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ON UNIVALENCE CRITERIA FOR ANALYTIC FUNCTIONS DEFINED BY A GENERALIZED DIFFERENTIAL OPERATOR

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Abstract In this paper we obtain sufficient conditions for univalence of analytic functions defined by a generalized differential operator introduced by the authors in (Far East J. Math. Sci., 33(3),(2009), 299-308).

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1. Introduction.

Let $\mathcal{H}$ be the class of functions analytic in $U := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U). \tag{1}$$

We will use the following operator which defined and studied by the authors (see [1]).

$$\begin{aligned} D^0 f(z) &= f(z) \\ &= z + \sum_{n=2}^{\infty} a_n z^n, \end{aligned}$$

$$\begin{aligned} D^1_{\alpha,\beta,\lambda} f(z) &= [1-\beta(\lambda-\alpha)]f(z) + \beta(\lambda-\alpha)zf'(z) \\ &= z + \sum_{n=2}^{\infty} [\beta(n-1)(\lambda-\alpha)+1]a_n z^n, \\ &\vdots \\ D^k_{\alpha,\beta,\lambda} f(z) &= D^1_{\alpha,\beta,\lambda}\left(D^{k-1}_{\alpha,\beta,\lambda} f(z)\right) \\ &= z + \sum_{n=2}^{\infty} [\beta(n-1)(\lambda-\alpha)+1]^k a_n z^n \end{aligned} \tag{2}$$

for $\beta > 0,\ 0 \le \alpha < \lambda$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with $D^k_{\alpha,\beta,\lambda} f(0) = 0$.

Remark 1.1.
(i) When $\alpha = 0, \beta = 1$, we receive Al-Oboudi$'$s differential operator (see [2]).
(ii) And when $\alpha = 0, \beta = 1$ and $\lambda = 1$ we get Sălăgean$'$s differential operator (see [3]).

Many differential operators studied by various authors can be seen in the literature (see for examples [4]-[8]).

Our considerations are based on the following results.

Lemma 1.1. (see [9]) *Let $f \in \mathcal{A}$. If for all $z \in U$*

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \tag{3}$$

then the function f is univalent in U.

Lemma 1.2. (see [10]) *Let $f \in \mathcal{A}$. If for all $z \in U$*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1, \tag{4}$$

then the function f is univalent in U.

Lemma 1.3. (see [11]) *Let μ be a real number, $\mu > \frac{1}{2}$ and $f \in \mathcal{A}$. If for all $z \in U$*

$$\left|(1-|z|^{2\mu})\frac{zf''(z)}{f'(z)} + 1 - \mu\right| \le \mu, \tag{5}$$

then the function f is univalent in U.

Lemma 1.4. (see [12]) *If $f(z) \in \mathcal{S}$ (the class of univalent functions) and*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \tag{6}$$

then

$$\sum_{n=1}^{\infty}(n-1)|b_n|^2 \leq 1.$$

Lemma 1.5. (see [13]) *Let $\nu \in \mathbb{C}$, $\Re\{\nu\} \geq 0$ and $f \in \mathcal{A}$. If for all $z \in U$*

$$\frac{(1-|z|^{2\Re(\nu)})}{\Re(\nu)}|\frac{zf''(z)}{f'(z)}| \leq 1, \tag{7}$$

then function

$$F_\nu(z) = \Big(\nu \int_0^z u^{\nu-1} f'(u) du\Big)^{1/\nu}$$

is univalent in U.

1. The Main Results

In this section, we establish the sufficient conditions to obtain a univalence for analytic functions involving the differential operator (2).

Theorem 2.1. *Let $f \in \mathcal{A}$. If for all $z \in U$*

$$\sum_{n=2}^{\infty}[\beta(n-1)(\lambda-\alpha)+1]^k[n(2n-1)]|a_n| \leq 1. \tag{8}$$

Then $D^k_{\alpha,\beta,\lambda} f(z)$ is univalent in U.

Proof. Let $f \in \mathcal{A}$. Then for all $z \in U$ we have

$$(1-|z|^2)|\frac{z[D^k_{\alpha,\beta,\lambda} f(z)]''}{[D^k_{\alpha,\beta,\lambda} f(z)]'}| \leq (1+|z|^2)|\frac{z[D^k_{\alpha,\beta,\lambda} f(z)]''}{[D^k_{\alpha,\beta,\lambda} f(z)]'}|$$

$$\leq \frac{2\sum_{n=2}^{\infty}[\beta(n-1)(\lambda-\alpha)+1]^k[n(n-1)]|a_n|}{1-\sum_{n=2}^{\infty}[\beta(n-1)(\lambda-\alpha)+1]^k n|a_n|}$$

the last inequality is less than 1 if the assertion (8) is hold. Thus in view of Lemma 1.1, $D^k_{\alpha,\beta,\lambda}f(z)$ is univalent in U.

Theorem 2.2. *Let $f \in \mathcal{A}$. If for all $z \in U$*

$$\sum_{n=2}^{\infty}[\beta(n-1)(\lambda-\alpha)+1]^k|a_n| \le \frac{1}{\sqrt{7}}. \tag{9}$$

Then $D^k_{\alpha,\beta,\lambda}f(z)$ is univalent in U.

Proof. Let $f \in \mathcal{A}$. It sufficient to show that

$$|\frac{z^2[D^k_{\alpha,\beta,\lambda}f(z)]'}{2[D^k_{\alpha,\beta,\lambda}f(z)]^2}| \le 1.$$

$$|\frac{z^2[D^k_{\alpha,\beta,\lambda}f(z)]'}{2[D^k_{\alpha,\beta,\lambda}f(z)]^2}| \le$$

$$\frac{1+\sum_{n=2}^{\infty}[\beta(n-1)(\lambda-\alpha)+1]^k n|a_n|}{2\{1-2\sum_{n=2}^{\infty}[\beta(n-1)(\lambda-\alpha)+1]^k|a_n| - (\sum_{n=2}^{\infty}[\beta(n-1)(\lambda-\alpha)+1]^k|a_n|)^2\}}$$

the last inequality is less than 1 if the assertion (9) is hold. Thus in view of Lemma 1.2, $D^k_{\alpha,\beta,\lambda}f(z)$ is univalent in U.

Theorem 2.3. *Let $f \in \mathcal{A}$. If for all $z \in U$*

$$\sum_{n=2}^{\infty} n[2(n-1)+(2\mu-1)][\beta(n-1)(\lambda-\alpha)+1]^k|a_n| \le 2\mu-1, \quad \mu > \frac{1}{2}. \tag{10}$$

Then $D^k_{\alpha,\beta,\lambda}f(z)$ is univalent in U.

Proof. Let $f \in \mathcal{A}$. Then for all $z \in U$ we have

$$|(1-|z|^{2\mu})\frac{z[D^k_{\alpha,\beta,\lambda}f(z)]''}{[D^k_{\alpha,\beta,\lambda}f(z)]'}+1-\mu| \le (1+|z|^2)|\frac{z[D^k_{\alpha,\beta,\lambda}f(z)]''}{[D^k_{\alpha,\beta,\lambda}f(z)]'}|+|1-\mu|$$

$$\le \frac{2\sum_{n=2}^{\infty}[\beta(n-1)(\lambda-\alpha)+1]^k[n(n-1)]|a_n|}{1-\sum_{n=2}^{\infty}[\beta(n-1)(\lambda-\alpha)+1]^k n|a_n|}+|1-\mu|$$

the last inequality is less than μ if the assertion (10) is hold. Thus in view of Lemma 1.3, $D^k_{\alpha,\beta,\lambda}f(z)$ is univalent in U.

As applications of Theorems 2.1, 2.2 and 2.3 we have the following result.

Theorem 2.4. *Let $f \in \mathcal{A}$. If for all $z \in U$ one of the inequalities (8-10) holds then*

$$\sum_{n=1}^{\infty}(n-1)|b_n|^2 \leq 1,$$

where

$$\frac{z}{D^k_{\alpha,\beta,\lambda}f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n.$$

Proof. Let $f \in \mathcal{A}$. Then in view of Theorems 2.1 , 2.2 or 2.3, $D^k_{\alpha,\beta,\lambda}f(z)$ is univalent in U. Hence by Lemma 1.4 we obtain the result.

Theorem 2.5. *Let $f \in \mathcal{A}$. If for all $z \in U$*

$$\sum_{n=2}^{\infty} n[2(n-1) + \Re(\nu)][\beta(n-1)(\lambda-\alpha)+1]^k|a_n| \leq \Re(\nu), \;\; \Re(\nu) > 0. \tag{11}$$

Then

$$G_\nu(z) = \Big(\nu \int_0^z u^{\nu-1}[D^k_{\alpha,\beta,\lambda}f(u)]'du\Big)^{1/\nu}$$

is univalent in U.

Proof. Let $f \in \mathcal{A}$. Then for all $z \in U$ we have

$$\frac{(1-|z|^{2\Re(\nu)})}{\Re(\nu)}|\frac{z[D^k_{\alpha,\beta,\lambda}f(z)]''}{[D^k_{\alpha,\beta,\lambda}f(z)]'}| \leq \frac{(1+|z|^{2\Re(\nu)})}{\Re(\nu)}|\frac{z[D^k_{\alpha,\beta,\lambda}f(z)]''}{[D^k_{\alpha,\beta,\lambda}f(z)]'}|$$

$$\leq \frac{2}{\Re(\nu)} \frac{\sum_{n=2}^{\infty}[\beta(n-1)(\lambda-\alpha)+1]^k[n(n-1)]|a_n|}{1-\sum_{n=2}^{\infty}[\beta(n-1)(\lambda-\alpha)+1]^k n|a_n|}$$

the last inequality is less than 1 if the assertion (11) is hold. Thus in view of Lemma 1.5, $G_\nu(z)$ is univalent in U.

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