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THE WEAK INVERTIBILITY IN THE UNIT BALL AND POLYDISK AND RELATED PROBLEMS

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Abstract. We will present an approach to deal with a problem of existence of (not) weakly invertible functions in various spaces of analytic functions in the unit ball and polydisk based on estimates for integral operators acting between functional classes of different dimensions.

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1. Introduction

Let $\mathbf{B}$ be as usual the unit ball and $\mathbf{S}$ be a unit sphere. Let also $U^n = \{z = (z_1, \dots, z_n) : |z_j| < 1,\ j = 1, \dots, n\}$ be the unit polydisk in n-dimensional complex space $\mathbf{C}^n$ and $\mathbf{T^n}$ be a boundary of U^n (see [3,15]). Let further X be some topological space of analytic functions in U^n or $\mathbf{B}$ in which the set of polynomials in $z_1, \dots, z_n$ is dense. We will assume that the operators $S(f)(z) = z_1 \cdots z_n f(z_1, \dots, z_n)$, $\Phi_z(f) = f(z)$, $z = (z_1, \dots, z_n) \in U^n$ or $z \in \mathbf{B}$ are continuous in X.

Definition 1. *Let $f \in X$ and assume there exists a sequence of polynomials $\{P_k\}_1^\infty$ such that*

$$\lim_{k\to\infty} P_k(z_1, \dots, z_n) f(z_1, \dots, z_n) = 1,$$

on the topology space X, then we say f is weakly invertible in X.

It is well-known that mentioned of definition of weak invertibility is closely related to problems of completeness of systems of polynomials in weighted classes of holomorphic functions.

Weak invertibility in one dimension has been studied by many authors (see [5,16,8], [8-10], [12-13] and references there). In this research area in spaces of function of one variable the most intensive investigation was done in fundamental work of Nikolski (see [8]).

The problem of extension of one variable results to higher dimension appears naturally. Recently some research was done in this direction (see for example [13] and references there). Our intension is to provide an approach that will work at the same time in the unit ball and polydisk and will solve some natural questions concerning the problem of generalization of one variable results to the case of unit ball and polydisk.

We use m_{2n} to denote the volume measure on U^n and m_n to denote the normalized Lebesgue measure on $\mathbf{T^n}$. When $n = 1$, we simply denote U^1 by U, $\mathbf{T^1}$ by $\mathbf{T}$, m_{2n} by m_2, m_n by m. Let dv denote the volume measure on $\mathbf{B}$, normalized so that $v(\mathbf{B}) = 1$, and let $d\sigma$ denote the surface measure on $\mathbf{S}$ normalized so that $\sigma(\mathbf{S}) = 1$. For $\alpha > -1$ the weighted Lebesgue measure dv_α is defined by $dv_\alpha = c_\alpha(1 - |z|^2)^\alpha dv(z)$ where $c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}$ is a normalizing constant so that $v_\alpha(\mathbf{B}) = 1$ (see [15]).

We denote by $H(\mathbf{B})$ the class of all holomorphic functions on $\mathbf{B}$. Let also $H(\mathbf{U})$ be a space of all holomorphic functions in the unit disk U, and similarly $H(\mathbf{U}^n)$ be a space of all holomorphic functions in U^n.

For $\alpha > -1$ and $p > 0$ the weighted Bergman space $A^p_\alpha(\mathbf{B})$ consists of holomorphic functions in $L^p(\mathbf{B}, dv_\alpha)$, that is, $A^p_\alpha(\mathbf{B}) = L^p(\mathbf{B}, dv_\alpha) \bigcap H(\mathbf{B})$.

As usual, we denote by $\overrightarrow{\alpha}$ the vector $(\alpha_1, \dots, \alpha_n)$.

For $\alpha_j > -1, j = 1, \dots, n, 0 < p < \infty$, recall that the weighted Bergman space $A^p_{\overrightarrow{\alpha}}(U^n)$ consists of all holomorphic functions on the polydisk satisfying the condition

$$\|f\|^p_{A^p_{\overrightarrow{\alpha}}} = \int_{U^n} |f(z)|^p \prod_{i=1}^{n} (1 - |z_i|^2)^{\alpha_i} dm_{2n}(z) < \infty.$$

When $\alpha_1 = \ldots = \alpha_n = \alpha$ then we use notation $A^p_\alpha(U^n)$.

For any f function from $L^1(\mathbf{B})$ we denote by $P[f]$ the Bergman projection of f function (see [15], Chapter 2).

2. Weakly invertible functions in higher dimensions

The goal of this section is to provide some generalizations known one dimensional assertions on weakly invertible functions in higher dimensions. First we present general arguments then we consider concrete situations. We will shows in particular that results in weak invertible functions are closely connected with estimates for analytic functions from spaces of different dimensions and we will provide two results in this direction. For the proofs of our main results we will need several lemmas.

Lemma A. **[15]** *There exists a positive integer N such that for any $0 < r \leq 1$ we can find a sequence $\{a_k\}$ in $\mathbf{B}$ with the following properties:*

(1) $B = \bigcup_k D(a_k, r)$*;*

(2)The sets $D(a_k, \frac{r}{4})$ *are mutually disjoint;*

(3) Each point $z \in \mathbf{B}$ *belongs to at most* N *of the sets* $D(a_k, 2r)$*.*

We are going to call as usual a_k an r-lattice in the Bergman metric or sampling sequence.

Lemma B. **[15]** *For every* $r > 0$ *there exists a positive constant* C_r *such that*

$$C_r^{-1} \leq \frac{1-|a|^2}{1-|z|^2} \leq C_r, \;\; C_r^{-1} \leq \frac{1-|a|^2}{|1-\langle a, z\rangle|} \leq C_r,$$

for all a *and* z *in* B *such that* $\beta(a, z) < r$*. Moreover, if* r *is bounded above, then we may choose* C_r *independent of* r*.*

Lemma C. ([7], page 126) *Let* $Q_m(\rho) = \{x = (x_1, \dots, x_n) \in \mathbb{R}^m : (\sum_{i=1}^m |x_i|^p)^{\frac{1}{p}} \leq \rho\}$*,* $p \in [1, \infty)$*,* h *be a positive measurable function and* $\rho \leq 1$*. Then*

$$\int_{Q_m(\rho)} h(\|x\|)dx = C(m)\int_0^\rho t^{m-1}h(t)dt,$$

where $\|x\| = (\sum_{i=1}^m |x_i|^p)^{\frac{1}{p}}$ *and* $C(m)$ *is a constant depending on* m*.*

Lemma 1. *Let* $f \in H(B)$ *and* $F(z_1, \dots, z_m) = C_\alpha \int_B \frac{f(z)(1-|z|)^\alpha dm_2(z)}{\prod_{k=1}^m (1-\langle z, z_k\rangle)^{\frac{\alpha+1+n}{m}}}$, $\alpha > -1$, C_α *is a constant of Bergman representation formula (see [15]).*

1) *Let* $p \leq 1$*. Then* $F \in H(B \times \dots \times B)$ *and*

$$|F(z_1, \dots, z_m)|^p \leq C\int_{\mathrm{B}} \frac{|f(\widetilde{w})|^p(1-|\widetilde{w}|)^t dv(\widetilde{w})}{\prod_{k=1}^m |1-\langle z_k, \widetilde{w}\rangle|^{\frac{\alpha+1+n}{m}p}},$$

where $t = p(n+\alpha+1)-(n+1)$, $z_j \in \mathrm{B}$, $j = 1, \dots, m$, $t > -1$*.*

2) *Let* $p > 1$, $\tau = p\,(\frac{n+1+\alpha}{mp'} - \tau_2)$, $\tau < 0$, $\tau_1 + \tau_2 = \frac{n+1+\alpha}{m}$, $\tau_1, \tau_2 > 0$, $\frac{1}{p} + \frac{1}{p'} = 1$*. Then* $F \in H(B \times \dots \times B)$ *and*

$$|F(z_1, \dots, z_m)|^p \lesssim \int_{\mathrm{B}} \frac{|f(w)|^p(1-|w|)^\alpha(1-|z_1|^2)^\tau \cdots (1-|z_m|^2)^\tau}{\prod_{k=1}^m |1-\langle z_k, w\rangle|^{p\tau_1}} dv(w),$$

where $z_j \in \mathrm{B}, j = 1, \dots, m$.

Proof. Using known properties of sampling sequence $\{a_k\}$ we get the following chain estimates ($p \leq 1$) from Lemma A and Lemma B and the fact that

$$\left(\sum_{k=1}^{\infty} a_k\right)^p \leq \sum_{k=1}^{\infty} a_k^p,\ a_k \geq 0,\ p \leq 1,$$

$$\begin{aligned}|F(z_1,\ldots,z_m)|^p &\lesssim \sum_{k\geq 0} \max_{D(a_k,\tau)} |f(w)|^p \left(\int_{D(a_k,\tau)} \frac{(1-|w|)^\alpha}{\prod_{j=1}^m |1-\langle w, z_j\rangle|^{\frac{\alpha+n+1}{m}}} dv(w)\right)^p \\ &\lesssim \sum_{k\geq 0} \max_{D(a_k,\tau)} |f(w)|^p \frac{(1-|a_k|)^{p\alpha}(v(D(a_k,\tau)))^p}{\prod_{j=1}^m |1-\langle a_k, z_j\rangle|^{\frac{\alpha+n+1}{m}p}}.\end{aligned}$$

Then using the relation

$$|1-\langle w, z\rangle| \asymp |1-\langle a_k, z\rangle|,\ w \in D(a_k,\tau),\ z \in \mathrm{B},\ \text{(see [15], page 63)}$$

and Lemma 2.24 from [15] and Lemma A we finally get

$$\begin{aligned}|F(z_1,\ldots,z_m)|^p &\leq C\sum_{k\geq 0}\int_{D(a_k,2\tau)} |f(\widetilde{w})|^p dv(w) \cdot \frac{(1-|a_k|)^{p\alpha}(1-|a_k|)^{p(n+1)}}{(1-|a_k|)^{(n+1)}\prod_{j=1}^m |1-\langle a_k, z_j\rangle|^{\frac{\alpha+n+1}{m}p}} \\ &\leq C\int_{\mathrm{B}} \frac{|f(\widetilde{w})|^p(1-|\widetilde{w}|)^t dv(\widetilde{w})}{\prod_{k=1}^m |1-\langle z_k, \widetilde{w}\rangle|^{\frac{\alpha+n+1}{m}p}},\end{aligned}$$

where $t = p(n+\alpha+1)-(n+1)$.

For $p > 1$ the proof is based on Hölder inequality applied twice and the estimate (see [15], Theorem 1.12):

$$\int_{\mathrm{B}} \frac{(1-|z|)^v}{|1-\langle w, z\rangle|^{s_1}} dv(z) \leq \frac{C}{(1-|w|)^{s_1-n-1-v}},\ w \in \mathrm{B},\ v > -1,\ s_1 > v+n+1,$$

applied m times for $s_1 = \tau_2 p' m$ we have

$$\begin{aligned}|F(z_1,\ldots,z_m)|^p &\leq C\left(\int_{\mathrm{B}} \frac{|f(w)|^p(1-|w|)^\alpha}{\prod_{k=1}^m |1-\langle z_k, w\rangle|^{p\tau_1}} dv(w)\right) \\ &\quad \cdot\left(\int_{\mathrm{B}} \frac{(1-|w|)^\alpha}{\prod_{k=1}^m |1-\langle z_k, w\rangle|^{p'\tau_2}} dv(w)\right)^{\frac{p}{p'}} = C(I_1 \times I_2) \\ &\leq C \cdot I_1 \prod_{k=1}^m \left(\int_{\mathrm{B}} \frac{(1-|w|)^\alpha dv(w)}{|1-\langle z_k, w\rangle|^{mp'\tau_2}}\right)^{\frac{p}{mp'}} \\ &\lesssim \int_{\mathrm{B}} \frac{|f(w)|^p(1-|w|)^\alpha(1-|z_1|^2)^\tau \cdots (1-|z_m|^2)^\tau}{\prod_{k=1}^m |1-\langle z_k,\, w\rangle|^{p\tau_1}} dv(w),\end{aligned}$$

where $z_j \in \mathrm{B}, j = 1, \ldots, m$, $\tau = p(\frac{\alpha+n+1}{mp'} - \tau_2)$, $\tau < 0$. Lemma 1 is proved.

Repeating arguments of proof of lemma 1 we can easily obtain it is complete analogue when integrals by ball will be replaced by integrals by polydisk.

The following lemma follows directly from classical Hölder's inequality for n functions and the well known one dimensional result ($n = 1$ case).

Lemma 2. *Let $t > -1$, $\beta_j > \frac{t+1}{n}$, $j = 1, \ldots, n$. Then*

$$\int_0^1 \frac{(1-|w|)^t d|w|}{\prod_{j=1}^n |1-|w|z_j e^{i\varphi}|^{\beta_j}} \leq \frac{C}{\prod_{j=1}^n |1-z_j e^{\overline{i\varphi}}|^{\beta_j - \frac{t+1}{n}}},\ z_j \in U,\ e^{i\varphi} \in \mathbf{T}.$$

Proposition 1. *Let X, Y be normed subspaces of $H(\mathbb{D}^2), f \in X, f(z) \neq 0, z \in \mathbb{D}^2$. Let also $X \subset H^\infty(\mathbb{D}^2)$, $X \subset Y$, assume also that the set of polynomials $\{p_n\}$ are dense in Y and all bounded functionals on Y can be represented by Cauchy duality (see [3]). Let*

$$f(z) = \sum_{k_1,k_2 \geq 0} a_{k_1,k_2} z_1^{k_1} z_2^{k_2}, |a_{m_1,m_2}| \leq C m_1^s m_2^s$$

for some fixed $s \in (0, +\infty)$. Let

$$\sum_{m_1,m_2=0}^{+\infty} a_{m_1,m_2} \varphi(z_1^{m_1} z_2^{m_2}) m_1^{k_1} m_2^{k_2} = 0;$$

$(k_1, k_2) \in \mathbb{Z}_+^2$ implies $\varphi(1) = 0$, and also $\|z_1^{m_1} z_2^{m_2}\|_Y \leq \frac{C}{m_1^q m_2^q}$, for some N large enough, $m_1, m_2 \geq 1, q \geq N$. Then f is weakly invertible in Y.

Proof. To prove our assertion it is enough to show that for every bounded linear functional φ on Y such that $\varphi(g) = 0$ for every $g \in \{p_n f\}$ we have $\varphi(1) = 0$. According to Lemma 2 of [19], we have $\mathcal{D}^{k_1,k_2} f = \psi_{k_1,k_2} f$ where

$$\psi_k \in H(\mathbb{D}^2),\ |\psi_k(z_1, z_2)| \leq \frac{C}{\prod_{j=1}^2 (1-|z_j|)^{2k_j+1}};$$

$$\mathcal{D}_{z_j} f = z_j \frac{\partial f}{\partial z_j}(z_1, z_2);\ \mathcal{D}^k f(z) = \mathcal{D}_{z_2}^{k_2} \mathcal{D}_{z_1}^{k_1} f(z),$$

hence for every sequence of polynomials $\{p_n\} \in \mathcal{P}$ ($\mathcal{P}$ is a set of polynomials) we have

$$\|p_m f - \mathcal{D}^k f\|_Y \leq C \|f\|_{H^\infty} \|p_m - \psi_k\|_Y,$$

hence choosing $\{p_m\}$ so that $p_m \to \psi_k$, we have if $\varphi \perp E(f)$ then $\varphi(\mathcal{D}^{k_1,k_2} f) = 0$, $(k_1, k_2) \in \mathbb{Z}_+^2$. To end the proof we note that since functionals on Y are represented by formulation of theorem as usual by Cauchy duality (see [3]) we will have the following chain of estimates and equalities.

First
$$\varphi(\mathcal{D}^{k_1,k_2}f)=\lim_{\rho\to 1^{-0}}\int_{\mathrm{T}^2}\mathcal{D}^{k_1,k_2}f(\rho\xi)g(\rho\xi)d\xi=0$$
where $g(z)=\varphi(e_z)$, $e_z(\xi)=\frac{1}{1-\xi z},\xi,z\in\mathbb{D}^2$. Then

$$\varphi(\mathcal{D}^{k_1,k_2}f)=\lim_{\rho\to 1}\sum_{m_1,m_2=0}^{\infty}a_{m_1,m_2}\varphi(z_1^{m_1}z_2^{m_2})m_1^{k_1}m_2^{k_2}\rho^{k_1+k_2}=0.$$

Now it is easy to see that the last series converges uniformly using conditions from formulation. Since

$$|\varphi(z_1^{m_1}z_2^{m_2})|\le\|\varphi\|\|z_1^{m_1}z_2^{m_2}\|_Y,$$

we will have by conditions of the proposition

$$\varphi(\mathcal{D}^{k_1,k_2}f)=\sum_{m_1m_2\ge 0}a_{m_1,m_2}\varphi(z_1^{m_1}z_2^{m_2})m_1^{k_1}m_2^{k_2}=0,$$

hence $\varphi(1)=0$ and Proposition 1 is proved.

Remark 1. Note this scheme based on duality in the unit disk appeared in [8] and were used before by Beurling see [8].

Remark 2. The same arguments based on duality we provided in Proposition 1 can be used also in the case of unit ball for determination of weak invertibility in unit ball case.

Theorem 1. *There exists a function $\widetilde{f}$, $\widetilde{f}\in H^\infty(\mathbb{D}^2)$, $\widetilde{f}(z)\neq 0, z\in\mathbb{D}^2$ which is not weakly invertible in $X\subset H(\mathbb{D}^2)$, where X is some normed subspace $H(\mathbb{D}^2)$, if for every $f\in X$, $f(z,z)\in Y(\mathbb{D})$, $Y\subset H(\mathbb{D})$ and if for any f_0*

$$f_0\in Y,\ f_0(z)=\sum_{k=0}^{\infty}b_kz^k,\ \sum_{k=0}^{\infty}\frac{|b_k|}{\alpha_k}<\infty$$

for some fixed positive sequence $\{\alpha_k\}_{k=0}^{\infty}$, $\alpha_k\le\alpha_{k+1}\le\dots$ such that $\sum_{k=1}^{\infty}\frac{\ln\alpha_k}{k^{\frac{3}{2}}}<+\infty$.

Proof. According to [20], there exists a function $f(z)=\frac{f_1(z)}{f_2(z)}$, $f_1\in H^\infty(\mathbb{D})$, $f_2\in H^\infty(\mathbb{D})$, $f_2\neq 0$, $z\in\mathbb{D}$ so that

$$f(e^{i\theta})=\sum_{k=1}^{\infty}C_{-k}e^{-ik\theta},\ \theta\in[-\pi,\pi],\ |C_{-k}|\le\frac{1}{\alpha_k},$$

where $k = 1, 2, \dots, \alpha_k \le \alpha_{k+1} \le \dots, \alpha_j > 0, j = 1, 2, \dots$ and the series for f absolutely converges. Moreover, $\sum_{k=1}^{\infty} \frac{\ln \alpha_k}{k^{\frac{3}{2}}} < +\infty$. Obviously

$$\widetilde{f} \in H^\infty(\mathbb{D}^2),\ f_2 \neq 0,\ z \in \mathbb{D}^2,\ \widetilde{f}(z_1, z_2) = f_2(\frac{z_1 + z_2}{2}),\ z = (z_1, z_2) \in \mathbb{D}^2.$$

Since $\widetilde{f}(z, z) = f_2(z)$ and for

$$\widetilde{g} = g(z, z),\ \|\widetilde{g}\|_{Y(\mathbb{D})} \le C\|g(z_1, z_2)\|_{X(\mathbb{D}^2)},\ g \in H(\mathbb{D}^2).$$

All we have to show that there exists such $\{\alpha_k\}$ so that $\widetilde{g}$ is not weakly invertible in $Y(\mathbb{D})$. For that reason we note only that obviously it is enough to present a linear bounded functional $\varphi \in Y^*$ so that $\varphi(z^k f_2) = 0$, $k = 0, 1, 2, \dots$, $z \in \mathbb{D}$ and $\varphi(1) \neq 0$.

Let $g(z) = \sum_{k=1}^{\infty} C_{-k} z^{k-1}$, $z \in \mathbb{D}$, then $g \in H^\infty(\mathbb{D})$. Obviously if

$$\varphi(G) = \lim_{\rho \to 1^{-0}} \frac{1}{2\pi} \int_{-\pi}^{\pi} G(\rho e^{i\xi}) g(\rho e^{-i\xi}) d\xi,\ G \in Y.$$

Then $\varphi \in Y^*$ (φ is a linear bounded functional on Y), moreover, it is an easy exercise to show that $\varphi(z^k f_2) = 0$, $k = 0, 1, 2, \dots$, $\varphi(1) = C_{-1} \neq 0$. Indeed it is almost obvious

$$\varphi(1) = \frac{1}{2\pi} \int_{\mathrm{T}} g(\xi) d\xi = C_{-1} \neq 0,\ \varphi(z^k f_2) = \int_{\mathrm{T}} e^{i(m+1)\xi} f_1(\xi) d\xi = 0,\ m = 0, 1, 2, \dots$$

and also by condition in formulation of the theorem we have

If

$$G \in Y,\ G(z) = \sum_{k=0}^{\infty} b_k z^k, \quad \text{then } |b_k| \cdot |C_{-k-1}| \le \frac{C}{\alpha_k} \cdot |b_k|.$$

So

$$\varphi(G) = \lim_{\rho \to 1^{-0}} \int_{\mathrm{T}} G(\rho\xi) g(\xi) d\xi = \lim_{\rho \to 1} \sum_{k=0}^{\infty} b_k C_{-k-1} \rho^k,\ G \in Y$$

converges uniformly and $\varphi \in Y^*$. The proof of Theorem 1 is complete.

Remark 3. General assertions we provided in Proposition 1 and Theorem 1 can provide various concrete examples of not weakly invertible functions from H^∞ for various concrete X classes in higher dimension.

Remark 4. The problem of weak invertibility of every nonzero function f, $f \in X \subset Y$, $X, Y \subset H(\mathbb{D})$ in one variable were considered intensively in [8].

The following proposition use ideas we indicate above for the study of weak invertibility of generalization of $\exp(\frac{z+1}{z-1})$ function on polydisk. In unit disk the weak invertibility of $\exp(\frac{z+1}{z-1})$ was studied in [8] systematically.

Proposition 2. *Let λ be an increasing positive function on $[0,1)$, $\log(\lambda(r)) = \varphi(\log\frac{1}{1-r})$, $\psi = \log\varphi$ and ψ is convex near $+\infty$ and let*

$$\int_0^1 \left(\frac{\log\lambda(r)}{1-r}\right)^{\frac{1}{2}} dr < \infty, \int_U \frac{\lambda(|z|)(1-|z|)^\alpha dm_2(z)}{|1-\langle w,\bar{z}\rangle|^{\alpha+2}} \le C\lambda(|w|),\ w\in U,\ \alpha>-1.$$

then the Φ function, $\Phi = T_n^\alpha(f_0)$ (this is expanded Bergman projection from [4]), $\alpha > -1, n \ge 1, f_0(z) = e^{\frac{z+1}{z-1}}, z \in U$, is not weakly invertible in

$$A_n(\lambda) = \{f \in H(U^n) : \sup_{z\in U^n} \frac{|f(z)|}{\tilde{\lambda}(|z|)} < \infty\},\ \tilde{\lambda}(z) = (\prod_{k=1}^n \lambda(|z_k|))^{\frac{1}{n}}.$$

Proof. Since $\Phi = T_n^\alpha(f_0)$, using Hölder's inequality for n functions we get

$$||\Phi||_{A_n(\lambda)} \le C||f_0||_{A_1(\lambda)}. \tag{6}$$

It is easy to note that

$$||\tilde{P}_n f_0 - 1||_{A_1(\lambda)} \le C_1||P_n\Phi - 1||_{A_n(\lambda)},\ \tilde{P}_n = P_n(z,\cdots,z),\ P_n = P_n(z_1,\cdots,z_n).$$

It remains to use the fact that f_0 is not weakly invertible in $A_1(\lambda)$(see [8]). Proposition 2 is proved.

Remark 5. For $n = 1$ Proposition 2 were obtained in [8].

Remark 6. Various other assertions can be obtained similarly for $\Phi = T_n^\alpha(f_0)$ function concerning it is weakly invertibility based on results of [8] for $n = 1$.

Remark 7. If $\exp(\frac{z+1}{z-1})$ is not weakly invertible in $X(U)$, then

$$f_0 = \exp(\frac{\sum_{k=0}^n \frac{z_k+1}{z_k-1}}{n}) \text{ and } f_0^1 = \exp(\frac{\frac{z_1+\cdots+z_n}{n}+1}{\frac{z_1+\cdots+z_n}{n}-1})$$

are not weakly invertible in $Y(U^n)$ or $Y(\mathbf{B})$ as soon as

$$||g(z,\cdots,z)||_X \le C||g(z_1,\cdots,z_n)||_Y,\ g\in H(\mathbf{B}) \text{ or } g\in H(U^n).$$

The Proposition 2 we formulated gives such an example.

We easily can note that any theorem on traces and diagonal map in A^p_α, H^p, Q_p, Bloch type classes (see [3],[4] and references there) can be applied in problems connected with weak invertibility in higher dimension.

In recent note [9] a result of the similar nature was proved. Namely it was proved by authors in [9] that $f \in H^2(U^n)$, $n \in \mathbb{N}$, then $f \in H^{2n}(\mathbf{B})$ and there holds

$$\sup_{0<r<1}\int_{\mathbf{S}}|f(r\xi)|^{2n}d\sigma_n(\xi)\leq C\sup_{0<r<1}\left(\int_{\mathrm{T}^n}|f(r\xi)|^2dm_n(\xi)\right)^n.$$

Note for $n=1$ the estimate we provided above is obvious. We add now new results in this direction connecting analytic spaces of different dimensions and at the same time generalizing some previously known one dimensional estimates.

Theorem 2. 1° *Let* $p\in(0,\infty)$, $t>-1$, $\alpha=-1+\frac{t+2}{n}$, $k=1,\ldots,n$, $f\in H(U^n)$. *Then*

$$\int_U|f(z,\ldots,z)|^p(1-|z|)^tdm_2(z)$$
$$\leq C\int_0^1\cdots\int_0^1\int_{\mathrm{T}}(1-\min r_k)^{-n}\prod_{k=1}^n(1-r_k)^\alpha|f(r_1\xi,\ldots,r_n\xi)|^pdm(\xi)dr_1\cdots dr_n;$$

2° *Let* $\beta=\alpha-n$, $\beta>-1$, $p\in(0,1]$, $f\in H(\mathbf{B})$. *Then*

$$\int_{Q_m(1)}\left(\sup_{\xi\in\mathrm{T}^n}(P[f])(R_1,\ldots,R_n)\right)^p\left(1-\left(\sum_{k=1}^nR_k^2\right)^{\frac12}\right)^\alpha dR_1\cdots dR_n$$
$$\lesssim C\int_{\mathbf{B}}|f(w)|^p(1-|w|)^\beta dv(w);$$

3° *Let* $p\leq1$, $\alpha>n-1$, $f\in H(U^n)$. *Then*

$$\int_0^1(1-r)^\alpha\sup_{\xi\in\mathbf{S}}|f(r\xi)|^pdr\leq C\int_{U^n}|f(w)|^p\prod_{k=1}^n(1-|w_k|)^{\frac{\alpha+1}{n}-2}dm_{2n}(w).$$

Proof. 1° The following dyadic decomposition of subframe and polydisk were introduced in [3] and will be also used by us.

$$U_{k_1,\ldots,k_n,l_1,\ldots,l_n}=U_{k_1,l_1}\times\cdots\times U_{k_n,l_n}=\Big\{(\tau_1\xi_1,\ldots,\tau_n\xi_n):\tau_j\in(1-\frac{1}{2^{k_j}},\frac{1}{2^{k_j+1}}],$$
$$k_j=0,1,2,\ldots;\ \frac{\pi l_j}{2^{k_j}}<\xi_j\leq\frac{\pi(l_j+1)}{2^{k_j}},\ l_j=-2^{k_j},\ldots,2^{k_j}-1,\ j=1,\ldots,n\Big\}.$$

We have

$$\int_U|f(z,\ldots,z)|^p(1-|z|)^tdm_2(z)\lesssim\sum_{k\geq0}\sum_{j=-2^k}^{2^k-1}\left(\int_{U_{j,k}}|f(z,\ldots,z)|^p(1-|z|)^tdm_2(z)\right)$$

$$\lesssim\ C\sum_{k_1,\ldots,k_n\geq0}\sum_{j=-2^{\min k_j}}^{2^{\min k_j}-1}\left(\sup_{z\in U_{j,k_1,\ldots,k_n}}|f(z)|^p\right)2^{-\frac{(k_1+\ldots+k_n)t}{n}}2^{-2\frac{k_1+\ldots+k_n}{n}}$$

$$\lesssim C \sum_{k_1,\ldots,k_n\geq 0} \sum_{j=-2^{\min k_j}}^{2^{\min k_j}-1} 2^{\min k_j n} 2^{k_1+\ldots+k_n} 2^{-(\sum_{k=1}^n k_j)(\frac{t}{n}+\frac{2}{n})} \int_{U_{j,k_1,\ldots,k_n}} |f(z)|^p dm_2(z)$$

$$\lesssim C \int_{U^n} |f(z)|^p \prod_{k=1}^n (1-|z_k|)^\alpha (1-\min_k |z_k|)^{-n} dm_{2n}(z),\ \alpha = -1+\frac{t+2}{n},$$

where we used the fact that

$$\sup_{z\in U_{j,k_1,\ldots,k_n}} |f(z)|^p \leq C 2^{\min k_j n} 2^{k_1+\ldots+k_n} \int_{U^*_{j,k_1,\ldots,k_n}} |f(z)|^p dm_{2n}(z)$$

which follows from subharmonicity of $|f(z)|^p$, $0<p<\infty$ and where $U^*_{j,k_1,\ldots,k_n}$ are enlarged dyadic cubes (see [3]). We used at the last step also the fact that $U^*_{j,k_1,\ldots,k_n}$ is a finite covering of polydisk U^n (see [3]).

2° According to Bergman representation formula (see [11,15]) in the unit ball we have

$$f(z_1,\cdots,z_n) = C_\beta \int_{\mathbf{B}} \frac{f(w)(1-|w|)^\beta dv(w)}{(1-\langle \overline{w}, z\rangle)^{\beta+n+1}},$$

where $\beta > -1, \beta$ can be large enough, $z=(z_1,\cdots,z_n)\in \mathbf{B}$.

For $p\leq 1$ consider the same integral but with $(z_1,\cdots,z_n)\in U^n$, $\sum_{j=1}^n |z_j|^2 < 1$, it follows from Lemma 1 we will have

$$\left(\sup_{\xi\in T^n} (P[f])(R_1,\cdots,R_n)\right)^p \lesssim \int_0^1 \int_{\mathbf{S}} \frac{|f(r\xi)|^p (1-r)^{\beta p+(p-1)(n+1)} d\sigma(\xi) dr}{(1-r(\sum_{k=1}^n R_k^2)^{\frac{1}{2}})^{(\beta+n+1)p}}.$$

Then using Lemma C for $p=2$ and

$$h(\|x\|) = \frac{(1-(\sum_{k=1}^n R_k^2)^{\frac{1}{2}})^\alpha}{(1-r(\sum_{k=1}^n R_k^2)^{\frac{1}{2}})^{(\beta+n+1)p}},\ \|x\| = (\sum_{k=1}^n R_k^2)^{\frac{1}{2}}, \|x\|\leq 1$$

we can easily get what we need.

3° can be obtained similarly as 2°, but instead of integral representation in the unit ball we have to use known integral representation in the polydisk. The proof of 3° follows from Lemma 1 for $p\leq 1$ and Lemma 2. Indeed,

$$f(z_1,\cdots,z_n) = C_\beta \int_{U^n} \frac{f(w_1,\cdots,w_n)\prod_{k=1}^n (1-|w_k|)^\beta dm_{2n}(w)}{\prod_{k=1}^n (1-\langle \overline{w_k}, z_k\rangle)^{\beta+2}}.$$

Let $z=(z_1,\cdots,z_n)$ be in $\mathbf{B}$ so that $|z_j|<1, j=1,2,\cdots,n$. Then by Lemma 1 for large enough β

$$\sup_{\xi\in\mathbf{S}} |f(z_1,\cdots,z_n)|^p \leq C_\beta \int_{U^n} \frac{|f(w)|^p \prod_{k=1}^n (1-|w_k|)^{\beta p+2p-2} dm_{2n}(w)}{\prod_{k=1}^n (1-r|w_k|)^{(\beta+2)p}},$$

$|z_j| = r, z_j = r\xi_j$. It remains to use Lemma 2. Theorem 2 is proved.
Remark 8. All estimates of Theorem 1 are obvious for $n = 1$ and $p = 1$ or coincide with classical assertions from the theory of analytic functions of one variable (see [15]).
We showed above that one dimensional results provide new assertions on weakly invertible function in standard Bergman spaces in the unit ball. Under some additional conditions on general positive Borel measure in the unit ball we may state that our functions are weakly invertible in general mixed norm spaces in the unit ball.

We denote by $(A^p_\alpha)_1$ the space of all holomorphic functions in the unit ball such that

$$\|f\|^p_{(A^p_\alpha)_1} = \int_{\mathrm{S}}\int_{\widetilde{\Gamma}_t(\xi)} \frac{|f(z)|^p(1-|z|)^\alpha}{(1-|z|)^n} dv(z)d\sigma(\xi) < \infty,\ n \in \mathbb{N},\ 0 < p < \infty,\ \alpha > -1,$$

where

$$\widetilde{\Gamma}_t(\xi) = \{z \in \mathrm{B} :\ |1 - \xi\overline{z}| < t(1-|z|)^{\frac{1}{n}},\ t > 1\}$$

enlarged approach region.

Lemma 3. **([15])** *Suppose $r > 0$, $p > 0$ and $\alpha > -1$. Then there exists a constant $C > 0$ such that*

$$|f(z)|^p \le \frac{C}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(w)|^p dv_\alpha(w)$$

for all $f \in H(\mathrm{B})$ and $z \in \mathrm{B}$.
Lemma 4. *Let $\sigma > 1$, $t > 0$, $\xi \in S$, $\widetilde{\Gamma}_\sigma(\xi) = \{z : |1 - \overline{\xi}z| < \sigma(1-|z|)^{\frac{1}{n}}\}$. Then there exist $\widetilde{\sigma}(\sigma, t) > 1$ such that $D(z, t) \subset \widetilde{\Gamma}_{\widetilde{\sigma}}(\xi)$ for all $z \in \Gamma_\sigma(\xi)$.*

Proof. Let $w \in D(z,t)$, $z \in \Gamma_\sigma(\xi)$. We will show that $w \in \widetilde{\Gamma}_{\widetilde{\sigma}}(\xi)$ for some $\widetilde{\sigma} > 1$. Since $z \in \Gamma_\sigma(\xi)$, then $|1 - \overline{\xi}z| < \sigma(1-|z|)$, hence

$$\begin{aligned} |1 - \langle \xi, w\rangle| &\le |1 - \langle \xi, z\rangle| + |\langle \overline{\xi}, z\rangle - \langle \overline{\xi}, w\rangle| \le \sigma(1-|z|) + |z-w| \\ &\le \sigma(1-|w|) + (\sigma+1)|z-w|. \end{aligned}$$

We will show $|z - w| \le \sigma_1(1-|w|)^{\frac{1}{2}}$ for some $\sigma_1 > 1$. This is enough since $w \in D(z,t)$ is the same to $z \in D(w,t)$ we have by exercise 1.1 from [8]:

$$\frac{|P_w(z) - c|^2}{R^2\sigma_1^2} + \frac{|Q_z(w)|^2}{R^2\sigma_1} < 1.$$

where

$$R = \tanh(t),\, c = \frac{(1-R^2)w}{1-R^2|w|^2},\, \sigma_1 = \frac{1-|w|^2}{1-R^2|w|^2},$$
$$P_w(z) = \frac{\langle z, w\rangle w}{|w|^2},\, Q_w(z) = z - \frac{\langle z, w\rangle w}{|w|^2}.$$

Hence
$$|z-w| \le C_1\left(|z-P_w| + \left|\frac{\langle z, w\rangle w}{|w|^2} - c\right| + |c-w|\right);$$
$$|c-w| \le |w|\left(1 - \frac{1-R^2}{1-R^2|w|^2}\right) \le \frac{C_2}{1-R^2}(1-|w|) = S_2.$$

It is enough to show

$$\left|z - \frac{\langle z, w\rangle w}{|w|^2}\right| + \left|\frac{\langle z, w\rangle w}{|w|^2} - c\right| \le R\left(\frac{1-|w|^2}{1-R^2|w|^2}\right)^{\frac{1}{2}}\widetilde{c}(R).$$

Note

$$|z-w|^2 \le C_3\left(|c-w|^2 + \left|z - \frac{\langle z, w\rangle w}{|w|^2}\right|^2 + \left|\frac{\langle z, w\rangle w}{|w|^2} - c\right|^2\right).$$

Hence

$$\begin{aligned} |z-w|^2 &\le C_4(|c-w|^2 + R^2\sigma_1) \le C_4\left(S_2(|w|, R)^2 + \frac{R^2(1-|w|^2)}{1-R^2|w|^2}\right) \\ &\le C_4\left(S_2 + \frac{R(1-|w|^2)^{\frac{1}{2}}}{1-R^2|w|^2}\right)^2. \end{aligned}$$

Hence $|z-w| \le C_5(1-|w|)^{\frac{1}{2}}$.

Lemma 5. *Let μ be a positive Borel measure in B. Let $D(w, t) \subset \widetilde{\Gamma}_\sigma(\xi)$, $w \in \Gamma_\tau(\xi)$, where τ, σ, ξ, t are from Lemma 4. Let $f \in H(B)$. Then*

$$\int_{\Gamma_\tau(\xi)} \frac{|f(z)| d\mu(z)}{(1-|z|)^n} \le C \int_{\widetilde{\Gamma}_\sigma(\xi)} |f(z)| \int_{D(z, t)} d\mu(w) \frac{1}{(1-|z|)^{2n+1}} d\upsilon(z).$$

Proof. Since $\chi_{D(z,t)}(w) = \chi_{D(w,t)}(z)$, $z, w \in$ B, $t > 0$. Using Lemma 3, Lemma

4 and Fubini theorem, we have

$$\begin{aligned}\int_{\Gamma_\tau(\xi)}\frac{|f(z)|d\mu(z)}{(1-|z|)^n} &\leq C\int_{\Gamma_\tau(\xi)}\frac{1}{(1-|z|)^{n+1}}\Big(\int_{D(z,t)}|f(w)|dv(w)\Big)\frac{d\mu(z)}{(1-|z|)^n}\\ &\lesssim \int_{\Gamma_\tau(\xi)}\frac{1}{(1-|z|)^{2n+1}}\int_{\mathrm{B}}|f(w)|\chi_{D(w,t)}(z)dv(w)d\mu(z)\\ &\lesssim \int_{\widetilde{\Gamma}_\sigma(\xi)}\int_{\Gamma_\tau(\xi)}\frac{1}{(1-|z|)^{2n+1}}|f(w)|\chi_{D(w,t)}(z)dv(w)d\mu(z)\\ &\lesssim \int_{\widetilde{\Gamma}_\sigma(\xi)}\frac{1}{(1-|w|)^{2n+1}}|f(w)|\Big(\int_{D(w,t)}d\mu(z)\Big)dv(w).\end{aligned}$$

Theorem 3. *Suppose $\alpha>-1$, $1<q<p\leq\infty$, $t>1$, μ be a nonnegative Borel measure in B. Then the following is true.*

$$\Big\{\int_S\Big(\int_{\Gamma_t(\xi)}\frac{|f(z)|d\mu(z)}{(1-|z|)^n}\Big)^q d\sigma(\xi)\Big\}^{\frac{1}{q}}\leq C\|f\|_{(A^p_\alpha)_1},$$

if

$$K_{p,q}=\int_{\widetilde{\Gamma}_{\widetilde{\sigma}}(\xi)}\Big(\int_{D(z,\delta)}d\mu(w)\Big)^{\frac{p}{p-1}}\frac{dv(z)}{(1-|z|)^{n+(n+1)\frac{p}{p-1}}}\in L^{\frac{(p-1)q}{p-q}}(S)$$

for any $\widetilde{\sigma}>1$, $\delta>0$.

Proof. Let us consider first $p=\infty$ case. The proof follows directly from Lemma 6 that we proved above. Let $f\in H(\mathrm{B})$. Then we easily have by Lemma 6

$$M_q(f,\mu)=\int_{\mathrm{S}}\Big(\int_{\Gamma_t(\xi)}\frac{|f(z)|d\mu(z)}{(1-|z|)^n}\Big)^q d\sigma(\xi)\leq C\int_{\mathrm{S}}\Big(\int_{\widetilde{\Gamma}_{\widetilde{\sigma}}(\xi)}\frac{|\widetilde{f}(z)|dv(z)}{(1-|z|)^{2n+1}}\Big)^q d\sigma(\xi),$$

where

$$|\widetilde{f}(z)|=\Big(\int_{D(z,\delta)}d\mu(w)\Big)\cdot|f(z)|.$$

Hence

$$M_q(f,\mu)\leq\Big(\sup_{z\in\mathrm{B}}|f(z)|\Big)\cdot K(\mu).$$

This is enough since $K_{\infty,q}(\mu)=K(\mu)$.

Let $q>1$, $1<p<\infty$. Then again using Lemma 5 and Hölder inequality, we have the following chain of estimates.

$$\begin{aligned}S(\mu,f,t) &= \int_{\Gamma_t(\xi)}\frac{|f(z)|d\mu(z)}{(1-|z|)^n}\leq C\int_{\widetilde{\Gamma}_\sigma(\xi)}\frac{|f(z)|\Big(\int_{D(z,\delta)}d\mu(w)\Big)dv(z)}{(1-|z|)^{2n+1}}\\ &\lesssim C\Big(\int_{\widetilde{\Gamma}_\sigma(\xi)}\frac{|f(z)|^p dv(z)}{(1-|z|)^n}\Big)^{\frac{1}{p}}\cdot\Big\{\int_{\widetilde{\Gamma}_\sigma(\xi)}\frac{1}{(1-|z|)^q}\Big(\int_{D(z,\delta)}d\mu(w)\Big)^{p'}\frac{dv(z)}{(1-|z|)^n}\Big\}^{\frac{1}{p'}}.\end{aligned}$$

Hence again using Hölder inequality, $q=(n+1)p'$, $\frac{1}{p}+\frac{1}{p'}=1$,

$$\int_{\mathrm{S}}(S(\mu,f,t))^q d\sigma(\xi)\leq C\|f\|^q_{(A^p_\alpha)_1}.$$

Lemma 6. **([15])** *For every $r>0$ there exists a positive constant C_r such that*

$$C_r^{-1}\leq\frac{1-|a|^2}{1-|z|^2}\leq C_r,\ \ C_r^{-1}\leq\frac{1-|a|^2}{|1-\langle a,z\rangle|}\leq C_r,$$

for all a and z in B such that $\beta(a,z)<r$.

Theorem 4. *Let $\alpha>-1$, $q<p\leq\infty$, $0<q<1$. Let μ be a nonnegative Borel measure in B. For any fixed $\sigma>0$,*

$$\left\{\int_S\left(\int_{\Gamma_\sigma(\xi)}\frac{|f(z)|d\mu(z)}{(1-|z|)^n}\right)^q d\sigma(\xi)\right\}^{\frac{1}{q}}\leq C\|f\|_{A^p_\alpha},\tag{1}$$

if

$$\int_S\left(\sum_{j,\,z_j\in\widetilde{\Gamma}_\tau(\xi)}|a_j|\int_{D(z_j,\delta)}d\mu(w)\frac{1}{(1-|z_j|)^s}\right)^q d\sigma(\xi)\leq\|\{a_j\}\|^q_{l^p}\tag{2}$$

for any $\tau>0$, $\delta-$lattice $\{z_j\}$, $s=\frac{\alpha+n+1}{p}+n$, where μ be a positive measure on B, $d\sigma$ the normalized rotation invariant Lebesgue measure on S, $\Gamma_\sigma(\xi)$ is the corresponding Koranyi approach region with vertex ξ on S, i.e.

$$\Gamma_\sigma(\xi)=\{z\in\mathrm{B}:\,|1-\langle z,\zeta\rangle|<\sigma(1-|z|^2),\ \sigma>1\}.$$

Proof. Let first condition (2) holds. $D(z_j,\delta)\bigcap\Gamma_\sigma(\xi)\neq\emptyset$ implies $z_j\in\widetilde{\Gamma}_\tau(\xi)$ for some $\tau>1$ (Note this for n=1 was proved in [30]). Hence for any function f, $f\in H(\mathrm{B})$, we have the following chain of estimates, let $D_j=D(z_j,\delta)$,

$$\begin{aligned}\int_{\Gamma_\sigma(\xi)}\frac{|f(z)|d\mu(z)}{(1-|z|)^n}&\lesssim C\sum_{j,\,D_j\bigcap\Gamma_\sigma(\xi)\neq\emptyset}\int_{D_j}\frac{|f(z)|d\mu(z)}{(1-|z|)^n}\lesssim C\sum_j\left(\sup_{z\in D_j}|f(z)|\right)\frac{\mu(D_j)}{(1-|z_j|)^n}\\&\lesssim C\sum_j\left(\sup_{z\in D_j}|f(z)|\right)(1-|z_j|)^{\frac{\alpha+n+1}{p}}\\&\quad\cdot\frac{1}{(1-|z_j|)^{(\alpha+n+1)(\frac{1}{p}-1)+n}}\left(\int_{D(z_j,\delta)}d\mu(w)\right)\cdot\frac{1}{(1-|z_j|)^{\alpha+n+1}}\\&\lesssim C\sum_{j,\,z_j\in\widetilde{\Gamma}_\tau(\xi)}\sup_{z\in D_j}|f(z)|(1-|z_j|)^{\frac{\alpha+n+1}{p}}\\&\quad\cdot\left(\int_{D(z_j,\delta)}d\mu(w)\right)\cdot\frac{1}{(1-|z_j|)^{\frac{\alpha+n+1}{p}+n}}.\end{aligned}$$

Hence we only need to show that

$$(1-|z_j|)^{\frac{\alpha+n+1}{p}}\sup_{z\in D_j}|f(z)|$$

is in l^p and it is norm dominated by $C\|f\|_{A^p_\alpha}$. This is obvious for $A^\infty_\alpha = H^\infty$, where H^∞ is the class of all bounded analytic functions in B. For $p<\infty$, we have using Lemma A, Lemma 3 and Lemma 6

$$\sum_j (1-|z_j|)^{\alpha+n+1}\left(\sup_{z\in D_j}|f(z)|^p\right) \le C\sum_j \int_{D(z_j,2\delta)} |f(w)|^p dv_\alpha(w) \lesssim C\|f\|^p_{A^p_\alpha}.$$

This is what we need.

Theorems 3 and 4 for n=1 unit disk case are known see [15].

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