ON WEAKLY SYMMETRIC AND SPECIAL WEAKLY RICCI SYMMETRIC LORENTZIAN β-KENMOTSU MANIFOLDS

G.T. Sreenivasa, Venkatesha , C.S.Bagewadi and K. Naganagoud

Abstract. In this paper we study weakly symmetric and special weakly Ricci symmetric Lorentzian β-Kenmotsu manifolds and obtained some interesting results.

2000 *Mathematics Subject Classification*: 53C10, 53C15.

1. Introduction

The notions of weakly symmetric and weakly Ricci-symmetric manifolds were introduced by L.Tamassy and T.Q.Binh in [12] and [13].

A non-flat $(2n+1)$-dimensional differentiable manifold (M^{2n+1}, g),$n > 2$, is called *pseudo symmetric* ([12], [13]) if there exists a 1-form α on M^{2n+1} such that

$$\begin{aligned}(\nabla_X R)(Y,Z,V) &= 2\alpha(X)R(Y,Z)V + \alpha(Y)R(X,Z)V + \alpha(Z)R(Y,X)V \\ &\quad +\alpha(V)R(Y,Z)X + g(R(Y,Z)V,X)A,\end{aligned}$$

where $X, Y, Z, V \in \chi(M^{2n+1})$ are vector fields and α is a 1-form on M^{2n+1}. $A \in \chi(M^{2n+1})$ is the vector field corresponding through g to the 1-form α which is given by $g(X, A) = \alpha(X)$.

A non-flat $(2n + 1)$-dimensional differentiable manifold (M^{2n+1}, g), $n > 2$ is called *weakly symmetric* ([12],[13]) if there exist 1-forms $\alpha, \acute{\beta}, \rho$ and γ such that the condition

$$\begin{aligned}(\nabla_X R)(Y,Z,V) &= \alpha(X)R(Y,Z)V + \acute{\beta}(Y)R(X,Z)V + \gamma(Z)R(Y,X)V \\ &\quad +\sigma(V)R(Y,Z)X + g(R(Y,Z)V,X)P, \qquad (1)\end{aligned}$$

holds for all vector fields $X, Y, Z, V \in \chi(M)$. A weakly symmetric manifold (M^{2n+1}, g) is pseudosymmetric if $\acute{\beta} = \gamma = \sigma = \frac{1}{2}\alpha$ and $P = A$, locally symmetric if $\alpha = \acute{\beta} = \gamma = \sigma = 0$ and $P = 0$. A weakly symmetric manifold is said to be *proper* if at least one of the 1-form $\alpha, \acute{\beta}, \gamma$ and σ is not zero or $P \neq 0$.

A non-flat $(2n+1)$-dimensional differentiable manifold (M^{2n+1}, g), $n > 2$ is called *weakly Ricci-symmetric* ([12],[13]) if there exist 1-forms ρ, μ and υ such that the condition

$$(\nabla_X S)(Y,Z) = \rho(X)S(Y,Z) + \mu(Y)S(X,Z) + \upsilon(Z)S(X,Y), \tag{2}$$

holds for all vector fields $X, Y, Z, V \in \chi(M)$. If $\rho = \mu = \upsilon$ then M^{2n+1} is called *pseudo Ricci-symmetric* ([5]).

If M is weakly symmetric, from (1), we have ([13])

$$\begin{aligned}(\nabla_X S)(Z,V) &= \alpha(X)S(Z,V) + \acute{\beta}(R(X,Z)V) + \gamma(Z)R(X,V) \\ &\quad + \sigma(V)S(X,Z) + g(R(X,V)Z),\end{aligned} \tag{3}$$

In [13], L. Tamassy and Q. Binh studied weakly symmetric and weakly Ricci-symmetric Einstein and Sasakian manifolds and in [6] and [**?**], the authors studied weakly symmetric and weakly Ricci-symmetric K-contact manifolds and Lorentzian para-Sasakian manifolds respectively.

The notion of special weakly Ricci symmetric manifold was introduced and studied by H. Singh and Q. Khan [11].

An n-dimensional Riemannian manifold (M^n, g) is called a *special weakly Ricci symmetric* $(SWRS)_n$ manifold if

$$(\nabla_X S)(Y,Z) = 2\alpha(X)S(Y,Z) + \alpha(Y)S(X,Z) + \alpha(Z)S(Y,X), \tag{4}$$

where α is a 1-form and is defined by

$$\alpha(X) = g(X, \rho), \tag{5}$$

where ρ is the associated vector field.

2.Preliminaries

A differentiable manifold of dimension $(2n+1)$ is called Lorentzian β-Kenmotsu manifold if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Riemannian metric g which satisfy ([1],[8],[9])

$$\eta\xi = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{6}$$

$$\phi^2 X = X + \eta(X)\xi, \quad g(X,\xi) = \eta(X), \tag{7}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{8}$$

for all $X, Y \in TM$.

Also an Lorentzian β-Kenmotsu manifold M^{2n+1} is satisfying ([2],[7])

$$\begin{aligned} \nabla_X \xi &= \beta[X - \eta(X)\xi], & (9) \\ (\nabla_X \eta)(Y) &= \beta[g(X,Y) - \eta(X)\eta(Y)], & (10) \end{aligned}$$

where ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g.
Further, on an Lorentzian β-Kenmotsu manifold M^{2n+1} the following relations hold ([1],[3], [7])

$$\begin{aligned} R(\xi, X)Y &= \beta^2[\eta(Y)X - g(X,Y)\xi], & (11) \\ R(X,Y)\xi &= \beta^2[\eta(X)Y - \eta(Y)X], & (12) \\ S(X,\xi) &= -2n\beta^2\eta(X), & (13) \end{aligned}$$

where S is the Ricci curvature and Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.

3.Weakly symmetric Lorentzian β-Kenmotsu manifolds

Assume that M^{2n+1} is a weakly symmetric Lorentzian β-Kenmotsu manifold. Taking covariant differentiation of the Ricci tensor S with respect to X we have

$$(\nabla_X S)(Z,V) = \nabla_X S(Z,V) - S(\nabla_X Z, V) - S(Z, \nabla_X V). \tag{14}$$

Replacing V with ξ in (14) and using (7), (10) and (13) we obtain

$$(\nabla_X S)(Z,\xi) = -2n\beta^2\beta g(X,Z) - \beta S(Z,X). \tag{15}$$

On the other hand replacing V with ξ in (3) and using (7), (11), (12) and (13) we get

$$\begin{aligned} (\nabla_X S)(Z,\xi) = & -2n\beta^2\alpha(X)\eta(Z) + \acute{\beta}\beta^2\eta(X)Z - \acute{\beta}\beta^2\eta(Z)X - 2n\beta^2\gamma(Z)\eta(X) \\ & +\sigma(\xi)S(X,Z) + \beta^2 g(X,Z)P(\xi) - \beta^2\eta(Z)P(X). \end{aligned} \tag{16}$$

Hence, comparing the right hand side of the equations (15) and (16) we have

$$\begin{aligned} & -2n\beta^2\beta g(X,Z) - \beta S(Z,X) \\ = & -2n\beta^2\alpha(X)\eta(Z) + \acute{\beta}\beta^2\eta(X)Z - \acute{\beta}\beta^2\eta(Z)X - 2n\beta^2\gamma(Z)\eta(X) \\ & +\sigma(\xi)S(X,Z) + \beta^2 g(X,Z)P(\xi) - \beta^2\eta(Z)P(X). \end{aligned} \tag{17}$$

Now putting $X = Z = \xi$ in (17) and using (6) and (13), we get

$$0 = 2n\beta^2[\alpha(\xi) + \gamma(\xi) + \sigma(\xi)]. \tag{18}$$

Since $2n\beta^2 \neq 0$, so we obtain

$$\alpha(\xi) + \gamma(\xi) + \sigma(\xi) = 0. \tag{19}$$

Now we will show that $\alpha + \gamma + \sigma = 0$ holds for all vector fields on M^{2n+1}. In (3) taking $Z = \xi$ similar to the previous calculations it follows that

$$\begin{aligned} 0 &= -2n\beta^2\alpha(X)\eta(V) + \beta^2 g(X,V)\acute{\beta}(\xi) - \beta^2\eta(V)\acute{\beta}(X) + \gamma(\xi)S(X,V) \\ &\quad -2n\beta^2\sigma(V)\eta(X) + \beta^2\eta(X)P(V) - \beta^2\eta(V)P(X). \end{aligned} \tag{20}$$

Putting $V = \xi$ in (20) and by virtue of (6) and (13), we get

$$\begin{aligned} 0 &= 2n\beta^2\alpha(X) + \beta^2\acute{\beta}(\xi)\eta(X) + \beta^2\acute{\beta}(X) - 2n\beta^2\gamma(\xi)\eta(X) \\ &\quad -2n\beta^2\sigma(\xi)\eta(X) + \beta^2\eta(X)P(\xi) + \beta^2 P(X). \end{aligned} \tag{21}$$

Now taking $X = \xi$ in (20), we have

$$0 = -2n\beta^2\alpha(\xi)\eta(V) - 2n\beta^2\gamma(\xi)\eta(V) + 2n\beta^2\sigma(V) - \beta^2 P(V) - \beta^2\eta(V)P(\xi). \tag{22}$$

Replacing V with X in (22) and summing with (21), in view of (19), we find

$$\begin{aligned} 0 &= 2n\beta^2\sigma(X) + 2n\beta^2\alpha(X) + \beta^2\acute{\beta}(\xi)\eta(X) \\ &\quad +\beta^2\acute{\beta}(X) - 2n\beta^2\gamma(\xi)\eta(X). \end{aligned} \tag{23}$$

Now putting $X = \xi$ in (17) we have

$$\begin{aligned} 0 &= -2n\beta^2\sigma(\xi)\eta(Z) - 2n\beta^2\alpha(\xi)\eta(Z) - \beta^2\acute{\beta}(Z) \\ &\quad -\beta^2\acute{\beta}(\xi)\eta(Z) + 2n\beta^2\gamma(Z). \end{aligned} \tag{24}$$

Replacing Z with X in (24) and taking the summation with (23), we have

$$0 = 2n\beta^2[\alpha(X) + \sigma(X) + \gamma(X)] - 2n\beta^2\eta(X)[\alpha(\xi) + \sigma(\xi) + \gamma(\xi)].$$

So in view of (19) we obtain $[\alpha(X) + \gamma(X) + \sigma(X)] = 0$ for all X on M^{2n+1}. Hence we can state the following:

Theorem 1 *In a weakly symmetric Lorentzian β-Kenmotsu manifold M^{2n+1}, the sum of 1-forms α, γ, and σ is zero everywhere.*

Suppose that M is a weakly Ricci-symmetric Lorentzian β-Kenmotsu manifold. Replacing Z with ξ in (2) and using (13) we have

$$(\nabla_X S)(Y,\xi) = -2n\beta^2[\rho(X)\eta(Y) + \mu(Y)\eta(X)] + \upsilon(\xi)S(X,Y). \tag{25}$$

In view of (25) and (15) we obtain

$$\begin{aligned} & -2n\beta^2 g(X,Y) - \beta S(Y,X) \\ = & -2n\beta^2[\rho(X)\eta(Y) + \mu(Y)\eta(X)] + \upsilon(\xi)S(X,Y). \end{aligned} \tag{26}$$

Taking $X = Y = \xi$ in (26) and by the use of (6) and (13) we get

$$0 = 2n\beta^2[\rho(\xi) + \mu(\xi) + \upsilon(\xi)], \tag{27}$$

which gives (since $2n\beta^2 \neq 0$)

$$\rho(\xi) + \mu(\xi) + \upsilon(\xi) = 0. \tag{28}$$

Now putting $X = \xi$ in (26) we have by virtue of (6) and (13) that

$$0 = -2n\beta^2\eta(Y)[\rho(\xi) + \upsilon(\xi)] + 2n\beta^2\mu(Y). \tag{29}$$

Using (28) this yields

$$0 = 2n\beta^2[\mu(\xi)\eta(Y) + \mu(Y)], \tag{30}$$

which gives (since $2n\beta^2 \neq 0$)

$$\mu(Y) = -\mu(\xi)\eta(Y). \tag{31}$$

Similarly taking $Y = \xi$ in (26) we also have

$$\rho(X) = \eta(X)[\mu(\xi) + \upsilon(\xi)]. \tag{32}$$

Hence applying (28) into the last equation we find

$$\rho(X) = -\rho(\xi)\eta(X). \tag{33}$$

Since $(\nabla_\xi S)(\xi, X) = 0$, from (2) we obtain

$$\eta(X)[\rho(\xi) + \mu(\xi)] = \upsilon(X). \tag{34}$$

By making use of (28) the last equation reduces to

$$\upsilon(X) = -\upsilon(\xi)\eta(X). \tag{35}$$

Therefore replacing Y with X in (31) and by the summation of the equations (31), (33) and (35) we obtain

$$\rho(X) + \mu(X) + \upsilon(X) = -[\rho(\xi) + \mu(\xi) + \upsilon(\xi)]\eta(X). \tag{36}$$

In view of (28) it follows that

$$\rho(X) + \mu(X) + \upsilon(X) = 0, \tag{37}$$

for all X, which implies $\rho + \mu + \upsilon = 0$ on M^{2n+1}.
Hence we can state:

Theorem 2 *In a weakly Ricci symmetric Lorentzian β-Kenmotsu manifold M^{2n+1}, the sum of 1-forms ρ, μ, and υ is zero everywhere.*

4.On special weakly Ricci symmetric Lorentzian β-Kenmotsu manifold

Taking cyclic sum of (4), we get

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\ = 4[\alpha(X)S(Y,Z) + \alpha(Y)S(Z,X) + \alpha(Z)S(X,Y)]. \tag{38}$$

Let M^{2n+1} admits a cyclic Ricci tensor. Then (38) reduces to

$$\alpha(X)S(Y,Z) + \alpha(Y)S(Z,X) + \alpha(Z)S(X,Y) = 0. \tag{39}$$

Taking $Z = \xi$ in (39) and then using (5) and (13), we get

$$-2n\beta^2[\alpha(X)\eta(Y) + \alpha(Y)\eta(X)] + \eta(\rho)S(X,Y) = 0. \tag{40}$$

Again, taking $Y = \xi$ in (40) and then using (5), (6) and (13), we get

$$2\eta(\rho)\eta(X) = \alpha(X). \tag{41}$$

Taking $X = \xi$ in (41) and using (5) and (6), we get

$$\eta(\rho) = 0. \tag{42}$$

Using (42) in (41), we have $\alpha(X) = 0$, $\forall X$.

This leads us to the following:

Theorem 3 *If a special weakly Ricci symmetric Lorentzian β-Kenmotsu manifold M^{2n+1} admits a cyclic Ricci tensor then the 1-form α must vanish.*

For an Einstein manifold, $(\nabla_X S)(Y,Z) = 0$ and $S(Y,Z) = kg(Y,Z)$, then (4) gives

$$2\alpha(X)g(Y,Z) + \alpha(Y)g(X,Z) + \alpha(Z)g(Y,Z) = 0. \tag{43}$$

Taking $Z = \xi$ in (43) and then using (5) and (6), we get

$$2\alpha(X)\eta(Y) + \alpha(Y)\eta(X) + \eta(\rho)g(X,Y) = 0. \tag{44}$$

Again, taking $X = \xi$ in (44) and using (5) and (6), we get

$$3\eta(\rho)\eta(Y) = \alpha(Y). \tag{45}$$

Taking $Y = \xi$ in (45) and using (5) and (6), we get

$$\eta(\rho) = 0. \tag{46}$$

Using (46) in (45), we get $\alpha(Y) = 0$, $\forall Y$. Thus we can state the following:

Theorem 4 *A special weakly Ricci symmetric Lorentzian β-Kenmotsu manifold M^{2n+1} can not be an Einstein manifold if the 1-form $\alpha \neq 0$.*

Next taking $Z = \xi$ in (4), we have

$$(\nabla_X S)(Y,\xi) = 2\alpha(X)S(Y,\xi) + \alpha(Y)S(X,\xi) + \alpha(\xi)S(Y,X). \tag{47}$$

The left-hand side can be written in the form

$$(\nabla_X S)(Y,\xi) = XS(Y,\xi) - S(\nabla_X Y,\xi) - S(Y,\nabla_X \xi). \tag{48}$$

In view of (5),(7), (10), (13) equation (47) becomes

$$\begin{aligned} & 2n\beta^2\beta g(X,Y) + \beta S(X,Y) \\ = \ & 2.2n\beta^2\alpha(X)\eta(Y) + 2n\beta^2\alpha(Y)\eta(X) + \eta(\rho)S(Y,X). \end{aligned} \tag{49}$$

Taking $Y = \xi$ in (49) and then using (5), (6) and (13), we get

$$\alpha(X) = 0. \tag{50}$$

Using (50) in (4), we get $(\nabla_X S)(Y,Z) = 0$, Thus we have the following:

Theorem 5 *A special weakly Ricci symmetric Lorentzian β-Kenmotsu manifold M^{2n+1} is an Einstein manifold.*

References

[1] C.S. Bagewadi, D.G. Prakasha and N.S. Basavarajappa, *On Lorentzian β-Kenmotsu manifolds,* Int. Jour. Math. Analysis., 19(2) (2008), 919-927.

[2] C.S. Bagewadi and E. Girish Kumar, *Note on Trans sasakian manifolds,* Tensor.N.S., 65(1)(2004), 80- 88.

[3] C.S. Bagewadi, D.G. Prakasha and Venkatesha, *On Lorentzian α-Sasakian Manifolds,* Communicated.

[4] M.C. Chaki, *On pseudo symmetric manifolds,* An. Stiint. Univ.Al.I.Cuza Iasi Sect.I a Mat., 33(1) (1987), 53-58.

[5] M.C. Chaki, *On pseudo Ricci symmetric manifolds,* Bulgar.J.Phys., 15(6) (1998), 526-531.

[6] U.C. De, T.Q.Binh and A.A.Shaikh, *On weakly symmetric and weakly Ricci-symmetric K-contact manifolds,* Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis., 16 (2000), 65-71.

[7] Q. Khan, *On Conharmonically and special weakly ricci symmetric sasakian Manifolds,* Novi Sad J. Math., 34(1),(2004), 71-77.

[8] K. Matsumoto, *On Lorentzian paracontact manifolds,* Bull. of Yamagata Univ. Nat. Sci., 12(2)(1989), 151-156.

[9] K. Matsumoto and I. Mihai, *On a certain transformation in a Lorentzian para- Sasakian manifold,* Tensor,N.S., 47(1988),189-197.

[10] C. Ozgur, *On weak symmetries of Lorentzian para-Sasakian manifolds,* Radovi Mathematicki., 11 (2002/03), 263-270.

[11] H. Singh and Q. Khan, *On special weakly symmetric Riemannian manifolds,* Publ.Debrecen, Hungary., 58(3)(2001),523-536.

[12] L. Tamassy and T.Q. Binh, *On weakly symmetric and weakly projective symmetric Riemannian manifolds,* Coll.Math.Soc.J.Bolyai., 56(1992), 663-670.

[13] L. Tamassy and T.Q. Binh, *On weak symmetries of Einstein and Sasakian manifolds,* Tensor,N.S., 53(1993), 140-148.

G.T. Sreenivasa, Venkatesha, C.S. Bagewadi
Department of P.G. Studies and Research in Mathematics,
Kuvempu University,
Shankaraghatta - 577 451,
Shimoga, Karnataka, INDIA.
email:*sreenivasgt@gmail.com, vensprem@gmail.com, csbagewadi@gmail.com*

K.Naganagoud
Department of Mathematics,
Sri Siddhartha Institute of Technology,
Tumkur - 572 105,
Karnataka, INDIA.
email:*kngoud15@gmail.com.*