CERTAIN CLASSES OF ANALYTIC FUNCTIONS OF COMPLEX ORDER

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Abstract.By making use of the Hadamard product, we define a new class of analytic functions of complex order. Coefficient inequalities, sufficient condition and an interesting subordination result are obtained.

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1. Introduction, Definitions And Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad a_k \geq 0, \tag{1}$$

which are analytic in the open disc $\mathcal{U} = \{z \in \mathbb{C} \setminus \mid z \mid < 1\}$ and $\mathcal{S}$ be the class of function $f \in \mathcal{A}$ which are univalent in $\mathcal{U}$.

The Hadamard product of two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is given by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k \, b_k \, z^k. \tag{2}$$

For a fixed function $g \in \mathcal{A}$ defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad (b_k \geq 0 \text{ for } k \geq 2) \tag{3}$$

We now define the following linear operator $D_{\lambda,g}^m f : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$D_{\lambda,g}^0 f(z) = (f * g)(z),$$

$$D^1_{\lambda,g}f(z) = (1-\lambda)(f(z)*g(z)) + \lambda\, z(f(z)*g(z))', \tag{4}$$

$$D^m_{\lambda,g}f(z) = D^1_{\lambda,g}(D^{m-1}_{\lambda,g}f(z)). \tag{5}$$

If $f \in \mathcal{A}$, then from (4) and (5) we may easily deduce that

$$D^m_{\lambda,g}f(z) = z + \sum_{k=2}^{\infty}\big[1+(k-1)\lambda\big]^m a_k\, b_k\, \frac{z^k}{(k-1)!}, \tag{6}$$

where $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\lambda \geq 0$.

Remark 1. It is interesting to note that several integral and differential operator follows as a special case of $D^m_{\lambda,g}f(z)$, here we list few of them.

1. When $g(z) = z/(1-z)$, $D^m_{\lambda,g}f(z)$ reduces to an operator introduced recently by F. Al-Oboudi [1].

2. Let the coefficients b_k be of the form

$$b_k = \frac{(\alpha_1)_{k-1}\ \ldots\ (\alpha_q)_{k-1}}{(\beta_1)_{k-1}\ \ldots\ (\beta_s)_{k-1}\,(k-1)!} \tag{7}$$

$(\alpha_1, \ldots\ \alpha_q,\ \beta_1, \ldots\ \beta_s \in \mathbb{C}$ and $\beta_j \neq 0, -1, -2, \ldots$ for j={1,2,...,s})

and if $m = 0$, then $D^m_{\lambda,g}f(z)$ reduces to the well-known Dziok-Srivastava operator. It is well known that Dziok-Srivastava operator includes as its special cases various other linear operator which were introduced and studied by Hohlov, Carlson and Shaffer and Ruscheweyh. For details we refer to [6, 7, 8]

Apart from these, the operator $D^m_{\lambda,g}f(z)$ generalizes the well-known operators like Sălăgean operator [14] and Bernardi-Libera-Livingston operator.

Using the operator $D^m_{\lambda,g}f(z)$, we define $\mathcal{H}^m_\lambda(g,\delta;A,B)$ to be the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$1 + \frac{1}{\delta}\left(\frac{D^{m+1}_{\lambda,g}f(z)}{D^m_{\lambda,g}f(z)} - 1\right) \prec \frac{1+Az}{1+Bz}, \quad z \in \mathcal{U}, \tag{8}$$

where $\delta \in \mathbb{C}\setminus\{0\}$, A and B are arbitrary fixed numbers, $-1 \leq B < A \leq 1$, $m \in \mathbb{N}_0$.

Remark 2. Several interesting well-known and new subclasses of analytic functions can be obtained by specializing the parameters in $\mathcal{H}^m_\lambda(g,\delta;A,B)$. Here we list a few of them.

1. If we let $g(z) = \dfrac{z}{1-z}$, $\lambda = 1$ in (8), then the class $\mathcal{H}_\lambda^m(g, \delta; A, B)$ reduces to the well- known class
$$\mathcal{H}^m(\delta; A, B) := \left\{ f : 1 + \frac{1}{\delta}\left(\frac{D^{m+1}f(z)}{D^m f(z)} - 1 \right) \prec \frac{1+Az}{1+Bz} \right\}$$
where $D^m f$ is the well- known Sălăgean operator. The class $\mathcal{H}^m(\delta; A, B)$ was introduced and studied by Attiya [4].

2. If we let $g(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{k-1} \, \ldots \, (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \, \ldots \, (\beta_s)_{k-1} \, (k-1)!} a_k z^k$ in (8), then we have following class of functions
$$\mathcal{H}_\lambda^m(\alpha_1, \beta_1; A, B) := \left\{ f : 1 + \frac{1}{\delta}\left(\frac{D^{m+1}(\alpha_1, \beta_1)f(z)}{D^m(\alpha_1, \beta_1)f(z)} - 1 \right) \prec \frac{1+Az}{1+Bz} \right\}$$
introduced and studied by C. Selvaraj and K.R.Karthikeyan in [17].

3. If we let $g(z) = \dfrac{z}{1-z}$, $\lambda = 1$, $m = 0$ in (8), then the class $\mathcal{H}_\lambda^m(g, \delta; A, B)$ reduces to the well- known class of starlike functions of complex order [12].

Further, we note that several subclass of analytic functions can be obtained by specializing the parameters in $\mathcal{H}_\lambda^m(g, \delta; A, B)$ (see for example [3, 11, 12]).

We use Ω to denote the class of bounded analytic functions $w(z)$ in $\mathcal{U}$ which satisfy the conditions $w(0) = 1$ and $| w(z) | < 1$ for $z \in \mathcal{U}$.

2.Coefficient estimates

Theorem 1. *Let the function $f(z)$ defined by (1) be in the class $\mathcal{H}_\lambda^m(g, \delta; A, B)$.*

(a) If $(A-B)^2 \mid \delta \mid^2 > (k-1)\big\{2B(A-B)\lambda \, Re\{\delta\} + (1-B^2)\lambda^2(k-1)\big\}$, let
$$G = \frac{(A-B)^2 \mid \delta \mid^2}{(k-1)\big\{2B(A-B)\lambda \, Re\{\delta\} + (1-B^2)\lambda^2(k-1)\big\}}, \quad k = 2, 3, \ldots, m-1,$$
$M = [G]$ (Gauss symbol) and $[G]$ is the greatest integer not greater than G. Then, for $j = 2, 3, \ldots, M+2$
$$| a_j | \leq \frac{1}{[1+(j-1)\lambda]^m \, \lambda^{j-1} \, (j-1)! b_j} \prod_{k=2}^{j} | (A-B)\delta - (k-2)B | \qquad (9)$$

and for $j > M+2$

$$| a_j |\leq \frac{1}{[1+(j-1)\lambda]^m\,\lambda^{j-1}(j-1)\,(M+1)!\,b_j}\prod_{k=2}^{M+3} | (A-B)\delta-(k-2)B | \,. \tag{10}$$

(b) If $(A-B)^2 \mid \delta \mid^2 \leq (k-1)\big\{2B(A-B)\lambda\, Re\{\delta\}+(1-B^2)\lambda^2(k-1)\big\}$*, then*

$$| a_j |\leq \frac{(A-B)\mid \delta \mid}{\lambda\,(j-1)\,[1+(j-1)\lambda]^m\,b_j} \quad j\geq 2. \tag{11}$$

The bounds in (9) and (11) are sharp for all admissible A, B, $\delta \in \mathbb{C}\setminus\{0\}$ *and for each* j.

Proof. Since $f(z)\in \mathcal{H}^m_\lambda(g,\delta;A,B)$, the inequality (8) gives

$$| D^{m+1}_{\lambda,\,g}f(z)-D^m_{\lambda,\,g}f(z) |=\big\{[(A-B)\delta+B]D^m_{\lambda,\,g}f(z)-BD^{m+1}_{\lambda,\,g}f(z)\big\}w(z). \tag{12}$$

Equation (12) may be written as

$$\sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\lambda(k-1)\,b_ka_kz^k \tag{13}$$

$$=\left\{(A-B)\delta z+\sum_{k=2}^{\infty}[(A-B)\delta-B(k-1)\lambda][1+(k-1)\lambda]^m b_k\,a_kz^k\right\}w(z).$$

Or equivalently

$$\sum_{k=2}^{j}[1+(k-1)\lambda]^m\lambda(k-1)\,b_ka_kz^k+\sum_{k=j+1}^{\infty}c_kz^k$$

$$=\left\{(A-B)\delta z+\sum_{k=2}^{j-1}[(A-B)\delta-B(k-1)\lambda][1+(k-1)\lambda]^m b_k\,a_kz^k\right\}w(z), \tag{14}$$

for certain coefficients c_k. Explicitly $c_k=[1+(k-1)\lambda]^m\lambda(k-1)b_ka_k-[(A-B)\delta-B(k-2)\lambda][1+(k-2)\lambda]^m b_{k-1}a_{k-1}z^{-1}$. Since $| w(z) |< 1$, we have

$$\left|\sum_{k=2}^{j}[1+(k-1)\lambda]^m\lambda(k-1)\,b_ka_kz^k+\sum_{k=j+1}^{\infty}c_kz^k\right| \tag{15}$$

$$\leq \left|(A-B)\delta z + \sum_{k=2}^{j-1}[(A-B)\delta - B(k-1)\lambda][1+(k-1)\lambda]^m b_k\, a_k z^k\right|.$$

Let $z = re^{i\theta}$, $r < 1$, applying the Parseval's formula (see [9] p.138) on both sides of the above inequality and after simple computation , we get

$$\sum_{k=2}^{j}[1+(k-1)\lambda]^{2m}\lambda^2(k-1)^2\, b_k^2 \mid a_k \mid^2 r^{2k} + \sum_{k=j+1}^{\infty} \mid c_k \mid^2 r^{2k}$$

$$\leq (A-B)^2 \mid \delta \mid^2 r^2 + \sum_{k=2}^{j-1} \mid (A-B)\delta - B(k-1)\lambda \mid^2 [1+(k-1)\lambda]^{2m} b_k^2 \mid a_k \mid^2 r^{2k}.$$

Let $r \longrightarrow 1^-$, then on some simplification we obtain

$$[1+(j-1)\lambda]^{2m}\lambda^2(j-1)^2\, b_j^2 \mid a_j \mid^2 \leq (A-B)^2 \mid \delta \mid^2$$

$$+\sum_{k=2}^{j-1}\left\{ \mid (A-B)\delta - B(k-1)\lambda \mid^2 - (k-1)^2\lambda^2\right\}[1+(k-1)\lambda]^{2m} b_k^2 \mid a_k \mid^2 \quad j \geq 2. \tag{16}$$

Now the following two cases arises:

(a) Let $(A-B)^2 \mid \delta \mid^2 > (k-1)\lambda\big\{2B(A-B)Re(\delta) + (1-B^2)\lambda(k-1)\big\}$, suppose that $j \leq M+2$, then for $j = 2$, (16) gives

$$\mid a_2 \mid \leq \frac{(A-B) \mid \delta \mid}{(1+\lambda)^m \lambda\, b_2},$$

which gives (9) for $j = 2$. We establish (9) for $j \leq M+2$, from (16), by mathematical induction. Suppose (9) is valid for $j = 2, 3, \ldots, (k-1)$. Then it follows from (16)

$$[1+(j-1)\lambda]^{2m}\lambda^2(j-1)^2\, b_j^2 \mid a_j \mid^2$$

$$\leq (A-B)^2 \mid \delta \mid^2$$
$$+\sum_{k=2}^{j-1}\left\{ \mid (A-B)\delta - B(k-1)\lambda \mid^2 - (k-1)^2\lambda^2\right\}[1+(k-1)\lambda]^{2m} b_k^2 \mid a_k \mid^2$$
$$\leq (A-B)^2 \mid \delta \mid^2$$
$$+\sum_{k=2}^{j-1}\left\{ \mid (A-B)\delta - B(k-1)\lambda \mid^2 - (k-1)^2\lambda^2\right\}[1+(k-1)\lambda]^{2m} b_k^2$$

$$\times\left\{\frac{1}{[1+(k-1)\lambda]^{2m}b_k^2\{\lambda^{k-1}(k-1)!\}^2}\prod_{n=2}^{k}\mid (A-B)\delta-(n-2)B\mid^2\right\}$$

$$=(A-B)^2\mid\delta\mid^2+\sum_{k=2}^{j-1}\left\{\mid (A-B)\delta-B(k-1)\lambda\mid^2-(k-1)^2\lambda^2\right\}$$

$$\times\left\{\frac{1}{\{\lambda^{k-1}(k-1)!\}^2}\prod_{n=2}^{k}\mid (A-B)\delta-(n-2)B\mid^2\right\}$$

$$=(A-B)^2\mid\delta\mid^2+$$

$$\left(\mid (A-B)\delta-B\lambda\mid^2-\lambda^2\right)\frac{1}{\lambda^2\,(1!)^2}(A-B)^2\mid\delta\mid^2+$$

$$\left(\mid (A-B)\delta-2B\lambda\mid^2-4\lambda^2\right)\frac{1}{\lambda^4\,(2!)^2}\left((A-B)^2\mid\delta\mid^2\mid (A-B)\delta-B\lambda\mid^2\right)$$

$$+\ldots\text{up to}\,(k=j-1)$$

$$=\frac{1}{\{\lambda^{j-2}(j-2)!\}^2}\prod_{k=2}^{j}\mid (A-B)\delta-(k-2)B\mid^2.$$

Thus, we get

$$\mid a_j\mid\le\frac{1}{[1+(j-1)\lambda]^m\,\lambda^{j-1}\,(j-1)!b_j}\prod_{k=2}^{j}\mid (A-B)\delta-(k-2)B\mid,$$

which completes the proof of (9).

Next, we suppose $j>M+2$. Then (16) gives

$$[1+(j-1)\lambda]^{2m}\lambda^2(j-1)^2\,b_j^2\mid a_j\mid^2\le(A-B)^2\mid\delta\mid^2$$

$$+\sum_{k=2}^{M-2}\left\{\mid (A-B)\delta-B(k-1)\lambda\mid^2-(k-1)^2\lambda^2\right\}[1+(k-1)\lambda]^{2m}b_k^2\mid a_k\mid^2$$

$$+\sum_{k=M+3}^{j-1}\left\{\mid (A-B)\delta-B(k-1)\lambda\mid^2-(k-1)^2\lambda^2\right\}[1+(k-1)\lambda]^{2m}b_k^2\mid a_k\mid^2.$$

On substituting upper estimates for a_2, a_3, $\ldots$, a_{M+2} obtained above and simplifying, we obtain (10).

(b) Let $(A-B)^2 \mid \delta \mid^2 \leq (k-1)\lambda\{2B(A-B)Re(\delta)+(1-B^2)\lambda(k-1)\}$, then it follows from (16)

$$[1+(j-1)\lambda]^{2m}\lambda^2(j-1)^2\, b_j^2 \mid a_j \mid^2 \leq (A-B)^2 \mid \delta \mid^2 \qquad (j \geq 2),$$

which proves (11).

The bounds in (9) are sharp for the functions $f(z)$ given by

$$D^m_{\lambda,\,g}f(z) = \begin{cases} z(1+Bz)^{\frac{(A-B)\delta}{B}} & if\, B \neq 0, \\ z\, exp(A\delta z) & if\, B = 0. \end{cases}$$

Also, the bounds in (11) are sharp for the functions $f_k(z)$ given by

$$D^m_{\lambda,\,g}f_k(z) = \begin{cases} z(1+Bz)^{\frac{(A-B)\delta}{B\lambda(k-1)}} & if\, B \neq 0, \\ z\, exp\left(\frac{A\delta}{\lambda(k-1)}\, z^{k-1}\right) & if\, B = 0. \end{cases}$$

Remark 2. If we let $\lambda = 1, g(z) = \dfrac{z}{1-z}$ in Theorem 1, we get the result due to Attiya [4].

3. A sufficient condition for a function to be in $\mathcal{H}^m_\lambda(g,\delta;\, A,B)$

Theorem 2. *Let the function $f(z)$ defined by (1) and let*

$$\sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\{(k-1)+\mid (A-B)\delta - B(k-1) \mid\}\lambda\, b_k \mid a_k \mid \leq (A-B) \mid \delta \mid \quad (17)$$

holds, then $f(z)$ belongs to $\mathcal{H}^m_\lambda(g,\delta;\, A,B)$.

Proof. Suppose that the inequality holds. Then we have for $z \in \mathcal{U}$

$$\mid D^{m+1}_{\lambda,\,g}f(z) - D^m_{\lambda,\,g}f(z) \mid - \big|(A-B)\delta D^m_{\lambda,\,g}f(z) - B\big[D^{m+1}_{\lambda,\,g}f(z) - D^m_{\lambda,\,g}f(z)\big]\big|$$

$$= \left|\sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\lambda(k-1)\, b_k\, a_k z^k\right| - \left|(A-B)\delta\left[z+\sum_{k=2}^{\infty}[1+(k-1)\lambda]^m b_k a_k z^k\right] - B\sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\lambda(k-1)b_k a_k z^k\right|$$

$$\leq \sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\big\{(k-1)\lambda+\mid (A-B)\delta-B(k-1)\lambda\mid\big\}b_k\mid a_k\mid r^k-(A-B)\mid\delta\mid r.$$

Letting $r \longrightarrow 1^-$, then we have

$$\mid D_{\lambda,g}^{m+1}f(z)-D_{\lambda,g}^{m}f(z)\mid -\Big|(A-B)\delta D_{\lambda,g}^{m}f(z)- B\big[D_{\lambda,g}^{m+1}f(z)-D_{\lambda,g}^{m}f(z)\big]\Big|$$

$$\leq \sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\big\{(k-1)\lambda+\mid (A-B)\delta-B(k-1)\lambda\mid\big\}b_k\mid a_k\mid -(A-B)\mid\delta\mid\leq 0.$$

Hence it follows that

$$\frac{\left|\frac{D_{\lambda,g}^{m+1}f(z)}{D_{\lambda,g}^{m}f(z)}-1\right|}{\left|B\left[\frac{D_{\lambda,g}^{m+1}f(z)}{D_{\lambda,g}^{m}f(z)}-1\right]-(A-B)\delta\right|}<1,\quad z\in\mathcal{U}.$$

Letting

$$w(z)=\frac{\frac{D_{\lambda,g}^{m+1}f(z)}{D_{\lambda,g}^{m}f(z)}-1}{B\left[\frac{D_{\lambda,g}^{m+1}f(z)}{D_{\lambda,g}^{m}f(z)}-1\right]-(A-B)\delta},$$

then $w(0)=0$, $w(z)$ is analytic in $\mid z\mid<1$ and $\mid w(z)\mid<1$. Hence we have

$$\frac{D_{\lambda,g}^{m+1}f(z)}{D_{\lambda,g}^{m}f(z)}=\frac{1+[B+\delta(A-B)]w(z)}{1+Bw(z)}$$

which shows that f belongs to $\mathcal{H}_{\lambda}^{m}(g,\delta;\,A,B)$.

3. Subordination Results For the Class $\mathcal{H}_{\lambda}^{m}(g,\delta;\,A,B)$

Definition 1. *A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever $f(z)$ is analytic, univalent and convex in $\mathcal{U}$, we have the subordination given by*

$$\sum_{k=1}^{\infty}b_k\,a_kz^k\prec f(z)\qquad(z\in\mathcal{U},\,a_1=1).\tag{18}$$

Lemma 1. *The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re\left\{1+2\sum_{k=1}^{\infty} b_k z^k\right\} > 0 \qquad (z \in \mathcal{U}). \tag{19}$$

For convenience, we shall henceforth

$$\begin{aligned}&\sigma_k(\delta,\, \lambda,\, m,\, A, B) \\ &\quad = [1+(k-1)\lambda]^m\, \lambda\big\{(k-1) + \mid (A-B)\delta - B(k-1) \mid \big\} b_k\end{aligned} \tag{20}$$

to be real.

Let $\widetilde{\mathcal{H}}_{\lambda}^{m}(g, \delta;\, A, B)$ denote the class of functions $f \in \mathcal{A}$ whose coefficients satisfy the conditions (17). We note that $\widetilde{\mathcal{H}}_{\lambda}^{m}(g, \delta;\, A, B) \subseteq \mathcal{H}_{\lambda}^{m}(g, \delta;\, A, B)$.

Theorem 3. *Let the function $f(z)$ defined by (1) be in the class $\widetilde{\mathcal{H}}_{\lambda}^{m}(g, \delta;\, A, B)$ where $-1 \le B < A \le 1$. Also let $\mathcal{C}$ denote the familiar class of functions $f \in \mathcal{A}$ which are also univalent and convex in $\mathcal{U}$. Then*

$$\frac{\sigma_2(\delta,\, \lambda,\, m,\, A, B)}{2\big[(A-B)|\delta| + \sigma_2(\delta,\, \lambda,\, m,\, A, B)\big]}(f * g)(z) \prec g(z) \qquad (z \in \mathcal{U};\, g \in \mathcal{C}), \tag{21}$$

and

$$\mathfrak{Re}(f(z)) > -\frac{(A-B)|\delta| + \sigma_2(\delta,\, \lambda,\, m,\, A, B)}{\sigma_2(\delta,\, \lambda,\, m,\, A, B)} \qquad (z \in \mathcal{U}). \tag{22}$$

The constant $\dfrac{\sigma_2(\delta,\, \lambda,\, m,\, A, B)}{2\big[(A-B)|\delta| + \sigma_2(\delta,\, \lambda,\, m,\, A, B)\big]}$ *is the best estimate.*

Proof. Let $f(z) \in \widetilde{\mathcal{H}}_{\lambda}^{m}(\delta;\, A, B)$ and let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{C}$. Then

$$\begin{aligned}&\frac{\sigma_2(\delta,\, \lambda,\, m,\, A, B)}{2\big[(A-B)|\delta| + \sigma_2(\delta,\, \lambda,\, m,\, A, B)\big]}(f * g)(z) \\ &\qquad = \frac{\sigma_2(\delta,\, \lambda,\, m,\, A, B)}{2\big[(A-B)|\delta| + \sigma_2(\delta,\, \lambda,\, m,\, A, B)\big]}\left(z + \sum_{k=2}^{\infty} a_k b_k z^k\right).\end{aligned}$$

Thus, by Definition 1, the assertion of the theorem will hold if the sequence

$$\left\{\frac{\sigma_2(\delta,\, \lambda,\, m,\, A, B)}{2\big[(A-B)|\delta| + \sigma_2(\delta,\, \lambda,\, m,\, A, B)\big]} a_k\right\}_{k=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this will be true if and only if

$$\mathfrak{Re}\left\{1 + 2\sum_{k=1}^{\infty} \frac{\sigma_2(\delta,\, \lambda,\, m,\, A, B)}{2\big[(A-B)|\delta| + \sigma_2(\delta,\, \lambda,\, m,\, A, B)\big]} a_k z^k\right\} > 0 \qquad (z \in \mathcal{U}). \tag{23}$$

Now

$$\mathfrak{Re}\left\{1+\frac{\sigma_2(\delta,\,\lambda,\,m,\,A,B)}{(A-B)|\delta|+\sigma_2(\delta,\,\lambda,\,m,\,A,B)}\sum_{k=1}^{\infty}a_k\,z^k\right\}$$

$$=\mathfrak{Re}\left\{1+\frac{\sigma_2(\delta,\,\lambda,\,m,\,A,B)}{(A-B)|\delta|+\sigma_2(\delta,\,\lambda,\,m,\,A,B)}a_1 z\right.$$

$$\left.+\frac{1}{(A-B)|\delta|+\sigma_2(\delta,\lambda,\,m,\,A,B)}\sum_{k=2}^{\infty}\sigma_2(\delta,\,\lambda,\,m,\,A,B)a_k z^k\right\}$$

$$\geq 1-\left\{\left|\frac{\sigma_2(\delta,\,\lambda,\,m,\,A,B)}{(A-B)|\delta|+\sigma_2(\delta,\,\lambda,\,m,\,A,B)}\right|r\right.$$

$$\left.+\frac{1}{|(A-B)|\delta|+\sigma_2(\delta,\,\lambda,\,m,\,A,B)|}\sum_{k=2}^{\infty}\sigma_k(\delta,\,\lambda,\,m,\,A,B)|a_k|r^k\right\}.$$

Since $\sigma_k(\delta,\,\lambda,\,m,\,A,B)$ is a real increasing function of k $(k\geq 2)$

$$1-\left\{\left|\frac{\sigma_2(\delta,\,\lambda,\,m,\,A,B)}{(A-B)|\delta|+\sigma_2(\delta,\,\lambda,\,m,\,A,B)}\right|r+\right.$$

$$\left.\frac{1}{|(A-B)|\delta|+\sigma_2(\delta,\,\lambda,\,m,\,A,B)|}\sum_{k=2}^{\infty}\sigma_k(\delta,\,\lambda,\,m,\,A,B)|a_k|r^k\right\}$$

$$>1-\left\{\frac{\sigma_2(\delta,\,\lambda,\,m,\,A,B)}{(A-B)|\delta|+\sigma_2(\delta,\,\lambda,\,m,\,A,B)}r+\frac{(A-B)|\delta|}{(A-B)|\delta|+\sigma_2(\delta,\,\lambda,\,m,\,A,B)}r\right\}$$
$$=1-r>0.$$

Thus (23) holds true in $\mathcal{U}$. This proves the inequality (21). The inequality (22) follows by taking the convex function $g(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k$ in (21). To prove the sharpness of the constant $\frac{\sigma_2(\delta,\lambda,m,A,B)}{2\left[(A-B)|\delta|+\sigma_2(\delta,\lambda,m,A,B)\right]}$, we consider $f_0(z)\in\widetilde{\mathcal{H}}_{\lambda}^{m}(g,\delta;\,A,B)$ given by

$$f_0(z)=z-\frac{(A-B)|\delta|}{\sigma_2(\delta,\,\lambda,\,m,\,A,B)}z^2\qquad(-1\leq B<A\leq 1).$$

Thus from (21), we have

$$\frac{\sigma_2(\delta,\,\lambda,\,m,\,A,B)}{2\left[(A-B)|\delta|+\sigma_2(b,\,\lambda,\,m,\,A,B)\right]}f_0(z)\prec\frac{z}{1-z}. \tag{24}$$

It can be easily verified that

$$\min\left\{Re\left(\frac{\sigma_2(\delta,\,\lambda,\,m,\,A,B)}{2\big[(A-B)|\delta|+\sigma_2(\delta,\,\lambda,\,m,\,A,B)\big]}f_0(z)\right)\right\}=-\frac{1}{2}\qquad(z\in\mathcal{U}).$$

This shows that the constant $\frac{\sigma_2(\delta,\lambda,m,A,B)}{2\big[(A-B)|\delta|+\sigma_2(\delta,\lambda,m,A,B)\big]}$ is best possible.

Remark 3. By specializing the parameters, the above result reduces to various other results obtained by several authors.

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