SOME RESULTS FOR GENERALIZED COCHAINS

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Abstract. In the first part of the paper a generalized p-cochain on $C^\infty(M)$ is defined, followed in the second part by some of its properties and applications in distributional symplectic geometry.

2000 *Mathematics Subject Classification*: 70E50.

1. Introduction

Let M be a smooth $2n$-dimmensional manifold and ω a symplectic structure on M. We denote by $C^\infty(M)$ (resp. $X'(M)$, resp. $\overset{p}{D}'(M)$) the space of smooth (C^∞) functions (resp. the spaced of generalized vector fields, resp. the space of p-De Rham currents) on M endowed with the uniform convergence topology. We remind that in local chart a generalized vector field (resp. an p-De Rham current) is a smooth vector field (resp. a smooth p-form) with distributions coefficients instead of smooth ones.

Definition 1.1. *A generalized p-cochain on $C^\infty(M)$ is an alternating p-linear map*

$$m : C^\infty(M) \times ... \times C^\infty(M) \longrightarrow \overset{0}{D}'(M).$$

We shall denote by $\overset{p}{C}'(C^\infty)$ the space of generalized p-cochains on $C^\infty(M)$.

Examples. 1) Each generalized vector field $X \in X'(M)$ defines in a natural way a generalized 1-cochain.

2) The map $\Im$ defined by

$$K : \overset{p}{D}'(M) \longrightarrow K(T) \in \overset{p}{C}'(C^\infty),$$
$$K(T)(f_1, ..., f_p) \overset{def}{=} T(\xi_{f_1}, ..., \xi_{f_p})$$

is a generalized 1-cochain.

3) There exists a linear map K from the space of De Rham currents into the space of generalized cochains, namely:

$$K : \overset{p}{D}'(M) \longrightarrow K(T) \in \overset{p}{C}'(C^\infty),$$
$$K(T)(f_1, ..., f_p) \overset{def}{=} T(\xi_{f_1}, ..., \xi_{f_p})$$

for any $f_1, ..., f_p \in C^\infty(M)$ and where ξ_{f_i} is the Hamiltonian vector field associated to f_i (i.e. $L_{\xi_{f_i}}\omega + df_i$).

The coboundary of generalized p-cochains is defined as usual; in the particular case of a generalized 1-cochain we have:

$$\partial m(f_1, f_2) = L_{\xi_{f_i}} m(f_2) - L_{\xi_{f_2}} m(f_1) - m\{f_1, f_2\}.$$

2. Some properties of generalized cochains

Some properties for generalized cochains are given in the following statements:

Proposition 2.1. i) *K is an injective map.*

ii) *If $T = T_\omega$ is the form like 2-current defind by the symplectic form ω (i.e. $T_\omega : \varphi \in \overset{2n-2}{D}(M) \longrightarrow < T_\omega, \varphi >= \int_M \omega \wedge \varphi$, where $\overset{p}{D}(M)$ denotes the space of p-forms with compact support on M), then*

$$K(T_\omega) = -\partial\Im.$$

iii) F*or each $\alpha \in \mathbb{R}$, $S \in \overset{1}{D}'(M)$ we have*

$$K(\alpha T_\omega + dS) = -\partial(\alpha\Im + K(S)).$$

The proof can be obtained immediately using the definitions of K and $\Im$.

Proposition 2.2. i) *If $\tilde{\omega}$ is the canonical isomorphism given by*

$$\tilde{\omega} : X \in X'(M) \longrightarrow \tilde{\omega}(X) \overset{def}{=} X \lrcorner\, \omega \in \overset{1}{D}'(M)$$

then

$$K\left|\overset{1}{D}{}'(M)\right. = -\tilde{\omega}^{-1}.$$

ii) *Let X be a generalized vector field on M. Then for each $f, g \in C^\infty(M)$ we have:*

$$X(f,g) = -L_X\omega(\xi_f, \xi_g).$$

iii) *$X \in X'(M)$ is a generalized 1-cocycle (i.e. $\partial X = 0$) if and only if $X \in X'_{loc}(M)$ (i.e. $L_X\omega = 0$).*

iv) *$X \in X'(M)$ is a generalized 1-coboundary if and only if $X \in X'_{glob}(M)$ (i.e. $X \lrcorner\, \omega + dH = 0$).*

Proof. i) Since the space $\overset{1}{D}{}'(M)$ can be identified with $\{X \lrcorner\, \omega \mid X \in X'(M)\}$, for any $f \in C^\infty(M)$, we get successively:

$$K\tilde{\omega}(X)(f) = \tilde{\omega}(X)(\xi_f) = (X \lrcorner\, \omega)(\xi_f) = -X(f).$$

so

$$K \circ \tilde{\omega} = -Id_{X'(M)},$$

or equivalent:

$$K = -\tilde{\omega}^{-1}.$$

ii) For any $f, g \in C^\infty(M)$ we can write:

$$\begin{aligned}\partial X(f,g) &= \partial\tilde{\omega}\tilde{\omega}^{-1}(X)(f,g)\\ &= -d\tilde{\omega}(X)(\xi_f,\xi_g)\\ &= -d(X \lrcorner\, \omega)(\xi_f,\xi_g)\\ &= -L_X\omega(\xi_f,\xi_g).\end{aligned}$$

Now the relation iii) and iv) can be derived immediately from ii)

Definition 2.1. *We say that $m \in \overset{1}{C}{}'(C^\infty)$ is a locally generalized p-cochain if, given an open set $U \subset M$ and p-functions $f_1, ..., f_p$ with*

$$f_1|_U = f_2|_U = ... = f_p|_U,$$

then

$$m(f_1, ..., f_p)|_U = 0.$$

Proposition 2.3. *Let $T \in \overset{2}{D}{}'(M)$ be an 2-De Rham current on M. Then*

i) *$K(T)$ is a locally generalized 2-cochain.*
ii) *For any $f, g \in C^\infty(M)$ the following equality holds:*

$$K(T)(f^2, g) = 2fK(T)(f, g).$$

Proof. i) Let U be an open set in M and $f \in C^\infty(M)$ such that $f|_U = 0$. Then for each $g \in C^\infty(M)$ we have:

$$K(T)(f, g)|_U = T(\xi_f, \xi_g)|_U.$$

Since

$$\xi_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} \right),$$

and

$$T|_U = \sum_{i,j=1}^{n} (T^{ij} dp_i \wedge dp_j + T_{ij} dq^i \wedge dq^j + ...),$$

it follows that:

$$T(\xi_f, \xi_g)|_U = \sum_{i,j=1}^{n} \left(T^{ij} \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^j} + T_{ij} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial p_j} + ... \right).$$

Since each term involves a partial derivative of f, clearly $T(\xi_f, \xi_g)$ vanishes on U, so $K(T)$ is a locally generalized 2-cochain.
ii) For any $f, g \in C^\infty(M)$ we can write successively:

$$\begin{aligned} K(T)(f^2, g) &= T(\xi_{f^2}, \xi_g) \\ &= T(2f\xi_f, \xi_g) \\ &= 2fT(\xi_f, \xi_g) \\ &= 2fK(T)(f, g). \end{aligned}$$

Also, as for classical cochains we can prove the following result:

Remark 2.1 *Let (M, ω) be a non-compact symplectic manifold and $m \in \overset{2}{C}'(C^\infty)$. Then m is a locally generalized 2-cochain if and only if m has the same property.*

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