Acta Universitatis Apulensis No. 23/2010
ISSN: 1582-5329 pp. 49-61

# A GENERALIZATION OF STRONGLY CESÀRO AND STRONGLY LACUNARY SUMMABLE SPACES

Iqbal H. Jebril

Abstract. In this paper we present a generalization of strongly Cesàro summable and strongly lacunary summable sequences by means of considering base space, a finite dimensional real 2-normed linear space and a generalized difference operator. We investigate the spaces under the action of different difference operators and show that these spaces become 2-Banach spaces when the base space is a 2-Banach space. We also prove that convergence and completeness in the 2-norm is equivalent to those in the derived norm as well as show that their topology can be fully described by using derived norm. Further we compute the 2-isometric spaces, investigate the relationship among the spaces and prove the Fixed Point Theorem for these 2-Banach spaces.

*Keywords and phrases:* 2-norm; Difference sequence space; Cesàro summable sequence; Lacunary summable sequence; Completeness; Fixed point theorem.

2000 *Mathematics Subject Classification*: 46B04; 40A05; 46A45.

## 1.Introduction

The concept of 2-normed spaces was initially developed by Gähler [8] in the mid of 1960's. Since then, Gunawan and Mashadi [9], Gürdal [10], Mazaheri and Kazemi [13] and many others have studied this concept and obtained various results.

Let $X$ be a real vector space of dimension $d$, where $d \geq 2$. A real valued function $\|\bullet, \bullet\|$ on $X^2$ satisfying the following four conditions:
$2N_1$ :$\|x_1, x_2\| = 0$ if and only if $x_1, x_2$ are linearly dependent,
$2N_2$ : $\|x_1.x_2\|$ is invariant under permutation,
$2N_3$ : $\|\alpha x_1, x_2\| = |\alpha|\ \|x_1, x_2\|$, for any $\alpha \in R$,
$2N_4$ : $\|x + x\prime, x_2\| \leq \|x, x_2\| + \|x\prime, x_2\|$,
is called a 2-norm on $X$ and the pair $(X, \|\bullet, \bullet\|)$ is called a 2-normed space.

A sequence $(x_k)$ in a 2-normed space $(X, \|\bullet, \bullet\|)$ is said to converge to some $L \in X$ in the 2-norm if $\lim\limits_{k\to\infty} \|x_k - L, u_1\| = 0$, for every $u_1 \in X$.

A sequence $(x_k)$ in a 2-normed space $(X, \|\bullet, \bullet\|)$ is said to be Cauchy sequence with respect to the 2-norm if $\lim\limits_{k,l\to\infty} \|x_k - x_l, u_1\| = 0$, for every $u_1 \in X$.

If every Cauchy sequence in $X$ converges to some $L \in X$, then $X$ is said to be complete with respect to the 2-norm. Any complete 2-normed space is said to be 2-Banach space.

The notion of difference sequence space was introduced by Kizmaz[11], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [4] by introducing the spaces $\ell_\infty(\Delta^s)$, $c(\Delta^s)$ and $c_0(\Delta^s)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [14], who studied the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$.

Tripathy, Esi and Tripathy [15] generalized the above notions and unified these as follows:

Let $m, s$ be non-negative integers, then for $Z$ a given sequence space we have $Z(\Delta_m^s)=\{x = (x_k) \in w : (\Delta_m^s x_k) \in Z\}$, where $\Delta_m^s x = (\Delta_m^s x_k) = (\Delta_m^{s-1} x_k - \Delta_m^{s-1} x_{k+m})$, and $\Delta_m^0 x_k = x_k$ for all $k \in N$, which is equivalent to the following binomial representation:

$$\Delta_m^s x_k = \sum_{v=0}^{s} (-1)^v \begin{pmatrix} s \\ v \end{pmatrix} x_{k+mv}$$

Let $m, s$ be non-negative integers, then for $Z$ a given sequence space we define: $Z(\Delta_{(m)}^s)=\{x = (x_k) \in w : (\Delta_{(m)}^s x_k) \in Z\}$, where $\Delta_{(m)}^s x = (\Delta_{(m)}^s x_k) = (\Delta_{(m)}^{s-1} x_k - \Delta_{(m)}^{s-1} x_{k-m})$, and $\Delta_{(m)}^0 x_k = x_k$ for all $k \in N$, which is equivalent to the following binomial representation:

$$\Delta_{(m)}^s x_k = \sum_{v=0}^{s} (-1)^v \begin{pmatrix} s \\ v \end{pmatrix} x_{k-mv}$$

We take $x_k = 0$, for non-positive values of $k$ (1.1) (see details [2])

Functional analytic studies of the spaces $|\sigma_1|$ of strongly Cesàro summable sequences can be found in Borwein [1], Freedman, Sember and Raphael [5] and Maddox [12].

The spaces $|\sigma_1|$ of strongly Cesàro summable sequence is defined as follows:

$$|\sigma_1| = \{x = (x_k) : \text{there exists } L \text{ such that } \tfrac{1}{p} \sum_{k=1}^{p} |x_k - L| \to 0\},$$

which is a Banach space normed by

$$\|x\| = \sup_p \left( \tfrac{1}{p} \sum_{k=1}^{p} |x_k| \right).$$

By a lacunary sequence $\theta = (k_p)$, $p = 1, 2, 3, \ldots$, where $k_0 = 0$, we mean an increasing sequence of non-negative integers with $h_p = (k_p - k_{p-1}) \to \infty$ as $p \to \infty$.

We denote $I_p = (k_{p-1}, k_p]$ and $\eta_p = \frac{k_p}{k_{p-1}}$ for $p = 1, 2, 3, \ldots$. The space of strongly lacunary summable seqauence $N_\theta$ was defined by Freedman, Sember adn Raphel [5] as follows:

$$N_\theta = \{x = (x_k) : \lim_{p\to\infty} \frac{1}{h_p} \sum_{k\in I_p} |x_k - L| = 0, \text{ for some } L\}.$$

The space $N_\theta$ is a Banach space with the norm

$$\|x\|_\theta = \sup_p \frac{1}{h_p} \sum_{k\in I_p} |x_k|.$$

## 2. Main results and discussions

In this section we define some new definitions, give some examples of 2-norm and investigate the main results of this article.

A divergent sequence has no limit in the usual sense. In summability theory, one aims at associating with certain divergent sequences a limit in a generalized sense. There are different types of summability techniques for different purposes. Our next aim is to extend the notion of two very famous summability methods to 2-normed linear space valued sequences.

Let $(X, \|\bullet, \bullet\|)_X$ be a finite dimensional 2-normed space and $w(X)$ denotes $X$-valued sequence space. Then for non-negative integer's $m$ and $s$, we define the following sequence spaces:

We denote $|\sigma_1|(\|\bullet, \bullet\|, \Delta^s_{(m)})$ the set of all $X$-valued strongly $\Delta^s_{(m)}$-Cesàro summable sequence is defined by

$|\sigma_1|(\|\bullet, \bullet\|, \Delta^s_{(m)}) = \{x \in w(X) : \lim_{p\to\infty} \frac{1}{p} \sum_{k=1}^{p} \|\Delta^s_{(m)} x_k - L, z_1\|_X = 0,$ for every $z_1 \in X$ and for some $L\}$,

For $L = 0$, we write this space as $|\sigma_1|^0(\|\bullet, \bullet\|, \Delta^s_{(m)})$.

Let $\theta$ be a lacunary sequence. Then we denote by $N_\theta(\|\bullet, \bullet\|, \Delta^s_{(m)})$ the set of all $X$-valued strongly $\Delta^s_{(m)}$-lacunary summable sequences and defined by

$N_\theta(\|\bullet, \bullet\|, \Delta^s_{(m)}) = \{x \in w(X) : \lim_{p\to\infty} \frac{1}{h_p} \sum_{k\in I_p} \|\Delta^s_{(m)} x_k - L, z_1\|_X = 0,$ for every $z_1 \in X$ and for some $L\}$.

For $L = 0$, we write this space as $N^0_\theta(\|\bullet, \bullet\|, \Delta^s_{(m)})$.

In the special case where $\theta = (2^p)$, we have $N_\theta(\|\bullet, \bullet\|, \Delta^s_{(m)}) = |\sigma_1|(\|\bullet, \bullet\|, \Delta^s_{(m)})$. Taking $s = 0$, the above spaces reduce to the spaces $|\sigma_1|(\|\bullet, \bullet\|)$ and $N_\theta(\|\bullet, \bullet\|)$ introduced and studied by Dutta [3].

Also $|\sigma_1|(\|\bullet, \bullet\|, \Delta^i_{(m)}) \subset |\sigma_1|(\|\bullet, \bullet\|, \Delta^s_{(m)})$ and $N_\theta(\|\bullet, \bullet\|, \Delta^i_{(m)}) \subset N_\theta(\|\bullet, \bullet\|, \Delta^s_{(m)})$, for every $i = 0, 1, \ldots, (s-1)$. Proof follows from the following inequality:

$\frac{1}{t}\sum_{k=1}^{t}\|\Delta^s_{(m)}x_k, z_1\|_X \leq \frac{1}{t}\sum_{k=1}^{t}\|\Delta^{s-1}_{(m)}x_k, z_1\|_X + \frac{1}{t}\sum_{k=1}^{t}\|\Delta^{s-1}_{(m)}x_{k-m}, z_1\|_X$ for each $t \in N$.

**Example 1.** Let us take $X = R^2$ and consider a 2-norm $\|\bullet,\bullet\|_X$ defined as:

$$\|x_1, x_2\|_X = abs\left(\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}\right)$$

where $x_i = (x_{i1}, x_{i2}) \in R^2$ for each $i = 1, 2$.

Consider the divergent sequence $x = \{\overline{1}, \overline{2}, \overline{3}, \ldots\} \in w(X)$, where $\overline{k} = (k, k)$, for each $k \in N$. Then $x$ belong to $|\sigma_1|(\|\bullet,\bullet\|, \Delta)$ and $N_\theta(\|\bullet,\bullet\|, \Delta)$, for $\theta = (2^p)$. Hence $x$ belongs to $|\sigma_1|(\|\bullet,\bullet\|, \Delta^s_{(m)})$ and $N_\theta(\|\bullet,\bullet\|, \Delta^s_{(m)})$, for $\theta = (2^p)$ for each $s, m > 1$.

Similarly we can have the summability spaces $N_\theta(\|\bullet,\bullet\|, \Delta^s_m)$ and $|\sigma_1|(\|\bullet,\bullet\|, \Delta^s_m)$.

**Proposition 2.1.** *The spaces* $N_\theta(\|\bullet,\bullet\|, \Delta^s_{(m)})$ *and* $|\sigma_1|(\|\bullet,\bullet\|, \Delta^s_{(m)})$, $N_\theta(\|\bullet,\bullet\|, \Delta^s_m)$ *and* $|\sigma_1|(\|\bullet,\bullet\|, \Delta^s_m)$ *are linear*

**Proof.** Proof is easy and so omitted.

Now we present some relationship between the spaces $|\sigma_1|(\|\bullet,\bullet\|, \Delta^s_{(m)})$ and $N_\theta(\|\bullet,\bullet\|, \Delta^s_{(m)})$ and also between the spaces $|\sigma_1|(\|\bullet,\bullet\|, \Delta^s_m)$ and $N_\theta(\|\bullet,\bullet\|, \Delta^s_m)$. We prove the results for the spaces $|\sigma_1|(\|\bullet,\bullet\|, \Delta^s_{(m)})$ and $N_\theta(\|\bullet,\bullet\|, \Delta^s_{(m)})$ and for the other two spaces it will follow on applying similar arguments.

**Proposition 2.2.** *Let* $\theta = (k_p)$ *be a lacunary sequence with* $\liminf_p \eta_p > 1$, *then* $|\sigma_1|(\|\bullet,\bullet\|, \Delta^s_{(m)}) \subseteq N_\theta(\|\bullet,\bullet\|, \Delta^s_{(m)})$.

**Proof.** Let $\liminf_p \eta_p > 1$. Then there exists a $\nu > 0$ such that $1 + \nu \leq \eta_p$ for all $p \geq 1$. Let $x \in |\sigma_1|(\|\bullet,\bullet\|, \Delta^s_{(m)})$. Then there exists some $L \in X$ such that $\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{t}\|\Delta^s_{(m)}x_k - L, z_1\|_X = 0$, for every $z_1 \in X$. Now we write

$$\frac{1}{h_p}\sum_{k\in I_p}\|\Delta^s_{(m)}x_k - L, z_1\|_X = \frac{1}{h_p}\sum_{1\leq i\leq k_p}\|\Delta^s_{(m)}x_i - L, z_1\|_X - \frac{1}{h_p}\sum_{1\leq i\leq k_{p-1}}\|\Delta^s_{(m)}x_i - L, z_1\|_X$$

$$= \frac{k_p}{h_p}\left(\frac{1}{k_p}\sum_{1\leq i\leq k_p}\|\Delta^s_{(m)}x_i - L, z_1\|_X\right) - \frac{k_{p-1}}{h_p}\left(\frac{1}{k_{p-1}}\sum_{1\leq i\leq k_{p-1}}\|\Delta^s_{(m)}x_i - L, z_1\|_X\right) \quad (2.1)$$

Now we have $\frac{k_p}{h_p} \leq \frac{1+\nu}{\nu}$ and $\frac{k_{p-1}}{h_p} \leq \frac{1}{\nu}$, since $h_p = k_p - k_{p-1}$. Hence using (2.1), we have $x \in N_\theta(\|\bullet,\bullet\|, \Delta^s_{(m)})$.

**Proposition 2.3.** *Let* $\theta = (k_p)$ *be a lacunary sequence with* $\limsup_p \eta_p < \infty$*, then* $N_\theta(\|\bullet,\bullet\|, \Delta^s_{(m)}) \subseteq |\sigma_1|(\|\bullet,\bullet\|, \Delta^s_{(m)})$.

**Proof.** Let $\limsup_p \eta_p < \infty$. Then there exists $M > 0$ such that $\eta_p < M$ for all $p \geq 1$. Let $x \in N^0_\theta(\|\bullet,\bullet\|, \Delta^s_{(m)})$ and $\epsilon > 0$. We can find $R > 0$ and $K > 0$ such that

$$\sup_{i\geq R} S_i = \sup_{i\geq R}\left(\frac{1}{h_i}\sum_{i=1}^{k_i}\|\Delta^s_{(m)}x_i, z_1\|_X - \frac{1}{h_i}\sum_{i=1}^{k_{i-1}}\|\Delta^s_{(m)}x_i, z_1\|_X\right) < \varepsilon$$

and $S_i < K$ for all $i = 1, 2, \ldots$. Then if $t$ is any integer with $k_{p-1} < t \leq k_p$, where $p > R$, we can write

$$\frac{1}{t}\sum_{i=1}^{t}\|\Delta^s_{(m)}x_i, z_1\|_X \leq \frac{1}{k_{p-1}}\sum_{i=1}^{k_p}\|\Delta^s_{(m)}x_i, z_1\|_X$$

$$= \frac{1}{k_{p-1}}\Big(\sum_{I_1}\|\Delta^s_{(m)}x_i, z_1\|_X + \sum_{I_2}\|\Delta^s_{(m)}x_i, z_1\|_X + \cdots + \sum_{I_p}\|\Delta^s_{(m)}x_i, z_1\|_X\Big)$$

$$= \frac{k_1}{k_{p-1}}S_1 + \frac{k_2 - k_1}{k_{p-1}}S_2 + \ldots + \frac{k_R - k_{R-1}}{k_{p-1}}S_R + \frac{k_{R+1} - k_R}{k_{p-1}}S_{R+1} + \ldots + \frac{k_p - k_{p-1}}{k_{p-1}}S_p$$

$$\leq \left(\sup_{i\geq 1} S_i\right)\frac{k_R}{k_{p-1}} + \left(\sup_{i\geq R} S_i\right)\frac{k_p - k_R}{k_{p-1}}$$

$$< K\frac{k_R}{K_{p-1}} + \epsilon M$$

Since $k_{p-1} \to \infty$ as $t \to \infty$, it follows that $x \in |\sigma_1|^0(\|\bullet,\bullet\|, \Delta^s_{(m)})$. The general inclusion $N_\theta(\|\bullet,\bullet\|, \Delta^s_{(m)}) \subseteq |\sigma_1|(\|\bullet,\bullet\|, \Delta^s_{(m)})$ follows by linearity.

**Proposition 2.4.** *Let* $\theta = (k_p)$ *be a lacunary sequence with* $1 < \liminf_p \eta_p \leq \limsup_p \eta_p < \infty$*, then* $|\sigma_1|(\|\bullet,\bullet\|, \Delta^s_{(m)}) = N_\theta(\|\bullet,\bullet\|, \Delta^s_{(m)})$.

**Proof.** Proof follows by combining Propostion 2.2 and Proposition2.3.

One may find it interesting to see the differences between the spaces $|\sigma_1|(\|\bullet,\bullet\|, \Delta^s_{(m)})$ and $|\sigma_1|(\|\bullet,\bullet\|, \Delta^s_m)$ as well as between the spaces $N_\theta(\|\bullet,\bullet\|, \Delta^s_{(m)})$ and $N_\theta(\|\bullet,\bullet\|, \Delta^s_m)$ through the following definition of 2-norm.

**Theorem 2.5.** (i) *Let* $Y$ *be the spaces* $|\sigma_1|(\|\bullet,\bullet\|,\Delta^s_{(m)})$. *We define the following function* $\|\bullet,\bullet\|_Y$ *on* $Y\times Y$ *by*
$\|x,y\|_Y=0$ *if* $x,y$ *are linearly dependent,*
$=\sup_p \frac{1}{p}\sum_{k=1}^{p}\|\Delta^s_{(m)}x_k,z_1\|_X$ *for every* $z_1\in X$, *if* $x,y$ *are linearly independent.*
*Then* $\|\bullet,\bullet\|_Y$ *is a 2-norm on* $Y$. (2.2)
*(ii) Let* $H$ *be the space* $|\sigma_1|(\|\bullet,\bullet\|,\Delta^s_m)$. *We define the following function* $\|\bullet,\bullet\|_H$ *on* $H\times H$ *by*
$\|x,y\|_H=0$ *if* $x,y$ *are linearly dependent,*
$=\sum_{k=1}^{ms}\|x_k,z_1\|_X+\sup_p\frac{1}{p}\sum_{k=1}^{p}\|\Delta^s_m x_k,z_1\|_X$ *for every* $z_1\in X$, *if* $x,y$ *are linearly independent.*
*Then* $\|\bullet,\bullet\|_H$ *is a 2-norm on* $H$. (2.3)

**Proof.**(i) If $x^1$, $x^2$ are linearly dependent, then $\|x^1,x^2\|_Y=0$. Conversely assume $\|x^1,x^2\|_Y=0$. Then using (2.2) we have

$$\sup_p\frac{1}{p}\sum_{k=1}^{p}\|\Delta^s_{(m)}x^1_k,z_1\|_X=0,$$

for every $z_1\in X$. Hence we have $\|\Delta^s_{(m)}x^1_k,z_1\|_X=0$, for all $k\geq 1$ and for every $z_1\in X$. Hence we must have $\Delta^s_{(m)}x^1_k=0$ for all $k\geq 1$. Let $k=1$, then

$$\Delta^s_{(m)}x^1_k=\sum_{i=0}^{s}(-1)^i\begin{pmatrix} s\\ v\end{pmatrix}x^1_{1-mi}=0$$

and so $x^1_1=0$, by putting $x^1_{1-mi}=0$ for $i=1,\ldots,s$ [See (1.1)]. Similarly taking $k=2,\ldots,ms$, we have $x^1_2=x^1_3=\ldots=x^1_{ms}=0$. Next let $k=ms+1$, then

$$\Delta^s_{(m)}x^1_{ms+1}=\sum_{i=0}^{s}(-1)^i\begin{pmatrix} s\\ v\end{pmatrix}x^1_{1+ms-mi}=0.$$

Since $x^1_1=x^1_2=\ldots=x^1_{ms}=0$, we have $x^1_{ms+1}=0$. Proceeding in this way we can conclude that $x^1_k=0$ for all $k\geq 1$. Hence $x^1=\theta$ and so $x^1,x^2$ are linearly dependent. It is obvious that $\|x^1,x^2\|_Y$ is invariant under permutation,
since $\|x^2,x^1\|_Y=\sup_p\frac{1}{p}\sum_{k=1}^{p}\|z_1,\Delta^s_{(m)}x^1_k\|_X$ and $\|\bullet,\bullet\|_X$ is a 2-norm.

Let $\alpha\in R$ be any element. If $\alpha x^1,x^2$ are linearly dependent then it is obvious that $\|\alpha x^1,x^2\|_Y=|\alpha|\ \|x^1,x^2\|_Y$. Otherwise,
$\|\alpha x^1,x^2\|_Y=\sup_p\frac{1}{p}\sum_{k=1}^{p}\|\Delta^s_{(m)}x^1_k,z_1\|_X=|\alpha|\ \sup_p\frac{1}{p}\sum_{k=1}^{p}\|\Delta^s_{(m)}x^1_k,z_1\|_X=|\alpha|\ \|x^1,x^2\|_Y.$

Lastly, let $x^1 = (x_k^1)$ and $y^1 = (y_k^1) \in Y$. Then clearly $\|x^1 + y^1, x^2\|_Y \leq \|x^1, x^2\|_Y + \|y^1, x^2\|_Y$. Thus we can conclude that $\|x^1, x^2\|_Y$ is a 2-norm on Y.

(ii) For this part we shall only show that $\|x^1, x^2\|_H = 0$ implies $x^1, x^2$ are linearly dependent.
Proof of other conditions of 2-norm is exactly same with that of part (i).
Let us assume that $\|x^1, x^2\|_H = 0$. Then using (2.3), we have for every $z_1 \in X$

$$\sum_{k=1}^{ms} \|x_k^1, z_1\|_X + \sup_p \frac{1}{p} \sum_{k=1}^{p} \|\Delta_m^s x_k^1, z_1\|_X = 0 \tag{2.4}$$

Hence $\sum\limits_{k=1}^{ms} \|x_k^1, z_1\|_X = 0$. Hence $x_k^1 = 0$ for $k = 1, 2, \ldots, ms$. Also we have from (2.4) for every $z_1 \in X$

$$\sup_p \frac{1}{p} \sum_{k=1}^{p} \|\Delta_m^s x_k^1, z_1\|_X = 0$$

Hence for every $z_1 \in X$, we have $\|\Delta_m^s x_k^1, z_1\|_X = 0$ for each $k \in N$. Thus we must have $\Delta_m^s x_k^1 = 0$ for each $k \in N$. Let $k = 1$, we have

$$\Delta_m^s x_1^1 = \sum_{v=0}^{s} (-1)^v \begin{pmatrix} s \\ v \end{pmatrix} x_{1+mv}^1 = 0 \tag{2.5}$$

we have $x_k^1 = 0$, for $k = 1 + mv$, for $v = 1, 2, \ldots, (s-1)$ (2.6)
Thus from (2.5) and (2.6) we have $x_{1+nm}^1 = 0$. Proceeding in this way inductively, we have $x_k^1 = 0$ for each $k \in N$. Hence $x^1 = \theta$ and so $x^1, x^2$ are linearly dependent.

**Theorem 2.6.** (i) *Let $M$ be the spaces $N_\theta(\|\bullet, \bullet\|, \Delta_{(m)}^s)$. We define the following function $\|\bullet, \bullet\|_M$ on $M \times M$ by*
$\|x, y\|_M = 0$ *if $x, y$ are linearly dependent,*
$= \sup\limits_p \frac{1}{h_p} \sum\limits_{k \in I_p} \|\Delta_{(m)}^s x_k, z_1\|_X$ *for every $z_1 \in X$, if $x, y$ are linearly independent.*
*Then $\|\bullet, \bullet\|_M$ is a 2-norm on $M$.* (2.7)
*(ii) Let $N$ be the space $N_\theta(\|\bullet, \bullet\|, \Delta_m^s)$. We define the following function $\|\bullet, \bullet\|_N$ on $N \times N$ by*
$\|x, y\|_N = 0$ *if $x, y$ are linearly dependent,*
$= \sum\limits_{k=1}^{ms} \|x_k, z_1\|_X + \sup\limits_p \frac{1}{h_p} \sum\limits_{k \in I_p} \|\Delta_m^s x_k, z_1\|_X$ *for every $z_1 \in X$, if $x, y$ are linearly independent.*
*Then $\|\bullet, \bullet\|_N$ is a 2-norm on $N$.* (2.8)

**Proof.** Proof follows by applying similar arguments as applied tao prove Theorem 2.5.

**Remark 1.** It is obvious that any sequence $x \in Z(\|\bullet,\bullet\|,\Delta^s_{(m)})$ if and only if $x \in Z(\|\bullet,\bullet\|,\Delta^s_m)$, for $Z = N_\theta$ and $|\sigma_1|$.

A 2-norm $\|\bullet,\bullet\|_1$ on a vector space $X$ is said to be equivalent to a norm $\|\bullet,\bullet\|_2$ on $X$ if there are positive numbers $A$ and $B$ such that for all $x, y \in X$ we have

$$A\|x,y\|_2 \leq ||x,y||_1 \leq B||x,y||_2$$

This concept is motivated by the fact that equivalent norms on $X$ define the same topology for $X$.

It is clear that the two 2-norms $\|\bullet,\bullet\|_Y$ and $\|\bullet,\bullet\|_H$ defined by (2.2) and (2.3) are equivalent. Also the two 2-norms $\|\bullet,\bullet\|_M$ and $\|\bullet,\bullet\|_N$ defined by (2.7) and (2.8) are equivalent.

Let $X$ and $Y$ be linear 2-normed spaces and $f : X \longrightarrow Y$ a mapping. We call $f$ an 2-isometry if $\|x_1 - y_1, x_2 - y_2\| = \|f(x_1) - f(y_1), f(x_2) - f(y_2)\|$, for all $x_1, x_2, y_1, y_2 \in X$.

**Theorem 2.7.** *For $Z = N_\theta$ and $|\sigma_1|$, the spaces $Z(\|\bullet,\bullet\|,\Delta^s_{(m)})$ and $Z(\|\bullet,\bullet\|,\Delta^s_m)$ are 2-isometric with the spaces $Z(\|\bullet,\bullet\|)$.*

**Proof.** We give the proof for the case $Z = |\sigma_1|$ and for the other case it will follow similarly. Let us consider the mapping $F : |\sigma_1|(\|\bullet,\bullet\|,\Delta^s_{(m)}) \longrightarrow |\sigma_1|(\|\bullet,\bullet\|)$, defined by $Fx = y = (\Delta^{(s)}_r x_k)$ for each $x = (x_k)$ in $|\sigma_1|(\|\bullet,\bullet\|,\Delta^s_{(m)})$. Then clearly $F$ is linear. Since $F$ is linear, to show $F$ is a 2-isometry, it is enough to show that $\|F(x^1), F(x^2)\|_1 = \|x^1, x^2\|_Y$, for every $x^1, x^2$ in $Z(\|\bullet,\bullet\|,\Delta^s_{(m)})$. Now using the definition of 2-norm (2.1), without loss of generality we can write

$$\|x^1, x^2\|_Y = \sup_p \frac{1}{p} \sum_{k=1}^{p} \|\Delta^s_{(m)} x^1_k, z_1\|_X = \|F(x^1), F(x^2)\|_1,$$

where $\|\bullet,\bullet\|_1$ is the 2-norm on $|\sigma_1|(\|\bullet,\bullet\|)$ which can be obtained from (2.2) by taking $s = 0$.
We can define same mapping on the spaces $|\sigma_1|(\|\bullet,\bullet\|)$, $|\sigma_1|(\|\bullet,\bullet\|,\Delta^s_m)$ and this completes the proof.

**Theorem 2.8.** *Let $Y$ be the spaces $|\sigma_1|(\|\bullet,\bullet\|,\Delta^s_{(m)})$. We define the following function $\| \bullet \|_\infty$ on $Y$ by*

$\|x^1\|_\infty = 0$ *if* $x^1$ *is linearly dependent,*
$= \sup\limits_p \frac{1}{p} \sum\limits_{k=1}^{p} \{\max\limits_l \|\Delta^s_{(m)} x^1_k, b_l\|_X\}$ *if* $x^1$ *are linearly independent,*
*where* $B = \{b_1, b_2, \ldots, b_d\}$ *is a basis of* $X$.
*Then* $\|\bullet\|_\infty$ *is a norm on* $Y$ *and we call this as derived norm on* $Y$. (2.9)

**Proof.** Proof is a routine verification and so omitted.

In a similar fashion we can define derived norm (norm) on each of the spaces $|\sigma_1|(\|\bullet,\bullet\|, \Delta^s_m)$, $N_\theta(\|\bullet,\bullet\|, \Delta^s_{(m)})$ and $N_\theta(\|\bullet,\bullet\|, \Delta^s_m)$. Hence we have the following Corollary.

**Corollary 2.9.** *The spaces* $|\sigma_1|(\|\bullet,\bullet\|, \Delta^s_{(m)})$, $|\sigma_1|(\|\bullet,\bullet\|, \Delta^s_m)$, $N_\theta(\|\bullet,\bullet\|, \Delta^s_{(m)})$ *and* $N_\theta(\|\bullet,\bullet\|, \Delta^s_m)$ *are normed linear spaces.*

**Theorem 2.10.** *If* $X$ *is a* 2*-Banach space then the spaces* $Z(\|\bullet,\bullet\|, \Delta^s_{(m)})$ *and* $Z(\|\bullet,\bullet\|, \Delta^s_m)$ *for* $Z = N_\theta$ *and* $|\sigma_1|$ *are* 2*-Banach spaces.*

**Proof.** We give the proof for the space $N_\theta(\|\bullet,\bullet\|, \Delta^s_{(m)})$ and for the other spaces it will follow similarly.
Let $(x^i)$ be any Cauchy sequence in $N_\theta(\|\bullet,\bullet\|, \Delta^s_{(m)})$. Let $\varepsilon > 0$ be given. Then there exists a positive integer $n_0$ such that $\|x^i - x^j, u^1\|_M < \varepsilon$, for all $i, j \geq n_0$ and for every $u^1$. Using the definition of 2-norm, we get

$$\sup_p \frac{1}{h_p} \sum_{k \in I_p} \|\Delta^s_{(m)}(x^i_k - x^j_k), z_1\|_X < \varepsilon$$

for all $i, j \geq n_0$ and for every $z_1 \in X$. It follows that

$$\|\Delta^s_{(m)}(x^i_k - x^j_k), z_1|_X < \varepsilon,$$

for all $i, j \geq n_0$, for all $k \in N$ and for every $z_1 \in X$. Hence $(\Delta^s_{(m)} x^i_k)$ is a Cauchy sequence in $X$ for all $k \in N$ and so it is convergent in $X$ for all $k \in N$, since $X$ is an 2-Banach space. For simplicity, let $\lim\limits_{i\to\infty} \Delta^s_{(m)} x^i_k = y_k$ exists for each $k \in N$. Taking $k = 1, 2, 3, \ldots, ms, \ldots$ we can easily conclude that $\lim\limits_{i\to\infty} x^i_k = x_k$ exists for each $k \in N$. Now for $i, j \geq n_0$, we have

$$\sup_p \frac{1}{h_p} \sum_{k \in I_p} \|\Delta^s_{(m)}(x^i_k - x^j_k), z_1\|_X < \varepsilon,$$

for every $z_1 \in X$. Hence we can have for every $z_1 \in X$,

$$\sup_p \frac{1}{h_p} \sum_{k \in I_p} \|\Delta^s_{(m)}(x^i_k - x_k), z_1\|_X < \varepsilon,$$

for all $i \geq n_0$ and as $j \to \infty$. It follows that $(x^i - x) \in N_\theta(\|\bullet, \bullet\|, \Delta^s_{(m)})$. Since $(x^i) \in N_\theta(\|\bullet, \bullet\|, \Delta^s_{(m)})$ and $N_\theta(\|\bullet, \bullet\|, \Delta^s_{(m)})$ is a linear space, so we have $x = x^i - (x^i - x) \in N_\theta(\|\bullet, \bullet\|, \Delta^s_{(m)})$. This completes the proof of the theorem.

Next results of this article give us important topological structure of the introduced spaces as well as highlight the importance of the concept of derived norm. We proof the next results only for the space $|\sigma_1|(\|\bullet, \bullet\|, \Delta^s_{(m)})$ for the other spaces it will follow on applying similar arguments.

**Theorem 2.11.** *A sequence $(x^i)$ converges to an $x$ in $|\sigma_1|(\|\bullet, \bullet\|, \Delta^s_{(m)})$ in the 2-norm if and only if $(x^i)$ also converges to $x$ in the derived norm.*

**Proof.** Let $(x^i)$ converges to $x$ in $|\sigma_1|(\|\bullet, \bullet\|, \Delta^s_{(m)})$ in the 2-norm. Then $\|x^i - x, u^1\|_Y \longrightarrow 0$ as $i \longrightarrow \infty$ for every $u^1 \in |\sigma_1|(\|\bullet, \bullet\|, \Delta^s_{(m)})$.
Using (2.2), we get

$$\sup_p \frac{1}{p} \sum_{k=1}^{p} \|\Delta^s_{(m)}(x^i_k - x_k), z_1\|_X \to 0$$

as $i \to \infty$ and for every $z_1 \in X$. Hence for any basis $\{b_1, b_2, \ldots, b_d\}$ of $X$, we have

$$\sup_p \frac{1}{p} \sum_{k=1}^{p} \left\{\max_l \|\Delta^s_{(m)}(x^1_k - x_k), b_l\|_X\right\} \to 0$$

as $i \to \infty$, for each $l = 1, 2, \ldots, d$. Thus it follows that $\|x^i - x\|_\infty \to 0$ as $i \to \infty$. Hence $(x^i)$ converges to $x$ in the derived norm.
Conversely assume $(x^i)$ converges to $x$ in the derived norm. Then we have $\|x^i - x\|_\infty \to 0$ as $i \to \infty$. Hence using (2.9), we get

$$\sup_p \frac{1}{p} \sum_{k=1}^{p} \left\{\max_l \|\Delta^s_{(m)}(x^1_k - x_k), b_l\|_X\right\} \to 0$$

as $i \to \infty$, $l = 1, 2, \ldots, d$. Therefore

$$\sup_p \frac{1}{p} \sum_{k=1}^{p} \|\Delta^s_{(m)}(x^1_k - x_k), b_l\|_X \to 0$$

as $i \to \infty$, for each $l = 1, 2, \ldots, d$. Let $y$ be the element of $|\sigma_1|(\|\bullet, \bullet\|, \Delta^s_{(m)})$. Then

$$\|x^i - x, y\|_Y = \sup_p \frac{1}{p} \sum_{k=1}^{p} \|\Delta^s_{(m)}(x^1_k - x_k), z_1\|_X.$$

Since $\{b_1, b_2, \ldots, b_d\}$ is a basis for $X$ and $z_1$ can be written as $z_1 = \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_d b_d$ for some $\alpha_1, \alpha_2, \ldots, \alpha_d \in R$. Now

$$\|x^i - x, y\|_Y = \sup_p \frac{1}{p} \sum_{k=1}^{p} \|\Delta^s_{(m)}(x^1_k - x_k), z_1\|_X$$

$$\leq |\alpha_1| \sup_p \frac{1}{p} \sum_{k=1}^{p} \|\Delta^s_{(m)}(x^i_k - x_k), b_1\|_X + \ldots + |\alpha_d| \sup_p \frac{1}{p} \sum_{k=1}^{p} \|\Delta^s_{(m)}(x^i_k - x_k), b_d\|_X$$

for each $i \in N$. Thus it follows that $\|x^i - x, y\|_Y \to 0$ as $i \to \infty$ for every $y$ in $|\sigma_1|(\|\bullet, \bullet\|, \Delta^s_{(m)})$. Hence $(x^i)$ converges to $x$ in $|\sigma_1|(\|\bullet, \bullet\|, \Delta^s_{(m)})$ in the 2-norm.

**Corollary 2.12.** *The space $|\sigma_1|(\|\bullet, \bullet\|, \Delta^s_{(m)})$ is complete with respect to the 2-norm if and only if it is complete with respect to the derived norm.*

**Remark 2.** *Associated to the derived norm $\| \bullet \|_\infty$, we can define balls (open) $S(x, \epsilon)$ centered at $x$ and radius $\epsilon$ as follows:*

$$S(x, \epsilon) = \{y : \|x - y\|_\infty < \epsilon\}.$$

Using these balls, Corrollary 2.11 becomes:

**Lemma 2.13.** *A sequence $(x_k)$ is convergent to $x$ in $|\sigma_1|(\|\bullet, \bullet\|, \Delta^s_{(m)})$ if and only if for every $\epsilon > 0$, there exists $n_0 \in N$ such that $x_k \in S(x, \epsilon)$ for all $k \geq n_0$.*

Hence we have the following important result.

**Theorem 2.14.** *The space $|\sigma_1|(\|\bullet, \bullet\|, \Delta^s_{(m)})$ is a normed space and its topology agrees with that generated by the derived norm $\| \bullet \|$.*

**Theorem 2.15. (Fixed Point Theorem)** *Consider the 2-Banach space $|\sigma_1|(\|\bullet, \bullet\|, \Delta^s_{(m)})$ under the 2-norm (2.2) and $T$ be a contractive mapping of $|\sigma_1|(\|\bullet, \bullet\|, \Delta^s_{(m)})$ into itself, that is, there exists a constant $C \in (0, 1)$ such that*

$$\|Ty^1 - Tz^1, x^2\|_Y \leq C\|y^1 - z^1, x^2\|_Y,$$

*for all $y^1, z^1, x^2$ in $Y$. Then $T$ has a unique fixed point in Y.*

**Proof.** If we can show that $T$ is also contractive with respect to norm, then we are done by Corollary 2.12 and the Fixed Point Theorem for Banach spaces. Now by hypothesis

$$\|Ty^1 - Tz^1, x^2\|_Y \le C\|y^1 - z^1, x^2\|_Y,$$

for all $y^1, z^1, x^2$ in $Y$. This implies that

$$\sup_p \frac{1}{p}\sum_{k=1}^{p} \|\Delta^s_{(m)}(Ty^1_k - Tz^1_k), u_1\|_X \le C \sup_p \frac{1}{p}\sum_{k=1}^{p} \|\Delta^s_{(m)}(y^1_k - z^1_k), u_1\|_X$$

for every $u_1$ in $X$.Then for a basis set $\{e_1, e_2, \dots, e_d\}$ of $X$, we get

$$\sup_p \frac{1}{p}\sum_{k=1}^{p} \|\Delta^s_{(m)}(Ty^1_k - Tz^1_k), e_i\|_X \le C \sup_p \frac{1}{p}\sum_{k=1}^{p} \|\Delta^s_{(m)}(y^1_k - z^1_k), e_i\|_X,$$

for all $y^1, z^1$ in $Y$ and $i = 1, 2, \dots, d$. Thus we have

$$\|Ty^1_k - Tz^1_k\|_\infty \le C\|y^1_k - z^1_k\|_\infty.$$

That is $T$ is contractive with respect to derived norm. This completes the proof.

## References


[1] D. Borwein, *Linear functionals connected with strong Cesàro summability*, J. London Math. Soc., 40, (1965), 628-634.

[2] H. Dutta, *Characterization of certain matrix classes involving generalized difference summability spaces*, Applied Sciences (APPS), 11, (2009), 60-67.

[3] H. Dutta, *Some results on 2-normed spaces*, Novi Sad J. Math., (In press).

[4] M. Et and R. Colak, *On generalized difference sequence spaces*, Soochow Jour. Math., 21, 4, (1995), 377-386.

[5] A.R. Freedman, J.J. Sember and M. Raphael, *Some Cesàro-type summability spaces*, Proc. Lond. Math. Soc., 37, 3, (1978), 508-520.

[6] S. Gähler, 2-*metrische Rume und ihre topologische Struktur*, Math. Nachr., 28, (1963), 115- 148.

[7] S. Gähler, *Linear* 2-*normietre Rume*, Math. Nachr., 28, (1965), 1-43.

[8] S. Gähler, *Uber der Uniformisierbarkeit* 2-*metrische Rume*, Math. Nachr., 28, (1965), 235 -244.

[9] H. Gunawan and Mashadi, *On finite Dimensional* 2-*normed spaces*, Soochow J. of Math., 27, 3, (2001), 631-639.

[10] M. Gürdal, *On Ideal Convergent Sequences in* 2*-Normed Spaces*, Thai J. Math., 4, 1, (2006), 85-91.

[11] H. Kizmaz, *On certain sequence spaces*, Canad. Math. Bull., 24, 2, (1981), 169-176.

[12] I. J. Maddox, *Spaces of strongly summable sequences*, Quart. J. Math. Oxford, 18, (1967), 345-355.

[13] H. Mazaheri and R. Kazemi, *Some results on* 2*-inner product spaces*, Novi Sad J. Math., 37, 2, (2007), 35-40.

[14] B. C. Tripathy and A. Esi, *A new type of difference sequence spaces*, International Journal of Science and Technology, 1, 1, (2006), 11-14.

[15] B .C. Tripathy, A. Esi and B. K. Tripathy, *On a new type of generalized difference Cesàro Sequence spaces*, Soochow J. Math., 31, 3, (2005), 333-340.

Iqbal H. Jebril
Department of Mathematics,
King Faisal University, Saudi Arabia
email:*iqbal501@yahoo.com*