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SANDWICH THEOREMS FOR ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION

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Abstract. Making use of the generalized Sălăgean operator, we obtain some subordination theorems for analytic functions defined by convolution.

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1. Introduction

Let H be the class of analytic functions in the unit disk $U = \{z \in C : |z| < 1\}$ and let $H[a, k]$ be the subclass of H consisting of functions of the form:

$$f(z) = a + a_k z^k + a_{k+1} z^{k+1} ... \ (a \in C). \tag{1.1}$$

Also, let A_1 be the subclass of H consisting of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.2}$$

If $f, \ g \in H$, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)), \ z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g.,[6] and [17]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $p, h \in H$ and let $\varphi(r, s, t; z) : C^3 \times U \to C$. If p and $\varphi(p(z), zp'(z),$ $z^2 p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z), \tag{1.3}$$

then p is a solution of the differential superordination (1.3). Note that if f is subordinate to g, then g is superordinate to f. An analytic function q is called a subordinant if $q(z) \prec p(z)$ for all p satisfying (1.3). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants of (1.3) is called the best subordinant. Recently Miller and Mocanu [18] obtained conditions on the functions h, q and φ for which the following implication holds:

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z). \tag{1.4}$$

Using the results of Miller and Mocanu [18], Bulboaca [5] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [6]. Ali et al. [1], have used the results of Bulboaca [5] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [30] obtained a sufficient condition for starlikeness of f in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [26] obtained sufficient conditions for the normalized analytic function f to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

For functions f given by (1.1) and $g \in A_1$ given by $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

For functions $f, g \in A_1$, we define the linear operator $D_\lambda^m : A_1 \to A_1$ $(\lambda \geqslant 0, m \in N_0 = N \cup \{0\}, N = \{1, 2, ...\})$ by:

$$D_\lambda^0(f * g)(z) = (f * g)(z)\,,$$

$$D_\lambda^1(f * g)(z) = D_\lambda(f * g)(z) = (1 - \lambda)(f * g)(z) + z\,((f * g)(z))'\,,$$

and (in general)

$$D_\lambda^m(f * g)(z) = D_\lambda(D_\lambda^{m-1}(f * g)(z))$$

$$= z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^m a_k b_k z^k \quad (\lambda \geqslant 0\,; m \in N_0). \tag{1.5}$$

From (1.5), we can easily deduce that

$$\lambda z \left(D_\lambda^m (f * g)(z)\right)' = D_\lambda^{m+1}(f * g)(z) - (1-\lambda) D_\lambda^m (f * g)(z) \; (\lambda > 0). \tag{1.6}$$

We observe that the function $(f * g)(z)$ reduces to several interesting functions for different choices of the function g.

(i) For $\lambda = 1$ and $b_k = 1$ (or $g(z) = \frac{z}{1-z}$), we have $D_1^m (f*g)(z) = D^m f(z)$, where D^m is the Sălăgean operator introduced and studiedby Sălăgean [24];

(ii) For $b_k = 1$ (or $g(z) = \frac{z}{1-z}$), we have $D_\lambda^m (f * g)(z) = D_\lambda^m f(z)$, where D_λ^m is the generalized Sălăgean operator introduced and studied by Al-Oboudi [2];

(iii) For $m = 0$ and

$$g(z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k \; (c \neq 0, -1, -2, ...), \tag{1.7}$$

where

$$(d)_k = \begin{cases} 1 & (k = 0; d \in C \backslash \{0\}) \\ d(d+1)...(d+k-1) & (k \in N; d \in C), \end{cases}$$

we have $D_\lambda^0 (f * g)(z) = (f * g)(z) = L(a, c) f(z)$, where the operator $L(a, c)$ was introduced by Carlson and Shaffer [8];

(iv) For $m = 0$ and

$$g(z) = z + \sum_{k=2}^{\infty} \left[\frac{1 + l + \lambda(k-1)}{1 + l}\right]^s z^k \; (\lambda \geqslant 0; l, s \in N_0), \tag{1.8}$$

we see that $D_\lambda^0 (f*g)(z) = (f*g)(z) = I(s, \lambda, l) f(z)$, where $I(s, \lambda, l)$ is the generalized multiplier transformation which was introduced and studied by Cătaş et al. [9]. The operator $I(s, \lambda, l)$, contains as special cases, the multiplier transformation (see [10]), the generalized Sălăgean operator introduced and studied by Al-Oboudi [2] which in turn contains as special case the Sălăgean operator (see [24]);

(v) For $m = 0$ and

$$g(z) = z + \sum_{k=2}^{\infty} \frac{(\alpha_1)_{k-1} ... (\alpha_l)_{k-1}}{(\beta_1)_{k-1} ... (\beta_s)_{k-1} (1)_{k-1}} z^k, \tag{1.9}$$

where, $\alpha_i, \beta_j \in C^* = C \backslash \{0\}$, $(i = 1, 2, ...l)$, $(j = 1, 2, ...s), l \leq s + 1, l, s \in N_0$, we see that, $D_\lambda^0 (f * g)(z) = (f * g)(z) = H_{l,s}(\alpha_1) f(z)$, where $H_{l,s}(\alpha_1)$ is the Dziok-Srivastava operator introduced and studied by Dziok and Srivastava [11] (see also

[12] and [13]). The operator $H_{l,s}(\alpha_1)$, contains in turn many interesting operators such as, Hohlov linear operator (see [14]), the Carlson-Shaffer linear operator (see [8] and [23]), the Ruscheweyh derivative operator (see [22]), the Barnardi-Libera-Livingston operator (see [4], [15] and [16]) and Owa-Srivastava fractional derivative operator (see [20]);

(vi) For $g(z)$ of the form (1.9), the operator $D_\lambda^m(f*g)(z) = D_\lambda^m(\alpha_1, \beta_1)f(z)$, inroduced and studied by Selvaraj and Karthikeyan [25].

In this paper, we will derive several subordination results involving the operator $D_\lambda^m(f*g)(z)$ and some of its special choises of the function $g(z)$.

2. Definitions and Preliminaries

To prove our results we shall need the following definition and lemmas.

Definition 1 [18]. *Let Q be the set of all functions f that are analytic and injective on $\overline{\mathrm{U}} \setminus E(f)$, where*

$$E(f) = \{\zeta \in \partial\mathrm{U} : \lim_{z\to\zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathrm{U} \setminus E(f)$.

Lemma 1[18]. *Let q be univalent in the unit disc U, and let θ and φ be analytic in a domain D containing $q(\mathrm{U})$, with $\varphi(w) \neq 0$ when $w \in q(\mathrm{U})$. Set $Q(z) = zq'(z)\varphi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(i) Q is a starlike function in U,

(ii) $\mathrm{Re}\,\dfrac{zh'(z)}{Q(z)} > 0,\ z \in \mathrm{U}$.

If p is analytic in U with $p(0) = q(0)$, $p(\mathrm{U}) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.1}$$

then $p(z) \prec q(z)$, and q is the best dominant of (2.1).

Lemma 2 [7]. *Let q be a univalent function in the unit disc U and let θ and φ be analytic in a domain D containing $q(\mathrm{U})$. Suppose that*

(i) $\mathrm{Re}\,\dfrac{\theta'(q(z))}{\varphi(q(z))} > 0$ for $z \in \mathrm{U}$,

(ii) $h(z) = zq'(z)\varphi(q(z))$ is starlike in U.

If $p \in H[q(0), 1] \cap Q$, with $p(\mathrm{U}) \subseteq D$, $\theta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U, and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.2}$$

then $q(z) \prec p(z)$, *and* q *is the best subordinant of (2.2).*

The following lemma gives us a necessary and sufficient condition for the univalence of a special function which will be used in some particular cases.

Lemma 3 [21] . *The function* $q(z) = (1-z)^{-2ab}$ *is univalent in* U *if and only if* $|2ab-1| \leq 1$ *or* $|2ab+1| \leq 1$.

3. Main Results

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\lambda > 0, \alpha, \delta, \zeta \in C$, $\mu, \beta \in C^* = C\backslash\{0\}$, $m \in N_0$ and the powers are understood as principle values.

Theorem 1. *Let* $\left(\frac{D_\lambda^{m+1}(f*g)(z)}{z}\right)^\mu \in H$ *and let* $q(z)$ *be analytic and univalent in* $U, q(z) \neq 0$ $(z \in U)$. *Suppose that* $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* U. *Let*

$$\text{Re}\left\{1 + \frac{\zeta}{\beta}q(z) + \frac{2\delta}{\beta}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0 \ (\beta \in C^*) \tag{3.1}$$

and

$$\chi(\alpha, \delta, \beta, \zeta, \mu, \lambda, f, g)(z) = \alpha + \zeta\left(\frac{D_\lambda^{m+1}(f*g)(z)}{z}\right)^\mu + \delta\left(\frac{D_\lambda^{m+1}(f*g)(z)}{z}\right)^{2\mu}$$
$$+\frac{\beta\mu}{\lambda}\left(\frac{D_\lambda^{m+2}(f*g)(z)}{D_\lambda^{m+1}(f*g)(z)} - 1\right). \tag{3.2}$$

If $q(z)$ *satisfies the following subordination:*

$$\chi(\alpha, \delta, \beta, \zeta, \mu, \lambda, f, g)(z) \prec \alpha + \zeta q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{D_\lambda^{m+1}(f*g)(z)}{z}\right)^\mu \prec q(z) \tag{3.3}$$

and $q(z)$ *is the best dominant.*

Proof. Define $p(z)$ by

$$p(z) = \left(\frac{D_\lambda^{m+1}(f * g)(z)}{z} \right)^\mu \quad (z \in U). \tag{3.4}$$

Differentiating (3.4) logarithmically with respect to z and using the identity (1.6) in the resulting equation, we have

$$\frac{zp'(z)}{p(z)} = \frac{\mu}{\lambda} \left(\frac{D_\lambda^{m+2}(f * g)(z)}{D_\lambda^{m+1}(f * g)(z)} - 1 \right).$$

Setting $\theta(w) = \alpha + \zeta w + \delta w^2$ and $\varphi(w) = \dfrac{\beta}{w}$, we can easily verify that θ is analytic in C, φ is analytic in C^* and $\varphi(w) \neq 0 (w \in C^*)$. Also, letting

$$Q(z) = zq'(z)\varphi(q(z)) = \beta \frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \alpha + \zeta q(z) + \delta(q(z))^2 + \beta \frac{zq'(z)}{q(z)}.$$

We can verify that $Q(z)$ is starlike univalent in U and

$$\mathrm{Re}\{\frac{zh'(z)}{Q(z)}\} = \mathrm{Re} \left\{ 1 + \frac{\zeta}{\beta} q(z) + \frac{2\delta}{\beta}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)} \right\} > 0.$$

The theorem follows by applying Lemma 1.

Taking $g(z)$ of the form (1.9) and using the identity (see [25])

$$z \left(D_\lambda^m(\alpha_1, \beta_1) f(z) \right)' = \alpha_1 D_\lambda^m(\alpha_1 + 1, \beta_1) f(z) - (\alpha_1 - 1) D_\lambda^m(\alpha_1, \beta_1) f(z), \tag{3.5}$$

we get the result obtained by Selvaraj and Karthikeyan [25, Theorem 2.1].

Taking $g(z)$ of the form (1.9) and using the identity (see [25])

$$\lambda z \left(D_\lambda^m(\alpha_1, \beta_1) f(z) \right)' = D_\lambda^{m+1}(\alpha_1, \beta_1) f(z) - (1 - \lambda) D_\lambda^m(\alpha_1, \beta_1) f(z), \tag{3.6}$$

we get the following result which corrects the result of Selvaraj and Karthikeyan [25, Theorem 2.2].

Corollary 1. *Let* $\left(\dfrac{D_\lambda^{m+1}(\alpha_1, \beta_1) f(z)}{z} \right)^\mu \in H$ *and let* $q(z)$ *be analytic and univalent in* $U, q(z) \neq 0$ $(z \in U)$. *Suppose that* $\dfrac{zq'(z)}{q(z)}$ *is starlike univalent in* U, (3.1) *holds and*

$$\chi_1(\alpha_1,\beta_1,\alpha,\delta,\beta,\zeta,\mu,\lambda,f)(z)=\alpha+\zeta\left(\frac{D_\lambda^{m+1}(\alpha_1,\beta_1)f(z)}{z}\right)^{\mu}$$

$$+\delta\left(\frac{D_\lambda^{m+1}(\alpha_1,\beta_1)f(z)}{z}\right)^{2\mu}+\frac{\beta\mu}{\lambda}\left(\frac{D_\lambda^{m+2}(\alpha_1,\beta_1)f(z)}{D_\lambda^{m+1}(\alpha_1,\beta_1)f(z)}-1\right). \qquad (3.7)$$

If $q(z)$ *satisfies the following subordination:*

$$\chi_1(\alpha_1,\beta_1,\alpha,\delta,\beta,\zeta,\mu,\lambda,f)(z)\prec\alpha+\zeta q(z)+\delta(q(z))^2+\beta\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{D_\lambda^{m+1}(\alpha_1,\beta_1)f(z)}{z}\right)^{\mu}\prec q(z)$$

and $q(z)$ *is the best dominant.*

Taking $m=0,\lambda=1$ and $g(z)$ of the form (1.7) and using the identity (see [23])

$$z\left(L(a,c)f(z)\right)'=aL(a+1,c)f(z)-(a-1)L(a,c)f(z), \qquad (3.8)$$

we get the following result which corrects the result of Shammugam et al.[27, Theorem 3].

Corollary 2. *Let* $\left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu}\in H$ *and let* $q(z)$ *be analytic and univalent in* $U, q(z)\neq 0$ $(z\in U)$*. Suppose that* $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* U*,* (3.1) *holds and*

$$\chi_2(a,c,\alpha,\delta,\beta,\zeta,\mu,f)(z)=\alpha+\zeta\left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu}+\delta\left(\frac{L(a+1,c)f(z)}{z}\right)^{2\mu}$$

$$+\beta\mu(a+1)\left(\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}-1\right). \qquad (3.9)$$

If $q(z)$ *satisfies the following subordination:*

$$\chi_2(a,c,\alpha,\delta,\beta,\zeta,\mu,f)(z)\prec\alpha+\zeta q(z)+\delta(q(z))^2+\beta\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu}\prec q(z)$$

and $q(z)$ is the best dominant.

Taking the function $q(z) = \dfrac{1+Az}{1+Bz}$ in Theorem 1, where $-1 \leq B < A \leq 1$, the condition (3.1) becoms

$$\text{Re}\left\{1 + \frac{\zeta}{\beta}\frac{1+Az}{1+Bz} + \frac{2\delta}{\beta}(\frac{1+Az}{1+Bz})^2 - \frac{2Bz}{1+Bz} - \frac{(A-B)z}{(1+Bz)(1+Az)}\right\} > 0 \ (\beta \in C^*), \tag{3.10}$$

hence, we have the following corollary.

Corollary 3. *Let* $q(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ and (3.10) *holds true. If* $f(z) \in A_1$ *and*

$$\chi(\alpha, \delta, \zeta, \beta, \mu, \lambda, f, g)(z) \prec \alpha + \zeta\left(\frac{1+Az}{1+Bz}\right) + \delta\left(\frac{1+Az}{1+Bz}\right)^2 + \beta\frac{(A-B)z}{(1+Az)(1+Bz)},$$

where $\chi(\alpha, \delta, \beta, \zeta, \mu, \lambda, f, g)(z)$ *is given by* (3.2), *then*

$$\left(\frac{D_\lambda^{m+1}(f*g)(z)}{z}\right)^\mu \prec \frac{1+Az}{1+Bz}$$

and $\dfrac{1+Az}{1+Bz}$ *is the best dominant.*

Taking the function $q(z) = \left(\dfrac{1+z}{1-z}\right)^\gamma$ $(0 < \gamma \leq 1)$, in Theorem 1, the condition (3.1) becoms

$$\text{Re}\left\{1 + \frac{\zeta}{\beta}\left(\frac{1+z}{1-z}\right)^\gamma + \frac{2\delta}{\beta}\left(\frac{1+z}{1-z}\right)^{2\gamma} + \frac{2z^2}{1-z^2}\right\} > 0 \ (\beta \in C^*), \tag{3.11}$$

hence, we have the following corollary.

Corollary 4. *Let* $q(z) = \left(\dfrac{1+z}{1-z}\right)^\gamma$ $(0 < \gamma \leq 1)$ and (3.11) *holds true. If* $f(z) \in A_1$ *and*

$$\chi(\alpha, \delta, \beta, \mu, \lambda, f, g)(z) \prec \alpha + \zeta\left(\frac{1+z}{1-z}\right)^\gamma + \delta\left(\frac{1+z}{1-z}\right)^{2\gamma} + \beta\frac{2\gamma z}{(1-z)^2},$$

where $\chi(\alpha, \delta, \beta, \zeta, \mu, \lambda, f, g)(z)$ *is given by* (3.2), *then*

$$\left(\frac{D_\lambda^{m+1}(f*g)(z)}{z}\right)^\mu \prec \left(\frac{1+z}{1-z}\right)^\gamma$$

and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best dominant.*

Taking $q(z) = e^{\mu A z}, |\mu A| < \pi$, Theorem 1, the condition (3.1) becoms

$$\operatorname{Re}\left\{1 + \frac{\zeta}{\beta}e^{\mu\gamma A z} + \frac{2\delta}{\beta}e^{2\mu\gamma A z}\right\} > 0 \ (\beta \in C^*), \tag{3.12}$$

hence, we have the following corollary.

Corollary 5. *Let* $q(z) = e^{\mu A z}, |\mu A| < \pi$ *and* (3.12) *holds. If* $f(z) \in A_1$ *and* $\chi(\alpha, \delta, \beta, \zeta, \mu, \lambda, f, g)(z) \prec \alpha + \zeta e^{\mu A z} + \delta e^{2\mu A z} + \beta A \mu z$*, where* $\chi(\alpha, \delta, \beta, \zeta, \mu, \lambda, f, g)(z)$ *is given by* (3.2)*, then*

$$\left(\frac{D_{\lambda}^{m+1}(f * g)(z)}{z}\right)^{\mu} \prec e^{\mu A z} \ (\mu \in C^*)$$

and $e^{\mu A z}$ *is the best dominant.*

Taking $m = 0, g(z) = \frac{z}{1-z}, \delta = \zeta = 0, \lambda = \alpha = \mu = 1, \beta = \frac{1}{b}(b \in C^*)$ and $q(z) = \frac{1}{(1-z)^{2b}}$ in Theorem 1, we obtain the result obtained by Srivastava and Lashin [29, Corollary 1].

Theorem 2. *Let* $\left(\frac{D_{\lambda}^{m}(f * g)(z)}{z}\right)^{\mu} \in H$ *and let* $q(z)$ *be analytic and univalent in* $U, q(z) \neq 0$ $(z \in U)$*. Suppose that* $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* U *and* (3.1) *holds and*

$$\eta(\alpha, \delta, \beta, \zeta, \mu, f, g)(z) = \alpha + \zeta\left(\frac{D_{\lambda}^{m}(f * g)(z)}{z}\right)^{\mu} + \delta\left(\frac{D_{\lambda}^{m}(f * g)(z)}{z}\right)^{2\mu}$$

$$+ \frac{\beta\mu}{\lambda}\left(\frac{D_{\lambda}^{m+1}(f * g)(z)}{D_{\lambda}^{m}(f * g)(z)} - 1\right).$$

If $q(z)$ *satisfies the following subordination:*

$$\eta(\alpha, \delta, \beta, \zeta, \mu, f, g)(z) \prec \alpha + \zeta q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{D_{\lambda}^{m}(f * g)(z)}{z}\right)^{\mu} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. The proof is similar to the proof of Theorem 1, and hence we omit it.

Taking $m=0, g(z)=\dfrac{z}{1-z}, \delta=\zeta=0, \lambda=\alpha=1, \mu=a, \beta=\frac{1}{ab}(a, b \in C^{*})$ and $q(z)=\dfrac{1}{(1-z)^{2ab}}$ in Theorem 2 and combining with Lemma 3, we obtain the result due to Obradović et al. [19, Theorem 1];

Taking $m=0, g(z)=\dfrac{z}{1-z}, \delta=\zeta=0, \lambda=\alpha=1, \mu=a, \beta=\dfrac{e^{i\lambda}}{ab\cos\lambda}(a, b \in C^{*}, |\lambda|<\frac{\pi}{2})$ and $q(z)=\dfrac{1}{(1-z)^{2ab\cos\lambda e^{-i\lambda}}}$ in Theorem 2 and combining with Lemma 3, we obtain the result due to Aouf et al. [3, Theorem 1].

Taking $m=0, \lambda=1$ and $g(z)$ of the form (1.7) and using the identity (3.8), we get the following result which corrects the result of Shammugam et al.[28, Theorem 3.1].

Corollary 6. *Let $\left(\dfrac{L(a,c)f(z)}{z}\right)^{\mu} \in H$ and let $q(z)$ be analytic and univalent in $U, q(z) \neq 0$ $(z \in U)$. Suppose that $\dfrac{zq'(z)}{q(z)}$ is starlike univalent in U, (3.1) holds and*

$$\chi_3(a,c,\alpha,\delta,\beta,\zeta,\mu,f)(z)=\alpha+\zeta\left(\frac{L(a,c)f(z)}{z}\right)^{\mu}+\delta\left(\frac{L(a,c)f(z)}{z}\right)^{2\mu}$$
$$+\beta\mu a\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}-1\right).$$

If $q(z)$ satisfies the following subordination:

$$\chi_3(a,c,\alpha,\delta,\beta,\zeta,\mu,f)(z) \prec \alpha+\zeta q(z)+\delta(q(z))^2+\beta\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu} \prec q(z)$$

and $q(z)$ is the best dominant.

Theorem 3. *Let $q(z)$ be convex, univalent in U, $q(z) \neq 0$ and $\dfrac{zq'(z)}{q(z)}$ be starlike univalent in U. Suppose that*

$$\operatorname{Re}\left\{\frac{2\delta}{\beta}(q(z))^2+\frac{\zeta}{\beta}q(z)\right\}q'(z)>0. \tag{3.11}$$

If $f(z) \in A, 0 \neq \left(\frac{D_{\lambda}^{m+1}(f*g)(z)}{z}\right)^{\mu} \in H[q(0),1] \cap Q$, *and* $\chi(\alpha,\delta,\beta,\zeta,\mu,\lambda,f,g)(z)$ *is univalent in* U, *then*

$$\alpha+\zeta q(z)+\delta(q(z))^2+\beta\frac{zq'(z)}{q(z)} \prec \chi(\alpha,\delta,\beta,\zeta,\mu,\lambda,f,g)(z)$$

implies

$$q(z) \prec \left(\frac{D_{\lambda}^{m+1}(f*g)(z)}{z}\right)^{\mu} \tag{3.12}$$

and $q(z)$ *is the best subordinant,* $\chi(\alpha,\delta,\beta,\zeta,\mu,\lambda,f,g)(z)$ *is given by* (3.2).

Proof. Let $\theta(w)=\alpha+\zeta w+\delta w^2$ and $\varphi(w)=\frac{\beta}{w}$, we can verify that θ is analytic in , φ is analytic in C^* and $\varphi(w)\neq 0 (w \in C^*)$. Since $q(z)$ is convex, it follows that

$$\operatorname{Re}\frac{\theta'(q(z))}{\varphi(q(z))}=\operatorname{Re}\left\{\frac{2\delta}{\beta}(q(z))^2+\frac{\zeta}{\beta}q(z)\right\}q'(z)>0.$$

The assertion (3.12) follows by an application of Lemma 2. This completes the proof of Theorem 3.

Combining Theorem 1 and Theorem 3, we get the following sandwich theorem.

Theorem 4. Let q_1 and q_2 be univalent in U such that $q_1(z)\neq 0$ and $q_2(z)\neq 0$ $(z \in U)$, $\frac{zq_1'(z)}{q_1(z)}$ and $\frac{zq_2'(z)}{q_2(z)}$ are starlike univalent. Suppose that q_1 and q_2 satisfies (3.11) and (3.1), respectively. If $f \in A_1$, $\left(\frac{D_{\lambda}^{m+1}(f*g)(z)}{z}\right)^{\mu} \in H[q(0),1] \cap Q$, *and* $\chi(\alpha,\delta,\beta,\zeta,\mu,\lambda,f,g)(z)$ *is univalent in* U, *then*

$$\alpha+\zeta q_1(z)+\delta(q_1(z))^2+\beta\frac{zq_1'(z)}{q_1(z)} \prec \chi(\alpha,\delta,\beta,\zeta,\mu,\lambda,f,g)(z)$$

$$\prec \alpha+\zeta q_2(z)+\delta(q_2(z))^2+\beta\frac{zq_2'(z)}{q_2(z)}$$

implies

$$q_1(z) \prec \left(\frac{D_{\lambda}^{m+1}(f*g)(z)}{z}\right)^{\mu} \prec q_2(z)$$

and q_1 *and* q_2 *are the best subordinant and the best dominant, respectively and* $\chi(\alpha,\delta,\beta,\zeta,\mu,\lambda,f,g)(z)$ *is given by* (3.2).

Remark. According to Corollary 2, Theorems 3 and 4 correct the results obtained by Shammugam et al.[27, Theorems 4 and 5, respectively] for $m = 0, \lambda = 1$ and $g(z)$ of the form (1.7).

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