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DIFFERENTIAL SUBORDINATION RESULTS FOR CLASSES OF THE FAMILY $\xi(\varphi, \vartheta)$

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Abstract. In this paper we introduce two new classes of family of complex valued function in a unit disk corresponding to particular proper rational numbers. Then by using the Hadamard product we deduced some interesting differential subordination results.

Keywords: Differential subordination, Hadamard product.

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1. Introduction and Preliminaries

Let $\mathcal{E}_\alpha^+$ denotes the family of all functions *F(z)* in the unit disk $U = \{z : |z| < 1\}$ satisfying $F'(0) = 1$ of the form

$$F(z) = z + \sum_{n=2}^{\infty} a_n z^{n-n/\alpha} \quad \alpha \in \mathcal{N} - \{1\} = \{2, 3, 4...\} \tag{1}$$

Similarly $\mathcal{E}_\alpha^-$ denotes the family of all functions *F(z)* in the unit disk $U = \{z : |z| < 1\}$ of the form

$$F(z) = z - \sum_{n=2}^{\infty} a_n z^{n-n/\alpha} \quad \alpha \in \mathcal{N} - \{1\} = \{2, 3, 4...\} \tag{2}$$

satisfying $F'(0) = 1$.

If two functions f and g analytic in U then f is called subordinate to g if there exist a Schwarz function *w(z)* analytic in U such that *f(z)=g(w(z))* and $z \in U = \{z : |z| < 1\}$. Then we denote this subordination by $f(z) \prec g(z)$ or simply $f \prec g$ but in a special case if g is univalent in U then the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $\phi : C^3 \times U \rightarrow C$ and let h analytic in U. Assume that p, ϕ are analytic and univalent in U and p satisfies the differential superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \tag{3}$$

Then p is called a solution of the differential superordination. An analytic function q is called a subordinant if $q \prec p$ for all p satisfying equation (3). A univalent function q such that $p \prec q$ for all subordinants p of equaction (3) is said to be the best subordinant.

Let $\mathcal{E}^+$ be the class of analytic functions of the form

$$f(z) = 1 + \sum_1^\infty a_n z^n \quad z \in U, a_n \geq 0. \tag{4}$$

Let $f, g \in \mathcal{E}^+$ where

$$f(z) = 1 + \sum_1^\infty a_n z^n \text{ and } g(z) = 1 + \sum_1^\infty b_n z^n$$

then their convolution or Hadamard product $f(z) * g(z)$ is defined by

$$f(z) * g(z) = 1 + \sum_1^\infty a_n b_n z^n, \quad a_n \geq 0, b_n \geq 0, z \in U$$

Juneja et al. [1] define the family $\epsilon(\phi, \psi)$, so that

$$R\Big(\frac{f(z) * \phi(z)}{f(z) * \psi(z)}\Big) > 0,\ z \in U,$$

where

$$\phi(z) = 1 + \sum_1^\infty \phi_n z^n, \psi(z) = 1 + \sum_1^\infty \psi_n z^n$$

are analytic in U with the conditions $\phi_n > 0$, $\psi_n > 0$, $\phi_n > \psi_n$, $f(z) * \psi(z) \neq 0$. During our study of convolution or Hadamard product, we have seen that many different authors investigated this functional for various types of classes of univalent functions, namely Barnard and Kellogg [4], Liu and Srivastava [5].

Definition 1. *Let $\xi_\alpha^+(\varphi, \vartheta)$ be the class of family of all $F(z) \in \mathcal{E}_\alpha^+$ such that*

$$R\Big(\frac{F(z) * \varphi(z)}{F(z) * \vartheta(z)}\Big) > 0,\ z \in U,$$

where

$$\varphi(z)=1+\sum_{1}^{\infty}\varphi_n z^{n-n/\alpha},\vartheta(z)=1+\sum_{1}^{\infty}\vartheta_n z^{n-n/\alpha}$$

are analytic in U *with specific conditions,* $\varphi_n \geq 0$, $\vartheta_n \geq 0$, $\varphi_n \geq \vartheta_n$, $F(z)*\vartheta(z)\neq 0$ *and for all* $n \geq 0$.

Definition 2. *Let* $\xi_\alpha^-(\varphi,\vartheta)$ *be the class of family of all* $F(z)\in\mathcal{E}_\alpha^+$ *such that*

$$R\Big(\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\Big)>0,\, z\in U,$$

where

$$\varphi(z)=1-\sum_{1}^{\infty}\varphi_n z^{n-n/\alpha},\vartheta(z)=1-\sum_{1}^{\infty}\vartheta_n z^{n-n/\alpha}$$

are analytic in U *with specific conditions,* $\varphi_n \geq 0$, $\vartheta_n \geq 0$, $\varphi_n \geq \vartheta_n$, $F(z)*\vartheta(z)\neq 0$ *and for all* $n \geq 0$.

The aim of the present paper is to propose some sufficient conditions for all functions *F(z)* belongs to the new classes $\mathcal{E}_\alpha^+$ and $\mathcal{E}_\alpha^-$ to satisfy

$$\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\prec h(z),\, z\in U$$

where *q(z)* is a given univalent function in U such that $q(0)=1$.

The classes $\mathcal{E}_\alpha^+$ and $\mathcal{E}_\alpha^-$ defined in above exhibits contains some interesting properties. To prove our main results we need the following lemmas.

Lemma 1 [3]. *Let q(z) be univalent in the unit disk* U *and* $\theta(z)$ *be analytic in a domain* D *containing* $q(U)$. *If* $zq'(z)\theta(z)$ *is starlike in* U, *and*

$$zp'(z)\theta(p(z))\prec zq(z)\theta(q(z))$$

then $p(z)\prec q(z)$ *and q(z) is the best dominant.*

Theorem 1. *Let the function q(z) be univalent in the unit disk* U *such that* $q'(z)\neq 0$ *and* $\frac{zq'(z)}{q(z)}\neq 0$ *is starlike in* U *if* $F(z)\in\mathcal{E}_\alpha^+$ *satisfies the subordination*

$$b\left(\frac{(F(z)*\varphi(z))'}{(F(z)*\varphi(z))}-\frac{(F(z)*\vartheta(z))'}{(F(z)*\vartheta(z))}\right)\prec\frac{bzq'(z)}{q(z)},$$

then

$$\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\prec q(z),$$

then q(z) is the best dominant.

Proof. First we defined the function *p(z)* that is

$$P(z) = \frac{F(z) * \varphi(z)}{F(z) * \vartheta(z)},$$

then

$$\frac{zP'(z)}{P(z)} = \frac{z(F(z) * \vartheta(z)((F(z) * \varphi(z))' - ((F(z) * \varphi(z))(F(z) * \vartheta(z))'}{((F(z) * \vartheta(z))((F(z) * \varphi(z))}$$

and

$$b\bigg(\frac{zP'(z)}{P(z)}\bigg) = b\bigg(\frac{(F(z) * \varphi(z))'}{F(z) * \varphi(z)} - \frac{(F(z) * \vartheta(z))'}{F(z) * \vartheta(z)}\bigg). \tag{5}$$

By setting $\theta(\omega) = b/\omega$ it can easily be observed that $\theta(\omega)$ is analytic in $C - \{0\}$. Then we obtain that

$$\theta(p(z)) = b/p(z), \quad and \quad \theta(q(z)) = b/q(z).$$

So from equation (5) we have

$$b\bigg(\frac{zP'(z)}{P(z)}\bigg) = zp'(z)\theta(p(z)) = b\bigg(\frac{(F(z) * \varphi(z))'}{F(z) * \varphi(z)} - \frac{(F(z) * \vartheta(z))'}{F(z) * \vartheta(z)}\bigg) \prec \frac{zq'(z)}{q(z)} \tag{6}$$

so we have

$$zp'(z)\theta(p(z)) \prec \frac{bzq'(z)}{q(z)} \prec zq'(z)\theta(q(z))$$

this implies

$$\frac{F(z) * \varphi(z)}{F(z) * \vartheta(z)} \prec q(z)$$

Corollary 1. If $F(z) \in \mathcal{E}^+_\alpha$ satisfies the subordination

$$b\bigg(\frac{(F(z) * \varphi(z))'}{(F(z) * \varphi(z))} - \frac{(F(z) * \vartheta(z))'}{(F(z) * \vartheta(z))}\bigg) \prec \frac{b(A-B)z}{(1+Az)(1+Bz)}$$

then

$$\frac{F(z) * \varphi(z)}{F(z) * \vartheta(z)} \prec \frac{(1+Az)}{(1+Bz)},$$

and $\frac{(1+Az)}{(1+Bz)}$ is the best dominant.

Corollary 2. *If $F(z) \in \mathcal{E}_{\alpha}^{+}$ satisfies the subordination*

$$b\left(\frac{(F(z)*\varphi(z))'}{(F(z)*\varphi(z))} - \frac{(F(z)*\vartheta(z))'}{(F(z)*\vartheta(z))}\right) \prec \frac{2bz}{(1-z)(1+z)}$$

then

$$\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)} \prec \frac{(1+z)}{(1-z)},$$

and $\frac{(1+z)}{(1-z)}$ *is the best dominant.*

Lemma 2 [2]. *Let q(z) be convex in the unit disk U with $q(0) = 1$ and $R(q(z)) > 1/2, z \in U$. If $0 \leq U < 1$, p is analytic function in U with $p(0) = 1$ and if*

$$(1-u)p^2(z) + (2u-1)p(z) - u + (1-u)zp'(z)$$

$$\prec (1-u)q^2(z) + (2u-1)q(z) - u + (1-u)zq'(z)$$

then $p(z) \prec q(z)$ and q(z) is the best dominant.

Theorem 2. *Let q(z) be convex in the unit disk U with $q(0) = 1$ and $R(q(z)) > 1/2$. If $F(z) \in \mathcal{E}_{\alpha}^{+}$ and $\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}$ is an analytic function in U satisfies the subordination*

$$(1-u)\left(\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\right)^2(z) + (2u-1)\left(\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\right) - u$$

$$+(1-u)\left(\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\right)\left(\frac{z(F(z)*\varphi(z))'}{(F(z)*\varphi(z))} - \frac{z(F(z)*\vartheta(z))'}{(F(z)*\vartheta(z))}\right)$$

$$\prec (1-u)q^2(z) + (2u-1)q(z) - u + (1-u)zq'(z)$$

then

$$\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)} \prec q(z).$$

Proof. Let the function *p(z)* be defined by

$$p(z) = \frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}, z \in U$$

since $p(0) = 1$ therefore

$$(1-u)p^2(z) + (2u-1)p(z) - u + (1-u)zp'(z)$$

$$= (1-u)\Big(\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\Big)^2(z) + (2u-1)\Big(\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\Big) - u + (1-u)\Big(\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\Big)'$$

$$= (1-u)\Big(\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\Big)^2(z) + (2u-1)\Big(\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\Big) - u + (1-u)\Big(\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\Big)$$
$$\Big(\frac{z(F(z)*\varphi(z))'}{(F(z)*\varphi(z))} - \frac{z(F(z)*\vartheta(z))'}{(F(z)*\vartheta(z))}\Big)$$

$\prec (1-u)q^2(z) + (2u-1)q(z) - u + (1-u)zq'(z).$
Now by using the lemma (2) we have

$p(z) \prec q(z)$ this implies that

$$\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Corollary 3. *If $F(z) \in \mathcal{E}_\alpha^+$ and and $\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}$ is an analytic function in satisfying the subordination*

$$(1-u)\Big(\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\Big)^2(z) + (2u-1)\Big(\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\Big) - u$$
$$+(1-u)\Big(\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\Big)\Big(\frac{z(F(z)*\varphi(z))'}{(F(z)*\varphi(z))} - \frac{z(F(z)*\vartheta(z))'}{(F(z)*\vartheta(z))}\Big)$$
$$\prec (1-u)\Big(\frac{1+Az}{1+Bz}\Big)^2(z)+(2u-1)\Big(\frac{1+Az}{1+Bz}\Big)-u+(1-u)\Big(\frac{1+Az}{1+Bz}\Big)\Big(\frac{(A-B)z}{(1+Az)(1+Bz)}\Big)$$

then

$$\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)} \prec \Big(\frac{1+Az}{1+Bz}\Big),\ -1 \le B \le A \le 1$$

and $\left(\frac{1+\mathrm{Az}}{1+\mathrm{Bz}}\right)$ *is the best dominant.*

Proof. Let the function *q(z)* be defined by

$$q(z) = \Big(\frac{(1+Az)}{(1+Bz)},\ z \in U$$

this implies that $q(0) = 1$ and $R(q(z)) > 1/2$ for arbitrary $A, B, z \in U$ where

$$\frac{zq'(z)}{q(z)} = \Big(\frac{(A-B)z}{(1+Az)(1+Bz)}\Big)$$

Then applying Theorem 2, we obtain the result.

Corollary 4. *If* $F(z) \in \mathcal{E}_\alpha^+$ *and* $\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}$ *is an analytic function in satisfying the subordination*

$$(1-u)\left(\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\right)^2(z)+(2u-1)\left(\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\right)-u$$

$$+(1-u)\left(\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)}\right)\left(\frac{z(F(z)*\varphi(z))'}{(F(z)*\varphi(z))}-\frac{z(F(z)*\vartheta(z))'}{(F(z)*\vartheta(z))}\right)$$

$$\prec (1-u)\left(\frac{1+z}{1-z}\right)^2(z)+(2u-1)\left(\frac{1+z}{1-z}\right)-u+(1-u)\left(\frac{1+z}{1-z}\right)\left(\frac{2z}{(1-z)(1+z)}\right)$$

then

$$\frac{F(z)*\varphi(z)}{F(z)*\vartheta(z)} \prec \left(\frac{1+z}{1-z}\right),$$

and $\left(\frac{\text{1+z}}{\text{1-z}}\right)$ *is the best dominant.*

Proof. Let the function $q(z)$ be defined by

$$q(z)=\left(\frac{1+z}{1-z}\right),\ z\in U$$

then in view of Theorem 2 we obtain the result.

Other interesting results on differential subordination and superordination can be seen in [6].

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