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MULTIPLIER TRANSFORMATION AND ITS APPLICATIONS TO CERTAIN MEROMORPHICALLY *P*-VALENT FUNCTIONS

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Abstract. The aim of the present investigation is to determine some results of meromorphically p-valent functions belonging to the class $\mathcal{M}_n(p;\alpha)$ defined by using a multiplier transformation operator defined in this work and then to point their certain geometric and analytic consequences out.

Keywords: Meromorphic function, p-valent function, punctured open unit disc, starlikeness, multiplier transformation, inequalities.

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1. Introduction and definitions

Let $\mathcal{M}(p)$ denote the class of functions $f(z)$ of the following form:

$$f(z) = a_{-p}z^{-p} + a_{1-p}z^{1-p} + a_{2-p}z^{2-p} + \cdots \tag{1.1}$$

$$(a_{-p} \neq 0, a_k \in C; k = -p, 1-p, 2-p, \cdots;\ p \in N = \{1, 2, 3, \cdots\}),$$

which are analytic and p-valent in the punctured unit disc $D := U - \{0\}$, where $U = \{z \in C\ :\ |z| < 1\}$ and also C is the set of complex numbers.

For any integer n and a function $f(z) \in \mathcal{M}(p)$, let use define an operator $I^n[f](z)$ by

$$I^n[f](z) = a_{-p}z^{-p} + \sum_{k=1-p}^{\infty} \left(\frac{k}{p} + 2\right)^{-n} a_k z^k \quad (p \in N). \tag{1.2}$$

We note that the above operator is also a generalization of the operator defined by Cho [3], and one can easily observe that

$$I^n\left\{I^m[f]\right\}(z) = I^{n+m}[f]\left(z\right)$$

for all integers n and m. For any integer n, a function $f(z)$ belonging to $\mathcal{M}(p)$ is said to be in the class $\mathcal{M}_n(p;\alpha)$ if it satisfies the following inequality:

$$\Re e\left(\frac{I^{n-1}[f](z)}{I^n[f](z)}-2\right)<-\frac{\alpha}{p} \quad (0\leq\alpha<p; p\in N; z\in D). \tag{1.3}$$

Let the functions $f(z)$ and $F(z)$ be analytic in the disc $U=\{z:|z|<1\}$. Then the function $f(z)$ is subordination to the function $F(z)$, written by $f(z)\prec F(z)$, if $F(z)$ is univalent, $f(0)=F(0)$ and $f(U)\subset F(U)$.

In this investigation, we note that, for any integer n and also a function $f(z)$ belonging to the class $\mathcal{M}(p)$, $\mathcal{M}_n(p;\alpha)\subset\mathcal{M}_{n+1}(p;\alpha)$ is true and some consequences of our results are also generalization of the ceratin results obtained by Cho [3]. One may check some of the papers in which used certain transformations, for those, see [1], [2], [3], [4], and also [9].

For the proofs of the main results we need the following well-known results obtained by Miller and Mocanu [7] below.

Lemma 1. *Let* $\phi=\phi(u,v)\ :\ \mathcal{D}\subset C^2\to C$ be *a complex valued function, and also let* $u=u_1+iu_2$ and $v=v_1+iv_2$. *Suppose that the function* $\phi(u,v)$ *satisfies the following conditions:*
(i) $\phi(u,v)$ *is continuous,*
(ii) $(1,0)\in\mathcal{D}$ and $\Re e\{\phi(1,0)\}>0$,
and
(iii) $\Re e\{\phi(iu_2,v_1)\}\leq 0$ *for all* (iu_2,v_1) *such that* $v_1\leq-\frac{1+u_2^2}{2}$.

Let a function $p(z)=1+p_1z+p_2z^2+\cdots$ *be regular in* U *such that* $(p(z),zp'(z))\in\mathcal{D}$ for all $z\in U$. *Then*

$$\Re e\{\phi(p(z),zp'(z))\}\leq 0 \quad\Rightarrow\quad \Re e\{p(z)\}>0 \ \ (z\in U).$$

2. The main results and their consequences

Theorem 1. *If* $f(z)\in\mathcal{M}_n(p,\alpha)$ *then* $f(z)\in\mathcal{M}_{n+1}(p,\gamma)$, *where*

$$\gamma:=\frac{4p+2\alpha+1-\sqrt{(4p-2\alpha-1)^2+8p}}{4}. \tag{2.1}$$

Proof. Define $p(z)$ by

$$\frac{I^n[f](z)}{I^{n+1}[f](z)}=\lambda+(1-\lambda)\,p(z), \tag{2.2}$$

where

$$\lambda := \frac{4p-2\alpha-1+\sqrt{(4p-2\alpha-1)^2+8p}}{4p} > 1. \tag{2.3}$$

Clearly, the above function $p(z)$ has the form in the Lemma, i.e., it is both analytic in U and $p(z) = 1 + p_1 z + p_2 z^2 \cdots$. By differentiating of (2.2) logarithmically and using the identity below:

$$z\Big(I^n[f](z)\Big)' = p\Big(I^{n-1}[f](z) - 2I^n[f](z)\Big), \tag{2.4}$$

we obtain that

$$\frac{I^{n-1}[f](z)}{I^n[f](z)} = \lambda + (1-\lambda)\,p(z) + \frac{(1-\lambda)\,zp'(z)}{p\,(\lambda + (1-\lambda)\,p(z))}. \tag{2.5}$$

Since $f(z) \in \mathcal{M}_n(p,\alpha)$, one easily arrive at:

$$-\Re e\left(\frac{I^{n-1}[f](z)}{I^n[f](z)} - 2 + \frac{\alpha}{p}\right) = 2 - \left(\frac{\alpha}{p} + \lambda\right) - (1-\lambda)\,p(z) - \frac{(1-\lambda)\,zp'(z)}{p\,(\lambda + (1-\lambda)\,p(z))} > 0.$$

If define $\phi(u,v)$ by

$$\phi(u,v) = 2 - \left(\frac{\alpha}{p} + \lambda\right) - (1-\lambda)\,u - \frac{(1-\lambda)\,v}{p\,(\lambda + (1-\lambda)\,u)}, \tag{2.6}$$

it is easily seen that the function $\phi(u,v)$ satiesfies the conditions of Lemma 1, and indeed;

(i) $\phi(u,v)$ is continuous in $\mathcal{D} = \left(C - \left\{\frac{\lambda}{\lambda-1}\right\}\right) \times C$,

(ii) $(1,0) \in \mathcal{D}$ and $\Re e\,(\phi(1,0)) = 1 - \frac{\alpha}{p} > 0$ (since $0 \le \alpha < p$),

(iii) for all (iu_2, v_1) such that $v_1 \le -\frac{1+u_2^2}{2}$,

$$\begin{aligned}
\Re e\,\{\phi(iu_2,v_1)\} &= \Re e\left\{2 - \left(\tfrac{\alpha}{p} + \lambda\right) - (1-\lambda)\,iu_2 - \tfrac{(1-\lambda)v_1}{p(\lambda+(1-\lambda)iu_2)}\right\} \\
&= 2 - \left(\tfrac{\alpha}{p} + \lambda\right) - \tfrac{\lambda(1-\lambda)v_1}{p\left(\lambda^2+(1-\lambda)^2u_2^2\right)} \\
&\le 2 - \left(\tfrac{\alpha}{p} + \lambda\right) + \tfrac{\lambda(1-\lambda)\left(1+u_2^2\right)}{2p\left(\lambda^2+(1-\lambda)^2u_2^2\right)} \\
&\le 0 \quad \left(\text{since } \lambda = \tfrac{4p-2\alpha-1+\sqrt{(4p-2\alpha-1)^2+8p}}{4}\right),
\end{aligned}$$

which yields the inequality $\Re e\,\{p(z)\} > 0$ for all $z \in U$. The definition of the function $p(z)$ defined in (2.2) gives us:

$$\Re e\left(\frac{I^n[f](z)}{I^{n+1}[f](z)}\right) < \lambda \quad (z \in U),$$

or, equivalently,

$$\Re e\left(\frac{I^n[f](z)}{I^{n+1}[f](z)}-2\right)<-\frac{\gamma}{p}\quad (z\in U),$$

where γ given by (2.1). This completes the desired proof.

Since $\gamma>\alpha$ in Theorem 1, the following corollary is clear.

Corollary 1. *$\mathcal{M}_n(p;\alpha)\subset\mathcal{M}_{n+1}(p;\alpha)$ for any integer n.*

Corollary 2. *By putting $p=1$ in the Corollary 1, we get the result obtained by Cho* [3].

Corollary 3. *By letting $n=\alpha=0$ in the Corollary 2, we then receive the result obtained by Bajpai* [1].

Remark 1. *The result in Corollary 2 is comparable with the result obtained by Uralegaddi and Somanatha* [6].

Theorem 2. *Let $f(z)\in\mathcal{M}_n(p,\alpha)$ and $F_p^c(z)$ defined by*

$$F_p^c(z)=\frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)\,dt\quad (c>0)\,.$$

Then, $F_p^c(z)\in\mathcal{M}_n(p,\gamma)$, where

$$\gamma:=\frac{2p+2\alpha+2c+1-\sqrt{(6p-2\alpha-2c-1)^2+8p\left(1+4c-4p+2\alpha-\frac{2\alpha c}{p}\right)}}{4}. \tag{2.7}$$

Proof. Let $f(z)\in\mathcal{M}_n(p,\alpha)$. We then have

$$\Re e\left(\frac{I^{n-1}[f](z)}{I^n[f](z)}-2\right)<-\frac{\alpha}{p}\quad (0\leq\alpha<p)\,. \tag{2.8}$$

In view of the related operator and the definition of $F_p^c(z)$, we obtain

$$z\left(I^n[F_p^c](z)\right)'=p\left(I^{n-1}[F_p^c](z)-2I^n[F_p^c](z)\right), \tag{2.9}$$

and also

$$z\left(I^n[F_p^c](z)\right)'=cI^n[f](z)-(c+p)\,I^n[F_p^c](z). \tag{2.10}$$

By making use of the identities (2.9) and (2.10), (2.8) may be written as

$$\Re e\left(\frac{I^{n-1}[f](z)}{I^n[f](z)}-2\right)=\Re e\left(\frac{p\frac{I^{n-2}[F_p^c](z)}{I^{n-1}[F_p^c](z)}+c-p}{p+(c-p)\,\frac{I^n[F_p^c](z)}{I^{n-1}[F_p^c](z)}}-2\right)<-\frac{\alpha}{p}. \tag{2.11}$$

Now define a function $p(z)$ by

$$\frac{I^{n-1}[F_p^c](z)}{I^n[F_p^c](z)} = \lambda + (1-\lambda)\, p(z), \tag{2.12}$$

where

$$\kappa := 6p - 2\alpha - 2c - 1 \text{ and } \lambda := \frac{\kappa + \sqrt{\kappa^2 + 8p\left(1 + 4c - 4p + 2\alpha - \frac{2\alpha c}{p}\right)}}{4p}. \tag{2.13}$$

Clearly, $p(z)$ is analytic in U and in the form $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. By differentiating the both sides of (2.12) logarithmically and then using the identities (2.9) and (2.10), we get that

$$\frac{p\frac{I^{n-2}[F_p^c](z)}{I^{n-1}[F_p^c](z)} + c - p}{p + (c-p)\frac{I^n[F_p^c](z)}{I^{n-1}[F_p^c](z)}} = \lambda + (1-\lambda)\, p(z) + \frac{(1-\lambda)\, zp'(z)}{c - p + p\left[\lambda + (1-\lambda)\, p(z)\right]} \tag{2.14}$$

It follows from the real part of (2.14) that

$$-\Re e\left(\frac{p\frac{I^{n-2}[F_p^c](z)}{I^{n-1}[F_p^c](z)} + c - p}{p + (c-p)\frac{I^n[F_p^c](z)}{I^{n-1}[F_p^c](z)}} - 2 + \frac{\alpha}{p}\right)$$

$$= \Re e\left\{2 - \left(\frac{\alpha}{p} + \lambda\right) - (1-\lambda)\, p(z) - \frac{(1-\lambda)\, zp'(z)}{c - p + p\left(\lambda + (1-\lambda)\, p(z)\right)}\right\} > 0,$$

since $f(z) \in \mathcal{M}_n(p, \alpha)$. Let

$$\phi(u, v) := 2 - \left(\frac{\alpha}{p} + \lambda\right) - (1-\lambda)\, u - \frac{(1-\lambda)v}{c - p + p(\lambda + (1-\lambda)u)}. \tag{2.15}$$

Then the function $\phi(u, v)$ satisfies the conditions of Lemma 1, i.e.,

(i) $\phi(u, v)$ is continuous in $\mathcal{D} = \left(C - \left\{\frac{p(1-\lambda) - c}{p(1-\lambda)}\right\}\right) \times C$,

(ii) $(1, 0) \in \mathcal{D}$ and $\Re e\left(\phi(1, 0)\right) = 1 - \frac{\alpha}{p} > 0$, since $0 \leq \alpha < p$,

(iii) for all (iu_2, v_1) such that $v_1 \leq -\frac{1+u_2^2}{2}$,

$$\Re e\left\{\phi(iu_2, v_1)\right\}$$

$$= \Re e\left\{2 - \left(\frac{\alpha}{p} + \lambda\right) - (1-\lambda)\, iu_2 - \frac{(1-\lambda)\, v_1}{c - p + p\left(\lambda + (1-\lambda)\, iu_2\right)}\right\}$$

$$= 2 - \left(\frac{\alpha}{p} + \lambda\right) - \frac{(c-p+p\lambda)(1-\lambda)\,v_1}{(c-p+p\lambda)^2 + p^2(1-\lambda)^2 u_2^2}$$
$$\leq 2 - \left(\frac{\alpha}{p} + \lambda\right) + \frac{(c-p+p\lambda)(1-\lambda)\left(1+u_2^2\right)}{2(c-p+p\lambda)^2 + p^2(1-\lambda)^2 u_2^2} \leq 0,$$

where λ is given by (2.13). Therefore $\Re e\{p(z)\} > 0 \ (z \in U)$. The definition of $p(z)$ defined (2.12) yields:

$$\Re e\left(\frac{I^{n-1}[F_p^c](z)}{I^n[F_p^c](z)}\right) < \lambda \quad (z \in U),$$

or, equivalently,

$$\Re e\left(\frac{I^{n-1}[F_p^c](z)}{I^n[F_p^c](z)} - 2\right) < -\frac{\gamma}{p} \quad (z \in U),$$

where γ given by (2.7). That is that $F_p^c(z) \in \mathcal{M}_n(p,\gamma)$.

Corollary 4. *Since* $\gamma > \alpha$, it is clear that $F_p^c(z) \in \mathcal{M}_n(p,\alpha)$.

Remark 2. *By taking* $p = 1$, *we have the result given by Cho* [3].

Theorem 3. *If* $f(z) \in \mathcal{M}_n(p,\alpha)$, *then*

$$\Re e\left\{\left[z^p I^n[f](z)\right]^{\frac{1}{2\beta(p-\alpha)}}\right\} > 2^{-\frac{1}{\beta}} \qquad (\beta \geq 1; 0 \leq \alpha < p; z \in U).$$

Proof. Let $f(z) \in \mathcal{M}_n(p,\alpha)$. Then we obtain from (1.3) and (2.4)

$$\Re e\left(\frac{z[I^n[f](z)]'}{I^n[f](z)}\right) < -\alpha.$$

That is, that

$$\frac{1}{2(p-\alpha)}\left(\frac{z\left(I^n[f](z)\right)'}{I^n[f](z)} + p\right) \prec \frac{z}{1+z}. \tag{2.16}$$

Now take

$$p(z) = \left[z^p I^n[f](z)\right]^{\frac{1}{2(p-\alpha)}}.$$

Then (2.16) can be written as

$$z\Big[log(p(z))\Big]' \prec z\left[log\left(\frac{1}{1+z}\right)\right]' \tag{2.17}$$

By using the well-known assertion obtained by Sulfridge [7] for (2.17), we then receive that

$$p(z) = [z^p I^n f(z)]^{\frac{1}{2(p-\alpha)}} \prec \frac{1}{1+z}.$$

Therefore

$$p(z) = \left[z^p I^n [f](z)\right]^{\frac{1}{2\beta(p-\alpha)}} = \left(\frac{1}{1+w(z)}\right)^{\frac{1}{\beta}},$$

where $w(z)$ is analytic in U, $w(z) = 0$ and $|w(z)| < 1$ for $z \in U$.
Since

$$\left[\Re e\{t\} > 0 \text{ and } \beta \geq 1\right] \Rightarrow \Re e\left\{t^{\frac{1}{\beta}}\right\} \geq (\Re e\{t\})^{\frac{1}{\beta}},$$

we then get

$$\Re e\left\{[z^p I^n [f](z)]^{\frac{1}{2\beta(p-\alpha)}}\right\} \geq \left(\Re e[1+w(z)]^{-1}\right)^{\frac{1}{\beta}} > 2^{-\frac{1}{\beta}} \quad (z \in U).$$

Theorem 4. *If* $f(z) \in \mathcal{M}_n(p, \alpha)$, *then*

$$\Re e\left\{\left[z^p I^n [F_p^c](z)\right]^{\frac{1}{2\beta(p-\alpha)}}\right\} > 2^{-\frac{1}{\beta}} \quad (\beta \geq 1; 0 \leq \alpha < p; z \in U).$$

Proof. Let $f(z) \in \mathcal{M}_n(p, \alpha)$. Then from Theorem 2, we have $F_p^c(z) \in \mathcal{M}_n(p, \alpha)$. By using of the identity (2.9) and the same method used in Theorem 3, one easily arrived at the desired result as similar result as in the Theorem 2. Hence

$$\Re e\left\{\left[z^p I^n [F_p^c](z)\right]^{\frac{1}{2\beta(p-\alpha)}}\right\} > 2^{-\frac{1}{\beta}} \quad (z \in U).$$

Finally, by choosing suitable real numbers for parameter(s) in the related theorems, several results which will be important or interesting for both analytic and geometric functions theory can be easily revealed. Because it is not to list all of them, they are omited in this investigation.

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