A COMBINED MONTE CARLO AND QUASI-MONTE CARLO METHOD WITH APPLICATIONS TO OPTION PRICING

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Abstract. In this paper, we apply a combined Monte Carlo and Quasi-Monte Carlo method, which we proposed in an earlier paper [32], to the evaluation of an European Call option and of an Asian Call option. We assume that the stock price of the underlying asset $S = S(t)$ is driven by a Lévy process $Z(t)$, with independent increments distributed according to a NIG distribution. We compare our method with the Monte Carlo and Quasi-Monte Carlo methods. The numerical results indicate that our method provides significant error reduction over these methods.

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1. Introduction

The evaluation of the European and Asian options is one of the most important problems in financial mathematics. The value of such an option can be expressed in terms of a risk-neutral expectation of a random payoff. In the case of European options, when the asset is modeled under the standard Black-Scholes assumptions, the expectation is explicitly computable. However, in general, if the log-returns of the asset prices are non-normally distributed, a closed expression for the price of an European option is not available, and so numerical methods are involved. In the case of arithmetic Asian options, even in the standard Black-Scholes framework, exact closed-form formulas for pricing such options have not been available and therefore, the price must be numerically computed.

Simulation techniques such as Monte Carlo (MC) and Quasi-Monte Carlo (QMC) methods play a key role in the evaluation of such derivatives. The first application of

MC methods in this field appeared in Boyle [2], who used simulation to estimate the value of a standard European option. Applications of the QMC method to option pricing problems can be found in [8], [14] and [17]. Majority of the work done on applying these simulation techniques to financial problems was in direction where one needs to simulate from the normal distribution.

Barndorff-Nielsen [1] was the one who proposed to model the log-returns, by using the normal inverse Gaussian (NIG) distribution, as this class of distributions has proven to fit the semi-heavily tails observed in financial time series of various kinds extremely well [7], [34]. A method for evaluating such derivatives is the one proposed by Raible [27], who considered a Fourier method to evaluate call and put options. Other alternative methods for evaluating such derivatives are the MC and QMC methods. In [15], Kainhofer proposes a QMC algorithm for generating NIG variables, based on a technique proposed by Hlawka and Mück [12, 13] for generating low-discrepancy sequences for general distributions.

In an earlier paper [32], we proposed a combined MC and QMC method, to estimate a multidimensional integral I of a function f, with respect to the probability measure induced by a distribution function G on $[0,1]^s$. Our method is based on random sampling from sequences with low G-discrepancy. Other methods that combine the ideas of MC and QMC methods and their applications to option pricing can be found in [19], [20], [22], [28], [9] and [29].

In this paper, we first recall the general setting of our combined method and give some important theoretical results. Next, we apply our method to the evaluation of an European Call option and of an Asian Call option. We assume that the stock price of the underlying asset $S = S(t)$ is driven by a Lévy process $Z(t)$, with independent increments distributed according to a NIG distribution. We compare the estimate produced by our method with the estimates given by MC and QMC methods. The numerical results indicate that our method provides significant error reduction over these methods.

2. Monte Carlo and Quasi-Monte Carlo methods

We consider an s-dimensional continuous distribution on $[0,1]^s$, with distribution function G and density function g (g is nonnegative and $\int_{[0,1]^s} g(u)du = 1$).

We consider the problem of approximating the multidimensional integral of a function $f : [0,1]^s \to \mathbb{R}$, of the form

$$I = \int_{[0,1]^s} f(x)dG(x) = \int_{[0,1]^s} f(x)g(x)dx. \tag{1}$$

Two frequently used approaches are the MC and QMC methods.

In the MC method, we generate N independent sample variables $X_1, \ldots, X_N$, with the density function g on $[0,1]^s$. The integral I is estimated by the sample mean

$$\bar{I}_{MC} = \frac{1}{N} \sum_{k=1}^{N} f(X_k).$$

The estimator $\bar{I}_{MC}$ is an unbiased estimator of the integral I. The strong law of large numbers tells us that

$$P\Big(\lim_{N\to\infty} \bar{I}_{MC} = I\Big) = 1.$$

In other words, the MC estimator converges almost surely to I, as $N \to \infty$.

The practical advantage of the MC method is that we can easily measure the accuracy of the MC estimate, by constructing confidence intervals for I. They use the sample standard deviation $\bar{\sigma}[f] = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\big(f(X_i) - \bar{I}_{MC}\big)^2}$, and are of the form [33]

$$\left(\bar{I}_{MC} - t_{N-1,1-\frac{\alpha}{2}} \frac{\bar{\sigma}[f]}{\sqrt{N}}, \quad \bar{I}_{MC} + t_{N-1,1-\frac{\alpha}{2}} \frac{\bar{\sigma}[f]}{\sqrt{N}}\right),$$

where $t_{N-1,1-\frac{\alpha}{2}}$ is the $\big(1-\frac{\alpha}{2}\big)$-th percentile of the Student's t-distribution with $N-1$ degrees of freedom, and $1-\alpha$ is a given confidence level, $\alpha \in (0,1)$.

In the MC method, by constructing confidence intervals for I, we get probabilistic error bounds of order $O\big(1/\sqrt{N}\big)$.

The QMC method can be defined by analogy with the MC method, by replacing the random samples by a sequence of "well distributed" deterministic points. This approach uses the so-called *sequences with low G-discrepancy* in $[0,1]^s$. We define these sequences, using the notions of *G-star discrepancy* and *G-discrepancy.*

Definition 1 (G-star discrepancy). *We consider a distribution on $[0,1]^s$, with distribution function G. Let λ_G be the probability measure induced by G. Let $P = (x_1, \ldots, x_N)$ be a set of points in $[0,1]^s$. The G-star discrepancy of P is defined as*

$$D^*_{N,G}(P) = D^*_{N,G}(x_1, \ldots, x_N) = \sup_{J \subseteq [0,1]^s} \left| \frac{1}{N} A_N(J,P) - \lambda_G(J) \right|,$$

where the supremum is calculated over all subintervals J of $[0,1]^s$ of the form $\prod_{i=1}^{s}[0,a_i]$, and $A_N(J,P)$ counts the number of elements of P falling into the interval J, i.e.,

$$A_N(J,P) = \sum_{k=1}^{N} 1_J(x_k),$$

where 1_J is the characteristic function of J.

Definition 2 (G-discrepancy). *Under the same conditions as in Definition 1, the G-discrepancy of $P = (x_1, \ldots, x_N)$ is defined as*

$$D_{N,G}(P) = D_{N,G}(x_1, \ldots, x_N) = \sup_{J \subseteq [0,1]^s} \left| \frac{1}{N} A_N(J, P) - \lambda_G(J) \right|,$$

where the supremum is calculated over all subintervals J of $[0,1]^s$ of the form $\prod_{i=1}^{s} [a_i, b_i]$.

The notions of G-star discrepancy and G-discrepancy are natural generalizations of the notions of star discrepancy and discrepancy, respectively, which are used in the uniform case [18].

For a sequence $P = (x_k)_{k \geq 1}$ of points in $[0,1]^s$, we write $D^*_{N,G}(P)$ for the G-star discrepancy and $D_{N,G}(P)$ for the G-discrepancy of the first N terms of sequence P.

Definition 3 (sequence of points with low G-discrepancy). *A sequence of points $P = (x_k)_{k \geq 1}$, with $x_k \in [0,1]^s$, $k \geq 1$, is said to be with low G-discrepancy if we have*

$$D_{N,G}(P) = O\left(\frac{(\log N)^s}{N} \right) \quad \text{for all } N \geq 2.$$

Sequences with low G-discrepancy are used in QMC integration to approximate the integral (1). The QMC integration formula is

$$I = \int_{[0,1]^s} f(x) dG(x) \approx \frac{1}{N} \sum_{k=1}^{N} f(x_k), \tag{2}$$

where $(x_k)_{k \geq 1}$ is a sequence with low G-discrepancy in $[0,1]^s$.

The non-uniform Koksma-Hlawka inequality gives an upper bound for the error of approximation in formula (2).

Theorem 4 (non-uniform Koksma-Hlawka inequality). *([3], [21])* *Let $f : [0,1]^s \to \mathbb{R}$ be a function of bounded variation in the sense of Hardy and Krause. We consider a distribution on $[0,1]^s$, with distribution function G. Then, for any $x_1, \ldots, x_N \in [0,1]^s$, we have*

$$\left| \frac{1}{N} \sum_{k=1}^{N} f(x_k) - \int_{[0,1]^s} f(x) dG(x) \right| \leq V_{HK}(f) D^*_{N,G}(x_1, \ldots, x_N). \tag{3}$$

Generation of sequences with low G-discrepancy in $[0,1]^s$ is the central issue in the QMC method for approximating the integral I. Several methods for generating such sequences are proposed in [10], [11], [30] and [31].

An important advantage of the QMC method is that we get deterministic upper bounds for the error of approximation. Since in QMC method we use sequences with low G-discrepancy, the approximation error is of order $\mathcal{O}((\log N)^s/N)$, which is better than the order of MC error. This is due to the fact that, for each dimension s, the inequality $(\log N)^s/N < 1/\sqrt{N}$ holds for sufficiently large N.

Nevertheless, the error bound, given by the non-uniform Koksma-Hlawka inequality, while possible in theory, is intractable in practice. This is mainly due to the difficulty of computing the factors $V_{HK}(f)$ and $D^*_{N,G}(x_1,\ldots,x_N)$.

In order to take advantage of both types of methods, in the last years several authors proposed a variety of methods, in which MC and QMC ideas are combined. We mention here the following methods: Owen's method based on (t,s)-sequences [23, 24], the method based on shifted low-discrepancy sequences [4], [35], the so-called "hybrid" method [20] based on random sampling from sequences with low-discrepancy, the method based on s-dimensional mixed sequences [19], [22], [36] and the method based on s-dimensional H-mixed sequences [28], [29] and [9].

In [32] (see also [33]), we proposed a combined MC and QMC method based on random sampling from sequences with low G-discrepancy in $[0,1]^s$. Next, we describe our method.

3. Estimation of integrals using random sampling from sequences with low G-discrepancy in $[0,1]^s$

Our combined MC and QMC method for estimating the multidimensional integral I, given by (1), consists of the following.

We consider a distribution on $[0,1]^s$, with distribution function G and density function g. We use the marginal density functions g_l, $l=1,\ldots,s$, and the marginal distribution functions G_l, $l=1,\ldots,s$, defined as follows.

Definition 5. *Consider a distribution on $[0,1]^s$, with density function g. For a point $u=\left(u^{(1)},\ldots,u^{(s)}\right)\in[0,1]^s$, the marginal density functions g_l, $l=1,\ldots,s$, are defined by*

$$g_l\left(u^{(l)}\right)=\underbrace{\int\cdots\int}_{[0,1]^{s-1}} g\left(t^{(1)},\ldots,t^{(l-1)},u^{(l)},t^{(l+1)},\ldots t^{(s)}\right)dt^{(1)}\ldots dt^{(l-1)}dt^{(l+1)}\ldots dt^{(s)},$$

and the marginal distribution functions G_l, $l = 1, \ldots, s$, are defined by

$$G_l\big(u^{(l)}\big) = \int_0^{u^{(l)}} g_l(t)dt.$$

We assume that $G(u) = \prod_{l=1}^{s} G_l(u^{(l)})$, $\forall u = \big(u^{(1)}, \ldots, u^{(s)}\big) \in [0,1]^s$ (independent marginals). Moreover, we assume that the functions G_l, $l = 1, \ldots, s$, are invertible on $[0,1]$.

Let $\Omega = \{\beta_1, \ldots, \beta_r\}$ be a space containing sets of points β_i, $i = 1, \ldots, r$, with low G-discrepancy in $[0,1]^s$, where the point set β_i, $i = 1, \ldots, r$, is of the form

$$\beta_i = (\beta_{1,i}, \ldots, \beta_{N,i}),$$

with $\beta_{k,i} = (\beta_{k,i}^{(1)}, \ldots, \beta_{k,i}^{(s)}) \in [0,1]^s$, $k = 1, \ldots, N$.

We define the random variable X_N on the space Ω as follows.

Definition 6. *([32]) For an arbitrary point set $\beta_i = (\beta_{1,i}, \ldots, \beta_{N,i}) \in \Omega$, the value of the random variable X_N is defined as*

$$X_N(\beta_i) = \frac{1}{N} \sum_{k=1}^{N} f(\beta_{k,i}),$$

and is taken with probability $1/r$.

Remark 7. *([32]) The distribution of the random variable X_N is*

$$X_N : \begin{pmatrix} \frac{1}{N} \sum_{k=1}^{N} f(\beta_{k,i}) \\ 1/r \end{pmatrix}_{\substack{\beta_i = (\beta_{1,i}, \ldots, \beta_{N,i}) \\ i = 1, \ldots, r}}.$$

Theorem 8. *([32]) The random variable X_N has the following properties:*

$$\lim_{N \to \infty} E(X_N) = I, \quad (4)$$

$$\lim_{N \to \infty} Var(X_N) = 0. \quad (5)$$

Once we have defined the random variable X_N, we select the integers $i_1, \ldots, i_M$ at random from the uniform distribution on $\{1, \ldots, r\}$, and consider the corresponding point sets $\beta_{i_1}, \ldots, \beta_{i_M}$. For each point set, we compute the value of the random variable X_N. The values $X_N(\beta_{i_1}), \ldots, X_N(\beta_{i_M})$ are values of the sample variables

$X_{N,i_1}, \ldots, X_{N,i_M}$ that are independent identically distributed random variables and have the same distribution as X_N.

We use the notation $\overline{X}_{N,M}$ for the sample mean of the random variables $X_{N,i_1}, \ldots,$ X_{N,i_M}, and $\overline{x}_{N,M}$ for its value, i.e.,

$$\overline{X}_{N,M} = \frac{X_{N,i_1} + \ldots + X_{N,i_M}}{M},$$

$$\overline{x}_{N,M} = \frac{\sum_{l=1}^{M} X_{N,i_l}(\beta_{i_l})}{M} = \frac{\sum_{l=1}^{M} \left(\frac{1}{N} \sum_{k=1}^{N} f(\beta_{k,i_l}) \right)}{M}.$$

Proposition 9. *([32]) For a fixed N, the estimator $\overline{X}_{N,M}$ has the following properties:*

$$E(\overline{X}_{N,M}) = E(X_N), \quad (\textit{unbiased estimator of } E(X_N)), \tag{6}$$

$$Var(\overline{X}_{N,M}) = \frac{Var(X_N)}{M}, \tag{7}$$

$$\lim_{M\to\infty} Var(\overline{X}_{N,M}) = 0, \tag{8}$$

$$P\Big(\lim_{M\to\infty} \overline{X}_{N,M} = E(X_N) \Big) = 1, \quad (\overline{X}_{N,M} \textit{ converges almost surely to } E(X_N)). \tag{9}$$

Proposition 10. *([32]) For a fixed M, we have the following properties of the estimator $\overline{X}_{N,M}$:*

$$\lim_{N\to\infty} E(\overline{X}_{N,M}) = I,$$

$$\lim_{N\to\infty} Var(\overline{X}_{N,M}) = 0.$$

Taking into account these properties, in our combined method the integral I is approximated by

$$I \approx \overline{x}_{N,M} = \frac{\sum_{l=1}^{M} X_{N,i_l}(\beta_{i_l})}{M} = \frac{\sum_{l=1}^{M} \left(\frac{1}{N} \sum_{k=1}^{N} f(\beta_{k,i_l}) \right)}{M}. \tag{10}$$

Hence, in our method we take a random sampling from a finite set of QMC approximations, and we consider the sample mean of that sample as an estimator for the integral I. Our combined method involves random sampling from sequences with low G-discrepancy in $[0,1]^s$ (**r**andom **s**ampling from **n**on-**u**niform sequences with low G-discrepancy). It constructs the estimator $\overline{X}_{N,M}$, which we call an RSNU estimator. We call the value $\overline{x}_{N,M}$ an RSNU estimate.

Next, we derive confidence intervals for $E(X_N)$ and then we give an important remark concerning the confidence intervals for the integral I.

We consider a given confidence level $1-\alpha$, $\alpha \in (0,1)$. We use the sample standard deviation

$$\overline{\sigma}_{X_N} = \sqrt{\frac{1}{M-1}\sum_{l=1}^{M}\left(X_{N,i_l} - \overline{X}_{N,M}\right)^2}.$$

Proposition 11. *([32]) A $(1-\alpha)\%$ confidence interval for $E(X_N)$ is*

$$\left(\overline{X}_{N,M} - t_{M-1,1-\frac{\alpha}{2}}\frac{\overline{\sigma}_{X_N}}{\sqrt{M}}, \quad \overline{X}_{N,M} + t_{M-1,1-\frac{\alpha}{2}}\frac{\overline{\sigma}_{X_N}}{\sqrt{M}}\right), \tag{11}$$

where $t_{M-1,1-\frac{\alpha}{2}}$ is the $\left(1-\frac{\alpha}{2}\right)$-th percentile of the Student's t-distribution with $M-1$ degrees of freedom.

Remark 12. *([32]) We proved that $E(X_N) \to I$, as $N \to \infty$ (property (4)). Therefore, for N sufficiently large, we consider $E(X_N) \cong I$. Consequently, for large enough values of N, the confidence interval for I is well approximated by the confidence interval for $E(X_N)$, given by (11).*

In what follows, we give deterministic upper bounds for the error of approximation in formula (10).

Theorem 13. *([32]) The error of approximation in the combined method is bounded by*

$$\left|I - \overline{x}_{N,M}\right| \leq \frac{1}{M}V_{HK}(f)\sum_{l=1}^{M} D^*_{N,G}(\beta_{i_l}).$$

Corollary 14. *([32]) For a fixed M, the RSNU estimate satisfies the following property:*

$$\lim_{N\to\infty} \overline{x}_{N,M} = I.$$

4. Application to finance: Evaluation of European Options

In the following, we apply our combined method to a problem from mathematical finance. We consider a Black-Scholes type model with one bank account B, which compounds continuously with a constant interest rate, i.e., $B(t) = B(0)e^{rt}$ and one stock, whose price $S = S(t)$ is driven by a Lévy process $Z(t)$

$$S(t) = S(0)e^{Z(t)}. \tag{12}$$

Lévy processes can be characterized by the distribution of the random variable $Z(1)$. Typical examples for the distribution of $Z(1)$ are normal inverse gaussian (NIG), hyperbolic [7], variance-gamma [16], and Meixner distribution.

According to the fundamental theory of asset pricing [6], the risk-neutral price of an S-derivative, $C(0)$, is given by

$$C(0) = e^{-rT}E^{Q}(C_T(S)), \tag{13}$$

where $C_T(S)$ is the so-called *payoff* of the derivative, which, in this setting, coincides with its value at expiration time T, and Q is an equivalent martingale measure.

In this paper, we consider exponential NIG-Lévy processes, meaning that $Z(t)$ has independent increments, distributed according to a NIG distribution. We consider the measure obtained by Esscher transform method [27], as this preserves the distribution of $Z(1)$ in the class of NIG distributions. For a comprehensive discussion of the NIG distribution and the corresponding Lévy process, we refer to [1] and [34].

Next, we evaluate by simulation the value of an European Call option. The payoff of such an option is

$$C_T(S) = \max\{S(T) - K, 0\} = (S(T) - K)_+, \tag{14}$$

where the constant K is called the strike price.

The risk-neutral price of such an option is

$$C(0) = e^{-rT}E^{Q}(\max\{S(T) - K, 0\}) = e^{-rT}E^{Q}((S(T) - K)_+). \tag{15}$$

Replacing the stock price by (12), we get

$$C(0) = e^{-rT}E^{Q}((S(0)e^{Z(T)} - K)_+). \tag{16}$$

It is known from practice that the evaluation of the stock price $S(t)$ is made at discrete times $0 = t_0 < t_1 < t_2 < \ldots < t_s = T$. We consider time intervals of equal length Δt, i.e., $t_i = t_{i-1} + \Delta t$, $i = 1, \ldots, s$. We notice that

$$X_i = Z(t_i) - Z(t_{i-1}) = Z(t_{i-1} + \Delta t) - Z(t_{i-1}) \sim Z(\Delta t), \quad i = 1, \ldots, s,$$

are independent and identically distributed NIG random variables with the same distribution as $Z(t_1)$. Dropping the discounted factor from the risk-neutral option price, we get the expected payoff under the Esscher transform measure of the European Call option

$$E^Q((S(0)e^{Z(T)} - K)_+) = E((S(0)e^{\sum_{i=1}^s X_i} - K)_+). \tag{17}$$

We want to evaluate the expected payoff (17). In order to do this, we rewrite the expectation (17) as a multidimensional integral on $\mathbb{R}^s$

$$I = \int_{\mathbb{R}^s} \underbrace{\left(S(0)e^{\sum_{i=1}^s x^{(i)}} - K\right)_+}_{E(x)} dH(x) = \int_{\mathbb{R}^s} E(x)dH(x) = \int_{\mathbb{R}^s} E(x)\prod_{i=1}^s h_i(x^{(i)})dx, \tag{18}$$

where $H(x) = \prod_{i=1}^s H_i(x^{(i)}),\ \forall x = (x^{(1)}, \ldots, x^{(s)}) \in \mathbb{R}^s$, and $H_i(x^{(i)})$ denotes the distribution function of the so-called log-returns induced by $Z(t_1)$, with the corresponding density function $h_i(x^{(i)})$. These log increments are independent and NIG distributed, having the common probability density function

$$f_{NIG}(x;\mu,\beta,\alpha,\delta) = \frac{\alpha}{\pi}\exp\left(\delta\sqrt{\alpha^2-\beta^2} + \beta(x-\mu)\right)\frac{\delta K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}}, \tag{19}$$

where $K_1(x)$ is the modified Bessel function of third type of order 1 (see [26]).

Notice that in integral (18) singularities appear on the upper integration boundary, i.e., $\lim_{x^{(i)}\to\infty} E(x) = \infty$, for $i = 1, \ldots, s$.

In order to approximate the integral (18), we transform it to an integral on $[0,1]^s$, using an integral transformation, as follows.

We consider the family of independent double-exponential distributions with the same parameter $\lambda > 0$, having the density functions $h_{\lambda,i}(x) = h_\lambda(x) = \lambda/2\exp(-\lambda|x|)$, $i = 1, \ldots, s$. The distribution functions $H_{\lambda,i} = H_\lambda : \mathbb{R} \to (0,1)$, $i = 1, \ldots, s$, are given by

$$H_\lambda(x) = \begin{cases} \frac{1}{2}e^{\lambda x} & ,x < 0 \\ 1 - \frac{1}{2}e^{-\lambda x} & ,x \geq 0, \end{cases} \tag{20}$$

and their inverses $H_{\lambda,i}^{-1} = H_\lambda^{-1} : (0,1) \to \mathbb{R},\ i = 1, \ldots, s$, are defined by

$$H_\lambda^{-1}(x) = \begin{cases} \frac{1}{\lambda}\log(2x) & ,0 < x < \frac{1}{2} \\ -\frac{1}{\lambda}\log(2-2x) & ,\frac{1}{2} \leq x < 1. \end{cases} \tag{21}$$

Next, we consider the substitutions $x^{(i)} = H_\lambda^{-1}(z^{(i)}),\ i = 1, \ldots, s$.

The integral (18) becomes

$$I = \int_{[0,1]^s}\left(S(0)e^{\sum_{i=1}^s H_\lambda^{-1}(z^{(i)})} - K\right)_+ \prod_{i=1}^s \frac{h_i(H_\lambda^{-1}(z^{(i)})}{h_\lambda(H_\lambda^{-1}(z^{(i)})}dz. \tag{22}$$

The integral I can be expressed as

$$I = \int_{[0,1]^s} \underbrace{\left(S(0)e^{\sum_{i=1}^{s} H_\lambda^{-1}(z^{(i)})} - K\right)_+}_{f(z)} dG(z) = \int_{[0,1]^s} f(z)dG(z), \tag{23}$$

where $G : (0,1)^s \to [0,1]$, defined by

$$G(z) = \prod_{i=1}^{s} (H_i \circ H_\lambda^{-1})(z^{(i)}),\ \forall z = (z^{(1)}, \dots, z^{(s)}) \in (0,1)^s, \tag{24}$$

is a distribution function on $(0,1)^s$, with independent marginals $G_i = H_i \circ H_\lambda^{-1}$, $i = 1, \dots, s$.

The integral (23) is an improper integral because the function f has singularities on the right boundary of the interval $[0,1]^s$, i.e.,

$$\lim_{z^{(i)} \to 1} f(z^{(1)}, \dots, z^{(s)}) = \infty, \tag{25}$$

for $i = 1, \dots, s$.

In the following, we compare numerically our combined method with the MC and QMC methods. As a measure of comparison, we use the absolute errors produced by these three methods in the approximation of integral (23).

The MC estimate is defined as follows:

$$\bar{I}_{MC} = \frac{1}{NM} \sum_{k=1}^{NM} f(x_k), \tag{26}$$

where $x_k = (x_k^{(1)}, \dots, x_k^{(s)})$, $k = 1, \dots, NM$, are independent identically distributed random points on $[0,1]^s$, with the common distribution function G defined in (24).

In order to generate such a point x_k, we proceed as follows. We first generate a random point $\alpha_k = (\alpha_k^{(1)}, \dots, \alpha_k^{(s)})$, where the component $\alpha_k^{(i)}$ is a point uniformly distributed on $[0,1]$, for $i = 1, \dots, s$. Then, for each component $\alpha_k^{(i)}$, $i = 1, \dots, s$, we apply the inversion method [5] and we obtain that $G_i^{-1}(\alpha_k^{(i)}) = (H_\lambda \circ H_i^{-1})(\alpha_k^{(i)})$ is a point with the distribution function G_i. As the s-dimensional distribution with the distribution function G has independent marginals, it follows that $x_k = (G_1^{-1}(\alpha_k^{(1)}), \dots, G_s^{-1}(\alpha_k^{(s)}))$ is a point with the distribution function G on $[0,1]^s$. We notice that we need to know the inverse of the distribution function of a NIG distributed random variable or, at least an approximation of it. As the inverse function is not explicitly known, an approximation of it is needed in our simulations.

In order to obtain an approximation of the inverse, we use the Matlab function "niginv" as implemented by R. Werner, based on a method proposed in [26].

The QMC estimate is defined as follows:

$$\bar{I}_{QMC} = \frac{1}{NM}\sum_{k=1}^{NM} f(x_k), \tag{27}$$

where $x = (x_k)_{k\geq 1}$ is a sequence with low G-discrepancy in $[0,1]^s$, with $x_k = (x_k^{(1)},\ldots,x_k^{(s)})$, $k \geq 1$.

In order to generate such a sequence, we apply a method proposed by Hlawka and Mück [12, 13]. Their method is based on the following result.

Theorem 15. *([11]) Consider a distribution on $[0,1]^s$, with distribution function G and density function $g(u) = \prod_{j=1}^{s} g_j(u^{(j)})$, $\forall u = \left(u^{(1)},\ldots,u^{(s)}\right) \in [0,1]^s$. Assume that the marginal density functions g_j, $j = 1,\ldots,s$, are continuous on $[0,1]$. Furthermore, assume that $g_j(t) \neq 0$, for almost every $t \in [0,1]$ and for all $j = 1,\ldots,s$. Denote by $M_g = \sup_{u\in[0,1]^s} g(u)$. Let $\alpha = (\alpha_1,\ldots,\alpha_N)$ be a set of points in $[0,1]^s$. Generate the set of points $\beta = (\beta_1,\ldots,\beta_N)$, with*

$$\beta_k^{(j)} = \frac{1}{N}\sum_{r=1}^{N}\left[1+\alpha_k^{(j)} - G_j\left(\alpha_r^{(j)}\right)\right] = \frac{1}{N}\sum_{r=1}^{N} 1_{[0,\alpha_k^{(j)}]}\left(G_j\left(\alpha_r^{(j)}\right)\right),$$

for all $k = 1,\ldots,N$ and all $j = 1,\ldots,s$, where $[a]$ denotes the integer part of a. Then the generated set of points has a G-discrepancy of

$$D_{N,G}(\beta_1,\ldots,\beta_N) \leq (2+6sM_g)D_N(\alpha_1,\ldots,\alpha_N).$$

During our experiments, we used the SQRT point set $\alpha = (\alpha_1,\ldots,\alpha_{NM})$ in $[0,1]^s$, defined by [25]

$$\alpha_k = (\{k\sqrt{p_1}\},\{k\sqrt{p_2}\},\ldots,\{k\sqrt{p_s}\}), \qquad k = 1,\ldots,NM,$$

where $p_1, p_2, \ldots, p_s$ are the first s prime numbers. The defined SQRT point set α is with low discrepancy in $[0,1]^s$.

In this case, all the values $\beta_k^{(j)}$, $k = 1,\ldots,NM$, $j = 1,\ldots,s$, generated with the Hlawka-Mück method, are of the form $\frac{i}{NM}$, $i = 0,\ldots,NM$. In particular, some values $\beta_k^{(j)}$ might assume a value of 1. A value of 1 is a singularity of function $f(z)$, as illustrated in (25). Hence, the sequence constructed with Hlawka-Mück method is not directly suited for unbounded problems. To overcome this problem, we define a new point set $\bar{\beta} = (\overline{\beta}_1, \ldots, \overline{\beta}_{NM})$, in which a value of 1 for $\beta_k^{(j)}$ is replaced by $1 - \frac{1}{NM}$. In other words,

$$\overline{\beta}_k^{(j)} = \begin{cases} \beta_k^{(j)} & , \beta_k^{(j)} \neq 1 \\ 1 - \frac{1}{NM} & , \beta_k^{(j)} = 1, \end{cases} \tag{28}$$

for $k = 1, \ldots, NM$ and $j = 1, \ldots, s$.

This slight modification of the sequence is shown to have a minor influence, as the transformed set does not lose its low G-discrepancy and can be used for QMC integration [15].

In the following, we apply our combined method to estimate the integral (23). In order to do this, we need to populate the space Ω. For this, we first generate a set A that contains the first 30 prime numbers

$$A = \{2, 3, 5, 7, \ldots, 113\}.$$

Next, we construct all the subsets with s elements of the set A. There are $r = C_{30}^s$ such subsets of A. For each subset $A_i = \{p_{i,1}, \ldots, p_{i,s}\}$, we consider the SQRT point set $\alpha_i = (\alpha_{1,i}, \ldots, \alpha_{N,i})$, defined by

$$\alpha_{k,i} = (\{k\sqrt{p_{i,1}}\}, \ldots, \{k\sqrt{p_{i,s}}\}), \qquad k = 1, \ldots, N.$$

The defined SQRT point sets α_i, $i = 1, \ldots, r$, are with low discrepancy in $[0,1]^s$.

Further, we construct the space Ω of point sets with low G-discrepancy in $[0,1]^s$, $\Omega = \{\beta_1, \ldots, \beta_r\}$, where β_i, $i = 1, \ldots, r$, is of the form

$$\beta_i = (\beta_{1,i}, \ldots, \beta_{N,i}),$$

with $\beta_{k,i} = \left(\beta_{k,i}^{(1)}, \ldots, \beta_{k,i}^{(s)}\right) \in [0,1]^s$, $k = 1, \ldots, N$.

An arbitrary point set β_i, $i = 1, \ldots, r$, is obtained from the point set α_i, using the Hlawka-Mück method. In this case, all the values $\beta_{k,i}^{(j)}$, $k = 1, \ldots, N$, $j = 1, \ldots, s$, generated with the Hlawka-Mück method, are of the form $\frac{l}{N}$, $l = 0, \ldots, N$. In particular, some values $\beta_{k,i}^{(j)}$ might assume a value of 1. Since, a value of 1 is a singularity of the function $f(z)$, we define a new point set $\overline{\beta}_i = (\overline{\beta}_{1,i}, \ldots, \overline{\beta}_{N,i})$, in which a value of 1 for $\beta_{k,i}^{(j)}$ is replaced by $1 - \frac{1}{N}$, i.e.,

$$\overline{\beta}_{k,i}^{(j)} = \begin{cases} \beta_{k,i}^{(j)} & , \beta_{k,i}^{(j)} \neq 1 \\ 1 - \frac{1}{N} & , \beta_{k,i}^{(j)} = 1, \end{cases} \tag{29}$$

for $i = 1, \ldots, r$, $k = 1, \ldots, N$ and $j = 1, \ldots, s$.

Next, we select the integers $i_1, \ldots, i_M$ at random from the uniform distribution on $\{1, \ldots, r\}$ and consider the corresponding point sets with low G-discrepancy $\beta_{i_1}, \ldots, \beta_{i_M}$.

We calculate the following estimate:

$$\bar{I}_{RSNU} = \frac{\sum_{l=1}^{M} \left(\frac{1}{N} \sum_{k=1}^{N} f(\beta_{k,i_l}) \right)}{M}. \tag{30}$$

In our numerical experiments, we consider that the parameters of the NIG-distributed log-returns under the equivalent martingale measure given by the Esscher transform are the ones that are given in [15]

$$\mu = 0.00079 * 5,\ \beta = -15.1977,\ \alpha = 136.29,\ \delta = 0.0059 * 5. \tag{31}$$

We notice that these parameters are relevant for daily observed stock price log-returns [34]. As the class of NIG distributions is closed under convolution, we can derive weekly stock prices by using a factor of 5 for the parameters μ and δ. Further, we suppose that the initial stock price is $S(0) = 100$, the strike price is $K = 100$ and the risk-free annual interest rate is $r = 3.75\%$. We choose the parameter of the double-exponential distribution $\lambda = 95.2271$.

The option is sampled at weekly time intervals. We also let the option to have maturities of 3 weeks. Hence, our problem is a 3-dimensional integral over the payoff function.

We consider the "exact" option price to be the average of 10 MC simulations, with $N = 100000$ for the initial integral (18).

We give the results for the case when $M = 5$. The following table contains the value of N and the absolute values of the errors $|I - \bar{I}_{MC}|$, $|I - \bar{I}_{QMC}|$, $|I - \bar{I}_{RSNU}|$.

N	$\lvert I - \bar{I}_{MC} \rvert$	$\lvert I - \bar{I}_{QMC} \rvert$	$\lvert I - \bar{I}_{RSNU} \rvert$
1500	0.035780	0.003482	0.001617
1600	0.027438	0.005332	0.001305
1700	0.012101	0.004052	0.000238
1800	0.030410	0.004104	0.001907
1900	0.023049	0.004610	0.001543
2000	0.014081	0.003958	0.001808
2500	0.020786	0.003679	0.000085
3000	0.026064	0.003032	0.001723
3500	0.020570	0.003279	0.000702

Table 1: European Call Option: Case $s = 3$ and $M = 5$.

The numerical results, presented in Table 1, indicate that our RSNU estimate converges faster than the MC and QMC estimates. The error in our combined method is smaller than the error in MC method by approximately a factor of 10. The error in our method gives approximately a factor of 3 improvement over the error in QMC method.

5. Application to finance: Evaluation of Asian Options

In this section, we apply our combined method to evaluate the so-called (discrete sampled) Asian option, driven by the asset dynamics $S(t)$, as defined in (12). The general framework remains the same as in the previous section, only the payoff function is changed. The payoff of an Asian call option is defined by

$$C_T(S) = \Big(\frac{1}{s}\sum_{i=1}^{s} S(t_i) - K\Big)_+ = \max\Big\{\frac{1}{s}\sum_{i=1}^{s} S(t_i) - K, 0\Big\}, \tag{32}$$

with $0 = t_0 < t_1 < t_2 < \ldots < t_s = T$. The constant $K \geq 0$ is called the strike price. Thus, we get the following integration problem:

$$I = \int_{\mathbb{R}^s} \underbrace{\Big(\frac{S(0)}{s}\sum_{i=1}^{s} e^{\sum_{j=1}^{i} x^{(j)}} - K\Big)_+}_{A(x)} dH(x) = \int_{\mathbb{R}^s} A(x)dH(x), \tag{33}$$

where $H(x) = \prod_{i=1}^{s} H_i(x^{(i)}),\ \forall x = (x^{(1)}, \ldots, x^{(s)}) \in \mathbb{R}^s$, and $H_i(x^{(i)})$ denotes the distribution function of the so-called log-returns induced by $Z(t_1)$, with the corresponding density function $h_i(x^{(i)})$. These log increments are independent and NIG distributed, having the common density function defined in (19).

In a similar way to the previous section, we transform the integral (33) to an integral on $[0,1]^s$. We get the following integration problem on $[0,1]^s$:

$$I = \int_{[0,1]^s} \underbrace{\Big(\frac{S(0)}{s}\sum_{i=1}^{s} e^{\sum_{j=1}^{i} H_\lambda^{-1}(z^{(j)})} - K\Big)_+}_{f(z)} dG(z) = \int_{[0,1]^s} f(z)dG(z), \tag{34}$$

where $G : (0,1)^s \to [0,1]$, defined by

$$G(z) = \prod_{i=1}^{s} (H_i \circ H_\lambda^{-1})(z^{(i)}),\ \forall z = (z^{(1)}, \ldots, z^{(s)}) \in (0,1)^s, \tag{35}$$

is a distribution function on $(0,1)^s$, with independent marginals $G_i = H_i \circ H_\lambda^{-1}$, $i = 1, \ldots, s$. The inverse function H_λ^{-1} is defined in (21).

The integral I is an improper integral because the function f has singularities on the right boundary of the interval $[0,1]^s$, i.e., $\lim_{z^{(i)} \to 1} f(z^{(1)}, \ldots, z^{(s)}) = \infty$ for $i = 1, \ldots, s$.

In the following, we compare numerically our combined method, with the MC and QMC methods, in terms of absolute errors.

We suppose that the parameters of the NIG-distributed log-returns under the equivalent martingale measure given by the Esscher transform are the same as in (31). We assume that the initial stock price is $S(0) = 100$, the strike price is $K = 100$ and the risk-free annual interest rate is $r = 3.75\%$. We choose the parameter of the double-exponential distribution $\lambda = 95.2271$.

The Asian call option is sampled weekly. We also let the option to have maturities of 3 weeks. Hence, our problem is a 3-dimensional integral over the payoff function.

In a similar way to the previous section, we compute the estimates $\bar{I}_{MC}$, $\bar{I}_{QMC}$ and $\bar{I}_{RSNU}$, given by (26), (27) and (30), respectively.

The "true" price is obtained as the average of 10 MC simulations, with $N = 100000$. We give the results for the case when $M = 7$. The following table contains: the value of N and the absolute values of the errors $|I-\bar{I}_{MC}|$, $|I-\bar{I}_{QMC}|$, $|I-\bar{I}_{RSNU}|$.

N	$\lvert I - \bar{I}_{MC}\rvert$	$\lvert I - \bar{I}_{QMC}\rvert$	$\lvert I - \bar{I}_{RSNU}\rvert$
1500	0.025739	0.003467	0.001190
1600	0.022690	0.003075	0.000178
1700	0.019234	0.003445	0.001081
1800	0.023788	0.003057	0.001627
1900	0.022007	0.003400	0.001782
2000	0.019711	0.002724	0.001939
2500	0.016860	0.003133	0.001595
3000	0.015537	0.002580	0.000333
3500	0.009145	0.002504	0.000810

Table 2: Asian Call Option: Case $s = 3$ and $M = 7$.

From Table 2, we notice that the proposed RSNU estimate converges faster than the MC and QMC estimates. The error in our combined method is smaller than the error in MC method by approximately a factor of 10. The error in our method gives approximately a factor of 3 improvement over the error in QMC method.

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