SOME SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract. In this paper we introduce a new class $M^*(\alpha,\beta,\gamma,A,\lambda)$ consisting analytic and univalent functions with negative coefficients. The object of the paper is to show some properties for the class $M^*(\alpha,\beta,\gamma,A,\lambda)$.

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1. Introduction

Let S denote the class of normalised analytic univalent function f defined by

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1}$$

for $z\in D=\{z:|z|<1\}$.
Let T denote the subclass of S consisting functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n. \tag{2}$$

Further, we define the class $M\,(\alpha,\beta,\gamma,A,\lambda)$ as follows:

Definition. *A function f given by (1.1) is said to be a member of the class $M(\alpha,\beta,\gamma,A,\lambda)$ if it satisfies*

$$\left|\frac{zf'(z)-f(z)}{\alpha zf'(z)-Af(z)-(1-\lambda)\,(1-A)\gamma f(z)}\right|<\beta$$

where $0\le\alpha\le1,\beta(0<\beta\le1)$, $-1\le A<1$, $0\le\lambda\le1$ *and* $0\le\gamma<1$ *for all* $z\in D$.

Let us write

$$M^{*}(\alpha,\beta,\gamma,A,\lambda)=T\cap\, M(\alpha,\beta,\gamma,A,\lambda). \tag{3}$$

We note that when $A=-1$ and $\lambda=\frac{1}{2}$ the class of functions was studied by Darus [5]. Under the same condition, if we replace $\frac{zf'(z)}{f(z)}$ with $f'(z)$ we get back to the class of $L^{*}(\alpha)$ and various other subclasses of L^{*} which have been studied rather extensively by Kim and Lee [4], Uralegaddi and Sarangi [1], and Al-Amiri [2]. If $\lambda=0$, $\beta=1$ and $A=-1$ the class of functions was given by Silverman [3].

Next, our first result will concentrate on the coefficient estimate for the classes $M(\alpha,\beta,\gamma,A,\lambda)$ and $M^{*}(\alpha,\beta,\gamma,A,\lambda)$.

2.Coefficient Inequalities

In this section we will prove a sufficient condition for a function analytic in D to be in $M(\alpha,\beta,\gamma,A,\lambda)$.

Theorem 1. *If $f\in S$ satisfies*

$$\sum_{n=2}^{\infty}\left(n-1+\beta\left(n\alpha-A-(1-\lambda)\left(1-A\right)\gamma\right)\right)|a_n|\leq$$

$$\beta\left(\alpha-A-(1-\lambda)\left(1-A\right)\gamma\right) \tag{4}$$

where $0\leq\alpha\leq1$, $0<\beta\leq1$, $-1\leq A<1$, $0\leq\lambda\leq1$ and $0\leq\gamma<1$, then $f(z)\in M(\alpha,\beta,\gamma,A,\lambda)$.

Proof. Let us suppose that

$$\sum_{n=2}^{\infty}\left(n-1+\beta\left(n\alpha-A-(1-\lambda)\left(1-A\right)\gamma\right)\right)|a_n|$$

$$\leq\beta\left(\alpha-A-(1-\lambda)\left(1-A\right)\gamma\right)\in S.$$

It suffices to show that

$$\left|\frac{\frac{zf'(z)}{f(z)}-1}{\alpha\frac{zf'(z)}{f(z)}-A-(1-\lambda)\left(1-A\right)\gamma}\right|<\beta,\qquad (z\in D). \tag{5}$$

$$\left|\frac{\frac{zf'(z)}{f(z)}-1}{\alpha\frac{zf'(z)}{f(z)}-A-(1-\lambda)(1-A)\gamma}\right|$$

$$=\left|\frac{\sum\limits_{n=2}^{\infty}(n-1)a_nz^n}{\alpha-A-(1-\lambda)(1-A)\gamma+\sum\limits_{n=2}^{\infty}(n\alpha-A-(1-\lambda)(1-A)\gamma)a_nz^n}\right|$$

$$<\frac{\sum\limits_{n=2}^{\infty}(n-1)|a_n|}{\alpha-A-(1-\lambda)(1-A)\gamma-\sum\limits_{n=2}^{\infty}(n\alpha-A-(1-\lambda)(1-A)\gamma)|a_n|}.$$

from (5), the last expression satisfies

$$\sum_{n=2}^{\infty}(n-1)|a_n|$$

$$\leq\beta\left(\alpha-A-(1-\lambda)(1-A)\gamma-\sum_{n=2}^{\infty}(n\alpha-A-(1-\lambda)(1-A)\gamma)|a_n|\right)$$

that is

$$\sum_{n=2}^{\infty}(n-1+\beta(n\alpha-A-(1-\lambda)(1-A)\gamma))|a_n|$$

$$\leq\beta(\alpha-A-(1-\lambda)(1-A)\gamma)$$

which is equivalent to our condition of the theorem.

So that $f\in M(\alpha,\beta,\gamma,A,\lambda)$. Hence the theorem.

Next we give a necessary and sufficient condition for a function $f\in T$ to be in the class $M^*(\alpha,\beta,\gamma,A,\lambda)$.

Theorem 2. *Let the function f be defined by (2) and let $f\in T$. Then $f\in M^*(\alpha,\beta,\gamma,A,\lambda)$. If and only if (4) is satisfied. The result (4) is sharp.*

Proof. With the aid of Theorem 1, it suffices to show the (only if) part. Assume that $f \in M^*(\alpha,\beta,\gamma,A,\lambda)$. Then

$$\left|\frac{\frac{zf'(z)}{f(z)}-1}{\alpha\frac{zf'(z)}{f(z)}-A-(1-\lambda)(1-A)\gamma}\right|$$

$$=\left|\frac{\sum\limits_{n=2}^{\infty}(n-1)a_nz^n}{\alpha-A-(1-\lambda)(1-A)\gamma-\sum\limits_{n=2}^{\infty}(n\alpha-A-(1-\lambda)(1-A)\gamma)a_nz^n}\right|$$

$$<\frac{\sum\limits_{n=2}^{\infty}(n-1)|a_n|}{\alpha-A-(1-\lambda)(1-A)\gamma-\sum\limits_{n=1}^{\infty}(n\alpha-A-(1-\lambda)(1-A)\gamma)|a_n|}$$

Similarly, the method in Theorem 1 applies and obtained the required result. The result is sharp for function f of the form

$$f_n(z)=z-\frac{\beta(\alpha-A-(1-\lambda)(1-A)\gamma)z^n}{(n-1+\beta(n\alpha-A-(1-\lambda)(1-A)\gamma))},\qquad n\geq 2. \tag{6}$$

Corollary 1. *Let the function f be defined by (2) and let $f \in M^*(\alpha,\beta,\gamma,A,\lambda)$, then*

$$a_n\leq\frac{\beta(\alpha-A-(1-\lambda)(1-A)\gamma)}{(n-1+\beta(n\alpha-(1-\lambda)(1-A)\gamma))}\qquad n\geq 2. \tag{7}$$

3. Growth and Distortion Theorem

Growth and distortion properties for functions f in the class $M^*(\alpha,\beta,\gamma,A,\lambda)$ are given as follows:

Theorem 3. *If the function f be defined by (4) is in the class $M^*(\alpha,\beta,\gamma,A,\lambda)$, then for $0<|z|=r<1$, we have*

$$r-\frac{\beta(\alpha-A-(1-\lambda)(1-A)\gamma)r^2}{(1+\beta(2\alpha-A-(1-\lambda)(1-A)\gamma))}\leq|f(z)|$$

$$\leq r+\frac{\beta\left(\alpha-A-\left(1-\lambda\right)\left(1-A\right)\gamma\right)r^{2}}{\left(1+\beta\left(2\alpha-A-\left(1-\lambda\right)\left(1-A\right)\gamma\right)\right)}$$

with equality for

$$f_{2}(z)=z-\frac{\beta\left(\alpha-A-\left(1-\lambda\right)\left(1-A\right)\gamma\right)z^{2}}{\left(1+\beta\left(2\alpha-A-\left(1-\lambda\right)\left(1-A\right)\gamma\right)\right)},\qquad\left(z=ir,r\right).$$

and

$$1-\frac{2\beta\left(\alpha-A-\left(1-\lambda\right)\left(1-A\right)\gamma\right)r}{\left(1+\beta\left(2\alpha-A-\left(1-\lambda\right)\left(1-A\right)\gamma\right)\right)}\leq\left|f'(z)\right|$$

$$\leq 1+\frac{2\beta\left(\alpha-A-\left(1-\lambda\right)\left(1-A\right)\gamma\right)r}{\left(1+\beta\left(2\alpha-A-\left(1-\lambda\right)\left(1-A\right)\gamma\right)\right)}$$

with equality for

$$f_{2}(z)=z-\frac{\beta\left(\alpha-A-\left(1-\lambda\right)\left(1-A\right)\gamma\right)z^{2}}{\left(1+\beta\left(2\alpha-A-\left(1-\lambda\right)\left(1-A\right)\gamma\right)\right)},\qquad\left(z=\pm ir,\pm r\right).$$

Proof. Since $f\in M^{*}(\alpha,\beta,\gamma,A,\lambda)$, Theorem 1 yields the inequality

$$\sum_{n=2}^{\infty}a_{n}\leq\frac{\beta\left(\alpha-A-\left(1-\lambda\right)\left(1-A\right)\gamma\right)}{\left(1+\beta\left(2\alpha-A-\left(1-\lambda\right)\left(1-A\right)\gamma\right)\right)}.\tag{8}$$

Thus, for $0<|z|=r<1$, and making use of (8), we have

$$\left|f(z)\right|\leq\left|z\right|+\sum_{n=2}^{\infty}a_{n}\left|z^{n}\right|\leq r+r^{2}\sum_{n=2}^{\infty}a_{n}$$

$$\leq r+\frac{r^{2}\beta\left(\alpha-A-\left(1-\lambda\right)\left(1-A\right)\gamma\right)}{\left(1+\beta\left(2\alpha-A-\left(1-\lambda\right)\left(1-A\right)\gamma\right)\right)}.$$

and

$$\left|f(z)\right|\geq\left|z\right|+\sum_{n=2}^{\infty}a_{n}\left|z^{n}\right|\geq r+r^{2}\sum_{n=2}^{\infty}a_{n}$$

$$\geq r+\frac{r^{2}\beta\left(\alpha-A-\left(1-\lambda\right)\left(1-A\right)\gamma\right)}{\left(1+\beta\left(2\alpha-A-\left(1-\lambda\right)\left(1-A\right)\gamma\right)\right)}.$$

Besides, from Theorem 1, it follow that

$$\sum_{n=2}^{\infty} na_n \leq \frac{2\beta\left(\alpha - A - (1-\lambda)(1-A)\gamma\right)}{\left(1+\beta\left(2\alpha - A - (1-\lambda)(1-A)\gamma\right)\right)}. \tag{9}$$

Thus

$$|f'(z)| \leq 1 + \sum_{n=2}^{\infty} na_n \left|z^{n-1}\right| \leq$$

$$1 + r\sum_{n=2}^{\infty} na_n \leq 1 + \frac{2r\beta\left(\alpha - A - (1-\lambda)(1-A)\gamma\right)}{\left(1+\beta\left(2\alpha - A - (1-\lambda)(1-A)\gamma\right)\right)}$$

and

$$|f'(z)| \geq 1 - \sum_{n=2}^{\infty} na_n \left|z^{n-1}\right| \geq$$

$$1 - r\sum_{n=2}^{\infty} na_n \geq 1 - \frac{2r\beta\left(\alpha - A - (1-\lambda)(1-A)\gamma\right)}{\left(1+\beta\left(2\alpha - A - (1-\lambda)(1-A)\gamma\right)\right)}$$

Hence completes the proof of Theorem 3.

4.Radii of Starlikeness and Convexity

The radii of starlikeness and convex for the class $M^*(\alpha,\beta,\gamma,A,\lambda)$ is given by the following theorem:

Theorem 4. *If the function f be defined by (2) is in the class $M^*(\alpha,\beta,\gamma,A,\lambda)$, then $f(z)$ is starlike of order $\rho(0 \leq \rho < 1)$ in the disk $|z| < r_1(\alpha,\beta,\gamma,A,\lambda,\rho)$ where $r_1(\alpha,\beta,\gamma,A,\lambda,\rho)$ is the largest value for which*

$$r_1 = r_1(\alpha,\beta,\gamma,A,\lambda,\rho)$$

$$= \inf_{n\geq 2}\left(\frac{(1-\rho)\left[(n-1)+\beta\left(n\alpha - A - (1-\lambda)(1-A)\gamma\right)\right]}{(n-\rho)\beta\left(\alpha - A - (1-\lambda)(1-A)\gamma\right)}\right)^{\frac{1}{n-1}}.$$

The result is sharp for function $f_n(z)$ given by (6).
Proof. It suffices to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1-\rho, \qquad for \quad |z|\leq r_1.$$

We have

$$\left|\frac{zf'(z)}{f(z)}-1\right|$$

$$\leq \frac{\sum_{n=2}^{\infty}(n-1)\frac{\beta(\alpha-A-(1-\lambda)(1-A)\gamma)|z|^{n-1}}{(n-1+\beta(n\alpha-A-(1-\lambda)(1-A)\gamma))}}{1-\sum_{n=2}^{\infty}\frac{\beta(\alpha-A-(1-\lambda)(1-A)\gamma)|z|^{n-1}}{(n-1+\beta(n\alpha-A-(1-\lambda)(1-A)\gamma))}}\leq 1-\rho \tag{10}$$

Hence (10) holds true if

$$\sum_{n=2}^{\infty}\frac{(n-1)\beta\left(\alpha-A-(1-\lambda)\left(1-A\right)\gamma\right)|z|^{n-1}}{\left(n-1+\beta\left(n\alpha-A-(1-\lambda)\left(1-A\right)\gamma\right)\right)}$$

$$\leq(1-\rho)\left(1-\sum_{n=2}^{\infty}\frac{\beta\left(\alpha-A-(1-\lambda)\left(1-A\right)\gamma\right)|z|^{n-1}}{\left(n-1+\beta\left(n\alpha-A-(1-\lambda)\left(1-A\right)\gamma\right)\right)}\right)$$

and it follows that

$$|z|^{n-1}\leq\frac{(1-\rho)\left[(n-1)+\beta\left(n\alpha-A-(1-\lambda)\left(1-A\right)\gamma\right)\right]}{(n-\rho)\,\beta\left(\alpha-A-(1-\lambda)\left(1-A\right)\gamma\right)}, \qquad (n\geq 2)\,.$$

as required.

Theorem 5. *If the function f defined by (2) is in the class $M^*(\alpha,\beta,\gamma,A,\lambda)$, then f is convex of order $\rho(0\leq\rho<1)$, in the disk $|z|<r_2(\alpha,\beta,\gamma,A,\rho)$, where $r_2(\alpha,\beta,\gamma,A,\lambda,\rho)$, is the largest value for which*

$$r_2=r_2(\alpha,\beta,\gamma,A,\rho)=\inf_{n\geq 2}\left(\frac{(1-\rho)\left[(n-1)+\beta\left(n\alpha-A-(1-\lambda)\left(1-A\right)\gamma\right)\right]}{n\,(n-\rho)\,\beta\left(\alpha-A-(1-\lambda)\left(1-A\right)\gamma\right)}\right)^{\frac{1}{n-1}}.$$

The result is sharp for function $f_n(z)$ given by (6).
Proof. By using the same techniques as in the proof of the Theorem 4, we can show that

$$\left|\frac{zf''(z)}{f'(z)}\right|<\rho-1 \quad for \quad |z|\leq r_2,$$

with the aid of Theorem 1. Thus we have the assertion of Theorem 5.

5.Convex Linear Combinations

Our next result involves a linear combination of function of the type (6).

Theorem 5.1. *Let*

$$f_1 = z \tag{11}$$

and

$$f_n(z) = z - \frac{\beta\left(\alpha - A - (1-\lambda)(1-A)\gamma\right) z^n}{\left(n - 1 + \beta\left(n\alpha - A - (1-\lambda)(1-A)\gamma\right)\right)}, \qquad (n \geq 2)\,. \tag{12}$$

Then $f \in M^*(\alpha, \beta, \gamma, A, \lambda)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \delta_n f_n(z) \tag{13}$$

Where $\delta_n \geq 0$ and $\sum\limits_{n=1}^{\infty} \delta_n = 1$.

Proof. From (11), (12) and (13), it is easy to see that

$$f(z) = \sum_{n=1}^{\infty} \delta_n f_n(z) = z - \sum_{n=2}^{\infty} \frac{\beta\left(\alpha - A - (1-\lambda)(1-A)\gamma\right)\delta_n z^n}{\left(n - 1 + \beta\left(n\alpha - A - (1-\lambda)(1-A)\gamma\right)\right)}. \tag{14}$$

Since

$$\sum_{n=2}^{\infty} \frac{\left(n - 1 + \beta\left(n\alpha - A - (1-\lambda)(1-A)\gamma\right)\right)}{\beta\left(\alpha - A - (1-\lambda)(1-A)\gamma\right)}$$

$$\cdot \frac{\beta\left(\alpha - A - (1-\lambda)(1-A)\gamma\right)\delta_n}{\left(n - 1 + \beta\left(n\alpha - A - (1-\lambda)(1-A)\gamma\right)\right)}$$

$$= \sum_{n=1}^{\infty} \delta_n = 1 - \delta_1 \leq 1.$$

It follows from Theorem 1 that the function $f \in M^*(\alpha, \beta, \gamma, A, \lambda)$. Since

$$a_n \leq \frac{\beta\left(\alpha - A - (1-\lambda)(1-A)\gamma\right)}{\left(n - 1 + \beta\left(n\alpha - A - (1-\lambda)(1-A)\gamma\right)\right)}, \qquad (n \geq 2)\,.$$

Setting

$$\delta_n = \frac{(n-1+\beta\,(n\alpha - A - (1-\lambda)\,(1-A)\,\gamma))}{\beta\,(\alpha - A - (1-\lambda)\,(1-A)\,\gamma)} a_n, \qquad (n \geq 2)$$

and

$$\delta_1 = 1 - \sum_{n=2}^{\infty} \delta_n$$

it follows that $f(z) = \sum_{n=2}^{\infty} \delta_n f_n(z)$. This completes the proof of the theorem.

Finally we prove the following:

Theorem 5.2. *The class $M^*(\alpha, \beta, \gamma, A, \lambda)$ is closed under convex linear combinations.*

Proof. Suppose that the functions $f_1(z)$ and $f_2(z)$ defined by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad (j = 1, 2\,;\, z \in D) \tag{15}$$

are in the class $M^*(\alpha, \beta, \gamma, A, \lambda)$. Setting

$$f(z) = \mu f_1(z) + (1-\mu) f_2(z), \qquad 0 \leq \mu \leq 1. \tag{16}$$

we find from (15) that

$$f(z) = z - \sum_{n=1}^{\infty} \{\mu a_{n,1} + (1-\mu)\, a_{n,2}\} z^n, \qquad (\,0 \leq \mu \leq 1\,;\, z \in D). \tag{17}$$

In view of Theorem 1, we have

$$\sum_{n=2}^{\infty} [n - 1 + \beta\,(n\alpha - A - (1-\lambda)\,(1-A)\,\gamma)\,\{\mu a_{n,1} + (1-\mu)\,a_{n,2}\}]$$

$$= \mu \sum_{n=2}^{\infty} [n - 1 + \beta\,(n\alpha - A - (1-\lambda)\,(1-A)\,\gamma)] a_{n,1}$$

$$+ (1-\mu) \sum_{n=2}^{\infty} [n - 1 + \beta\,(n\alpha - A - (1-\lambda)\,(1-A)\,\gamma)] a_{n,2}$$

$$\leq \mu\beta\left(\alpha - A - (1-\lambda)(1-A)\gamma\right) + (1-\mu)\beta\left(\alpha - A - (1-\lambda)(1-A)\gamma\right)$$

$$= \beta\left(\alpha - A - (1-\lambda)(1-A)\gamma\right).$$

which shows that $f \in M^*(\alpha, \beta, \gamma, A, \lambda)$. Hence the theorem.

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