $U_x \neq X$ and $U_y \neq Y$. Let (x_1, y_1), (x_2, y_2) be a pair of different points in E. For each point $(z, w) \in X \times Y$ we consider a continuum

$$E_{zw} = \{(z, y) : y \in Y\} \cup \{(x, w) : x \in X\}.$$

Claim 1. *For each point $(x', y') \in E$ there exists a point $(z, w) \in E$ such that $(x', y') \in E_{zw}$ and $E_{zw} \cap U = \emptyset$.* If $E_{x'y'} \cap U = \emptyset$ the proof is completed. In the opposite case we have either $\{(x', y) : y \in Y\} \cap U \neq \emptyset$ or $\{(x, y') : x \in X\} \cap U \neq \emptyset$. Suppose that $\{(x', y) : y \in Y\} \cap U \neq \emptyset$. Then $\{(x, y') : x \in X\} \cap U = \emptyset$. There exists a point $z \in X$ such that $z \notin U$. Setting $y' = w$, we obtain a point $(z, w) \in E$ such that $(x', y') \in E_{zw}$ and $E_{zw} \cap U = \emptyset$. The proof in the case $\{(x, y') : x \in X\} \cap U \neq \emptyset$ is similar.

Now, by Claim 1, for (x_1, y_1) there exists a continuum E_{z_1,w_1} such that $E_{z_1w_1} \cap U = \emptyset$ and $(x_1, y_1) \in E_{z_1,w_1}$. Similarly, there exist a continuum E_{z_2,w_2} such that $E_{z_2w_2} \cap U = \emptyset$ and $(x_2, y_2) \in E_{z_2,w_2}$.

Claim 2. *The union $E_{z_1,w_1} \cup E_{z_2,w_2}$ is a continuum which contains the points (x_1, y_1), (x_2, y_2) and is contained in $E = X \times Y \setminus U$.* Obvious.

Finally, we infer that $E = X \times Y \setminus U$ is connected. Hence, $X \times Y$ is colocally connected. From Lemma 2.3 it follows that $X \times Y$ is a C-continuum. The proof is completed.

Now we shall prove the main theorem of this Section.

Theorem 2.5. *A C-continuum X is metrizable if and only if it admits a Whitney map for $C(X)$.*

Proof. If X is metrizable, then it admits a Whitney map for $C(X)$ ([7, pp. 24-26], [4, p. 106]). Conversely, let X admits a Whitney map $\mu : C(X) \to [0, +\infty)$. Suppose that X is non-metrizable. The remaining part of the proof is broken into several steps.

Step 1. *There exists a σ-directed directed inverse system $X = \{X_a, p_{ab}, A\}$ of metric compact spaces X_a such that X is homeomorphic to* $\lim \mathbf{X}$ [6, p. 397, Theorem 1.8].

Step 2. *There exists a cofinal subset $B \subset A$ such that for every $b \in B$ the projection $p_b : \lim X \to X_b$ is hereditarily irreducible.* This follows from Theorem 1.3.

Step 3. *If* $\lim \mathbf{X}$ *is a C-continuum, then for every pair C, D of disjoint non-degenerate subcontinua of* $\lim \mathbf{X}$ *there exists a non-degenerate subcontinuum $E \subset$* $\lim \mathbf{X}$ *such that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$.* Let $x \in C$ and $y, z \in D$. There exists a continuum E such that either $x, y \in E, z \in \lim \mathbf{X} \setminus E$ or $x, z \in E$, $y \in \lim \mathbf{X} \setminus E$, respectively since $\lim X$ is a C-continuum. We assume that $x, y \in E$, and $z \in \lim \mathbf{X} \setminus E$. It is clear that $C \cap E \neq \emptyset \neq D \cap E$ *and* $(C \cup D) \setminus E \neq \emptyset$ since $x \in C \cap E$, $y \in D \cap E$ and $z \in (C \cup D) \setminus E$.

In the remaining part of the proof we denote $\lim \mathbf{X}$ by X since X is homeomorphic to $\lim \mathbf{X}$.

Step 4. *Every restriction $C(p_a)|(C(X) \setminus X(1)) : (C(X) \setminus X(1)) \to C(p_a)\,(C(X)) \subset C(X_a)$ is one-to-one.* Consider the inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ whose limit is $C(X)$. From Theorem 1.3 it follows that there exists a subset B cofinal in A such that the projections p_b are hereditarily irreducible and $C(p_b)$ are light for every $b \in B$, see [7, p. 204, (1.212.3)]. Since $\lim \mathbf{X}$ is homeomorphic to $\lim\{X_b, p_{bc}, B\}$, we may assume that $B = A$. Let $Y_a = C(p_a)(C(X))$. Furthermore, $C(p_a)^{-1}(X_a(1)) = X(1)$ since from the hereditary irreducibility of p_a it follows that no non-degenerate subcontinuum of X maps under p_a onto a point. We infer that $C(p_a)^{-1}[Y_a \setminus X_a(1)] = C(X) \setminus X(1)$. Let us prove that the restriction

$C(p_a)|[C(X)\setminus X(1)]$ is one-to-one. Suppose that $C(p_a)|[C(X)\setminus X\ (1)]$ is not one-to-one. Then there exists a continuum C_a in X_a and two continua C, D in X such that $p_a(C) = p_a(D) = C_a$. It is impossible that $C \subset D$ or $D \subset C$ since p_a is hereditarily irreducible. Otherwise, if $C \cap D \neq \emptyset$, then for a continuum $Y = C \cup D$ we have that C and D are subcontinua of Y and $p_a(Y) = p_a(C) = p_a(D) = C_a$ which is impossible since p_a is hereditarily irreducible. We infer that $C \cap D = \emptyset$. There exists a non-degenerate subcontinuum $E \subset \lim X$ such that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D)\setminus E \neq \emptyset$ since $\lim \mathbf{X}$ is a C-continuum (Step 3). Moreover, we may assume that $E \cap D \neq D$. Now $p_a(E \cup D \cup C) = p_a(E \cup C)$ which is impossible since p_a is hereditarily irreducible. It follows that the restriction $P_a = C(p_a)|(C(X)\setminus X(1))$ is one-to-one and closed [2, p. 95, Proposition 2.1.4].

Step 5. *$C(X)\setminus X(1)$ is metrizable and $w(C(X)\setminus X(1)) \leq \aleph_0$.* From Step 4 it follows that P_a is a homeomorphism and $C(X)\setminus X(1)$ is metrizable. Moreover, $w(C(X)\setminus X(1)) \leq \aleph_0$ since Y_a as a compact metrizable space is separable and, consequently, second-countable [2, p. 320].

Step 6. *X is metrizable.* If X is a continuum., then $w(C(X)\setminus X(1)) = \aleph_0$ if and only if $w(X) = \aleph_0$. If $w(X) = \aleph_0$, then $w(C(X)) = \aleph_0$. Hence, $w(C(X)\setminus X(1)) = \aleph_0$. Conversely, if $w(C(X)\setminus X(1)) = \aleph_0$, then there exists a countable base $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ of $C(X)\setminus X(1)$. For each B_i let $C_i = \cup\{x \in X : x \in B,\ B \in B_i\}$, i.e., the union of all continua B contained in B_i.

Claim 1. *The family $\{C_i : i \in N\}$ is a network of X.* Let X be a point of X and let U be an open subsets of X such that $x \in U$. There exists and open set V such that $x \in V \subset \mathrm{Cl}V \subset U$. Let K be a component of $\mathrm{Cl}V$ containing x. By Boundary Bumping Theorem [8, p. 73, Theorem 5.4] K is non-degenerate and, consequently, $K \in C(X)\setminus X(1)$. Now, $\langle U\rangle \cap (C(X)\setminus X(1))$ is a neighbourhood of K in $C(X)\setminus X(1)$. It follows that there exists a $B_i \in \mathcal{B}$ such that $K \in B_i \subset \langle U\rangle \cap (C(X)\setminus X(1))$. It is clear that $C_i \subset U$ and $x \in C_i$ since $x \in K \subset U$. Hence, the family $\{C_i : i \in N\}$ is a network of X.

Claim 2. $nw(X) = \aleph_0$. Apply Claim 1 and the fact that $\mathcal{B}$ is countable.

Claim 3. $w(X) = \aleph_0$. By Claim 1 we have $nw(X) = \aleph_0$. Moreover, by [2, p. 171, Theorem 3.1.19] $w(X) = \aleph_0$.

Step 6 contradicts the assumption that X is non-metrizable. The proof is completed.

Corollary 2.6. *Let X be an (arcwise connected, aposyndetic, colocally connected or cartesian product $Y \times Z$ of continua) continuum. Then X is metrizable if and only if it admits a Whitney map for $C(X)$.*

Corollary 2.7. *Let a continuum X be the countable union of C-continua. Then X is metrizable if and only if it admits a Whitney map for $C(X)$.*

Proof. If X is metrizable, then it admits a Whitney map for $C(X)$ [4, p. 106]. Conversely, if X admits a Whitney map for $C(X)$ and if X is the countable union of C-continua $X_i : i \in \mathbb{N}$, then each X_i admits a Whitney map for $C(X_i)$.By Theorem 2.5 we infer that each X_i is metrizable. It is known [2, Corollary 3.1.20, p. 171] that if a compact space X is the countable union of its subspaces $X_n, n \in \mathbb{N}$, such that $w(X_n) \leq \aleph_0$, then $w(X) \leq \aleph_0$. Hence, X is metrizable.

Theorem 2.8. *A continuum X is metrizable if and only if $C(X)$ admits a Whitney map for $C^2(X) = C(C(X))$.*

Proof. If X is metrizable, then $C(X)$ is metrizable and admits a Whitney map for $C(C(X)) = C^2(X)$. Conversely, let $C(X)$ admits a Whitney map for $C(C(X))$. Now, $C(X)$ is an arcwise connected continuum. By Corollary 2.6, $C(X)$ is metrizable. Thus, X is metrizable since X is homeomorphic to $X(1) \subset C(X)$.

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