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PERIODIC SOLUTION OF A DISCRETE LOTKA-VOLTERRA SYSTEM WITH DELAY AND DIFFUSION

Zhi-jie Nan, Wei-jun Chen, Ming-xing Luo and Lin Li

Abstract. In this paper, we investigate the existence of periodic solution for a discrete Lotka-Volterra system on time scale. The result is also a generalization of the system in [9].

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1. Introduction

Let $\mathbb{T}$ be a time scale i.e., $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. Various mathematical models have been proposed in the study of population dynamics, one of the famous models for population dynamics is the following Lotka-Volterra system

$$\begin{cases} x_1'(t) = x_1(t)(r_{11}(t) + r_{12}(t)x_1(t) + r_{13}(t)x_2(t)) \\ x_2'(t) = x_2(t)(r_{21}(t) + r_{22}(t)x_1(t) + r_{23}(t)x_2(t)) \end{cases} \tag{1}$$

When $r_{13}(t)r_{22}(t) < 0$, system (1) is a Predator-Prey system, while when $r_{13}(t) < 0, r_{22}(t) < 0$, it is a competition system, and when $r_{13}(t) > 0, r_{22}(t) > 0$, the one is a reciprocal system. In order to reflect the seasonal fluctuations, some researchers considered the Lotka-Volterra system with periodic coefficients [15]. Moreover, some authors took into account the time delay effect, density regulation and diffusion between patches in many ecological systems [13, 19, 20].

After the theory of time scales was introduced by Stefan Hilger in 1988, there are some results related to the study of dynamic equations on time scale [1, 3, 4, 12, 16]. In this paper, to consider both the periodic variations of the environment and the density regulation of the predators with delay effect and diffusion between patches, we investigate the existence of positive periodic solutions of the following periodic competition Lotka-Volterra dynamic system with time delay and diffusion on time scale

$$\begin{cases} x_1(n+1) &=& x_1(n)\exp[r_1(n) - f_1(n)x_1(n) - \frac{g_1(n)y_1(n-\tau)}{x_1(n-\tau)+\beta_1(n)y_1(n-\tau)} \\ & & +\frac{p_1(n)x_2(n)-p_1(n)x_1(n)}{x_1(n)}] \\ x_2(n+1) &=& x_2(n)\exp[r_2(n) - f_2(n)x_2(n) + \frac{p_2(n)x_1(n)-p_2(n)x_2(n)}{x_2(n)}] \\ y_1(n+1) &=& y_1(n)\exp[r_3(n) - f_3(n)y_1(n) - \frac{g_2(n)x_1(n-\tau)}{x_1(n-\tau)+\beta_1(n)y_1(n-\tau)} \\ & & -\frac{h_1(n)y_2(n)}{y_1(n)+\beta_2(n)y_2(n)}] \\ y_2(n+1) &=& y_2(n)\exp[r_4(n) - f_4(n)y_2(n) + \frac{h_2(n)y_1(n-\tau)}{y_1(n-\tau)+\beta_2(n)y_2(n-\tau)}] \end{cases} \tag{2}$$

where $y_1(n)$ and $x_1(n)$ represent the population density of species y and x in patch 1, while $y_2(n)$ and $x_2(n)$ represent the density of species y and x in patch 2 respectively. Species x and y can diffuse between two patches. The species y is confined to compete with species x. $\tau > 0$ is a delay due to gestation. $p_i(n)(i = 1, 2)$ are strictly positive ω-periodic functions. Denote the dispersal rate of species y in the i-th patch $(i = 1, 2)$. $r_k(n)$, $f_k(n)(k = 1, 2, 3, 4)$, and $g_j(n), h_j(n), \beta_j(n)(j = 1, 2)$ are strictly positive ω-periodic functions.

The system (2) is a generalized discrete system of

$$\begin{cases} X_1'(t) &=& X_1(t)(r_1(t) - f_1(t)X_1(t) - \frac{g_1(t)Y(t-\tau)}{X_1(t-\tau)+\beta(t)Y(t-\tau)}) \\ & & -p_1(t)X_1(t) - p_1(t)X_2(t) \\ X_2'(t) &=& X_2(t)r_2(t) - f_2(t)X_2^2(t) + p_2(t)X_1(t) - p_2(t)X_2(t) \\ Y'(t) &=& Y(t)r_3(t) - f_3(t)Y^2(t) - \frac{g_2(t)X_1(t-\tau)Y(t)}{X_1(t-\tau)+\beta(t)Y(t-\tau)} \end{cases} \tag{3}$$

with transition of variables $X_i(t) = e^{x_i(n)}, Y_i(t) = e^{y_i(n)}(i = 1, 2)$. Therefore, our aim is to find $x_i, y_i \in \mathbb{R}^+$. Recently, some properties such as the uniform persistence, global asymptotic stability and the periodicity of system (3) have discussed in [8, 17]. Also, by using various continuation theorems in coincidence degree, it is of great interest for many authors to study the existence of periodic solutions [2, 5, 6, 7, 10, 14, 17, 18]. However, they are rare results related to study dynamic equations on time scale. Actually, the discrete models are governed by difference equations, the most direct and relevant biological issues are domains of simple (global stability) and complex (chaotic) dynamics and possible data fitting to field or lab data, see [11]. In this paper, we discuss the existence of positive ω-periodic solutions of a class of nonautonomous discrete time predator-prey systems (2) on time scale by using the well-known Mawhin's theorem. The criteria established here are pretty general since they can be applied to many famous predator-prey systems of form (2).

This paper is organized as follows. In Section 2, we present some basic definitions and results of topological degree theory. Section 3 is contributed to the proof of the main result while some examples were given in the last section.

2. Preliminaries

Let $\omega > 0$. Throughout this paper, the time scale is always assumed to be ω-periodic (i.e., $n \in \mathbb{Z}$ implies $n \pm \omega \in \mathbb{Z}$) and unbounded. We denote $\mathbb{I}_\omega = [0, \omega - 1] \cap \mathbb{Z}$.

Now, we introduce some results related to the topology degree theory which are important in our arguments.

Let X and Y be two Banach spaces. Consider an operator equation

$$\mathcal{L}x = \lambda \mathcal{N}x, \quad \lambda \in (0,1), \quad \forall x \in X, \tag{4}$$

where $\mathcal{L} : \mathrm{Dom}\mathcal{L} \cap X \to Y$ is a linear operator, $\mathcal{N} : X \to Y$ is a continuous operator and λ is a parameter. Let P and Q be two projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}P = \ker\mathcal{L}$ and $\mathrm{Im}\mathcal{L} = \ker Q = \mathrm{Im}(I - Q)$. It follows that $\mathcal{L}|_{\mathrm{Dom}\mathcal{L} \cap \ker P} : (I - P)X \to \mathrm{Im}\mathcal{L}$ is invertible and thus we denote the inverse of this map by Φ. If Ω is a bounded open subset of X, the mapping $\mathcal{N}$ is called *$\mathcal{L}$-compact* on $\overline{\Omega}$ if $Q \circ \mathcal{N}(\overline{\Omega})$ is bounded and $\Phi \circ (I - Q) \circ \mathcal{N} : \overline{\Omega} \to X$ is compact. Moreover, there exists an isomorphism $\Psi : \mathrm{Im}Q \to \ker\mathcal{L}$, since $\mathrm{Im}Q$ is isomorphic to $\ker\mathcal{L}$.

Notice that operator $\mathcal{L}$ is called a *Fredholm operator* of index zero if $\dim(\ker\mathcal{L}) = \mathrm{codim}(\mathrm{Im}\mathcal{L}) < \infty$ and $\mathrm{Im}\mathcal{L}$ is closed in Y.

Lemma 1 *[8] (Gains and Mawhin's theorem) Let $\mathcal{L}$ be a Fredholm mapping of index zero and let $\mathcal{N}$ be $\mathcal{L}$-compact on $\overline{\Omega}$. Suppose*

- *(c1) for each $\lambda \in (0,1)$, every solution $x \in \partial\Omega \cap \mathrm{Dom}\mathcal{L}$ of $\mathcal{L}x = \lambda\mathcal{N}(x, \lambda)$ is such that $x \notin \partial\Omega$;*
- *(c2) $Q \circ \mathcal{N}x \neq 0$ for each $x \in \partial\Omega \cap \ker\mathcal{L}$;*
- *(c3) $\deg(\Psi \circ Q \circ \mathcal{N}, \Omega \cap \ker\mathcal{L}, 0) \neq 0$.*

Then the equation $\mathcal{L}x = \mathcal{N}x$ has at least one solution lying in $\mathrm{Dom}\mathcal{L} \cap \overline{\Omega}$.

For convenience, we take the following notations, and the others are also be defined analogously.

$$\overline{f} = \frac{1}{\omega} \sum_{n \in \mathbb{I}_\omega} f(n), f^s = \min_{n \in \mathbb{I}_\omega} f(n), f^M = \max_{n \in \mathbb{I}_\omega} f(n),$$

where f is an ω-periodic function with $f(n + \omega) = f(n)$ for any $n \in \mathbb{Z}$.

3. Periodic solution

The main result about the existence of a ω-periodic solution is stated as follows.

Theorem 1 *Suppose that*

1) $\bar{r}_2 - \bar{p}_2 > 0$;

2) $\bar{r}_1 - \overline{(g_1/\beta_1)} > 0$;

3) $\bar{r}_1\beta^s - \bar{g}_1 > 0$;

4) $\frac{\bar{r}_2}{\bar{f}_4} - \frac{\bar{g}_2 C}{\bar{f}_4\beta_1^s} - \frac{\bar{h}_1}{\bar{f}_4\beta_2^s} > 0$, here $C = \max\{|\ln\frac{\bar{r}_1\beta_1^s - \bar{g}_1}{\bar{f}_1\beta_1^s}|, |\ln\frac{\bar{r}_1}{\bar{f}_1}|, |\ln\frac{\bar{r}_2}{\bar{f}_2}|\}$.

Then the difference system (2) has at least one ω-periodic solution.

By replacing variables $x_i(n), y_i(n)$ with new variables $\exp(u_i(n)), \exp(v_i(n))$ respectively (then replace u, v by x, y respectively), system (2) is rewrited in a more direct form

$$\begin{cases} x_1(n+1) - x_1(n) &= r_1(n) - p_1(n) + p_1(n)e^{x_2(n) - x_1(n)} \\ & \qquad -f_1(n)e^{x_1(n)} - \frac{g_1(n)e^{y_1(n-\tau)}}{e^{x_1(n-\tau)} + \beta_1(n)e^{y_1(n-\tau)}}] \\ x_2(n+1) - x_2(n) &= r_2(n) - f_2(n)e^{x_2(n)} - p_2(n) + p_2(n)e^{x_1(n) - x_2(n)} \\ y_1(n+1) - y_1(n) &= r_3(n) - f_3(n)e^{y_1(n)} - \frac{h_1(n)e^{y_2(n)}}{e^{y_1(n)} + \beta_2(n)e^{y_2(n)}} - \frac{g_2(n)e^{x_1(n-\tau)}}{e^{x_1(n-\tau)} + \beta_1(n)e^{y_1(n-\tau)}} \\ y_2(n+1) - y_2(n) &= r_4(n) - f_4(n)e^{y_2(n)} + \frac{h_2(n)e^{y_1(n-\tau)}}{e^{y_1(n-\tau)} + \beta_2(n)e^{y_2(n-\tau)}}. \end{cases} \tag{5}$$

Therefore, we are going to consider the ω-periodic solutions of system (5). In fact, we have the following priori bounds.

Lemma 2 *Suppose $\lambda \in (0,1)$, $\bar{r}_2 - \bar{p}_2 > 0$ and $\bar{r}_1 - \overline{(g_1/\beta_1)} > 0$. Then any solution of (5) is uniformly bounded by some constant.*

Proof. Corresponding to the operator equation (4), we have

$$\begin{cases} x_1(n+1) - x_1(n) &= \lambda[r_1(n) - p_1(n) + p_1(n)e^{x_2(n) - x_1(n)} \\ & \qquad -f_1(n)e^{x_1(n)} - \frac{g_1(n)e^{y_1(n-\tau)}}{e^{x_1(n-\tau)} + \beta_1(n)e^{y_1(n-\tau)}}] \\ x_2(n+1) - x_2(n) &= \lambda[r_2(n) - f_2(n)e^{x_2(n)} - p_2(n) + p_2(n)e^{x_1(n) - x_2(n)}] \\ y_1(n+1) - y_1(n) &= \lambda[r_3(n) - f_3(n)e^{y_1(n)} - \frac{h_1(n)e^{y_2(n)}}{e^{y_1(n)} + \beta_2(n)e^{y_2(n)}} - \frac{g_2(n)e^{x_1(n-\tau)}}{e^{x_1(n-\tau)} + \beta_1(n)e^{y_1(n-\tau)}}] \\ y_2(n+1) - y_2(n) &= \lambda[r_4(n) - f_4(n)e^{y_2(n)} + \frac{h_2(n)e^{y_1(n-\tau)}}{e^{y_1(n-\tau)} + \beta_2(n)e^{y_2(n-\tau)}}]. \end{cases} \tag{6}$$

Define

$$\begin{aligned} \Gamma &= \{\mathbf{u} = (x_1(n), x_2(n), y_1(n), y_2(n))^T | x_i, y_i : \mathbb{Z} \mapsto \mathbb{R}^+, \\ & \qquad x_i(n+\omega) = x_i(n), y_i(n+\omega) = y_i(n)\} \end{aligned}$$

with the norm

$$\|\mathbf{u}\| = \sum_{i=1}^{2} \max_{n\in\mathbb{I}_\omega} |x_i(n)| + \sum_{i=1}^{2} \max_{n\in\mathbb{I}_\omega} |y_i(n)|.$$

Then Γ is a Banach space. Take $X = Y = \Gamma$. Assume that $\mathbf{u} \in \Gamma$ is a solution of (6) for some fixed $\lambda \in (0,1)$, we claim that there exists a constant $M_1 > 0$ such that $\|\mathbf{u}\| < M_1$.

By summing the both sides of (6) from 0 to $\omega - 1$, we obtain

$$\begin{cases} \sum_{\mathbb{I}_\omega}(x_1(n+1) - x_1(n)) &= \lambda \sum_{\mathbb{I}_\omega}[r_1(n) - p_1(n) + p_1(n)e^{x_2(n)-x_1(n)} \\ & \quad -f_1(n)e^{x_1(n)} - \frac{g_1(n)e^{y_1(n-\tau)}}{e^{x_1(n-\tau)}+\beta_1(n)e^{y_1(n-\tau)}}] \\ \sum_{\mathbb{I}_\omega}(x_2(n+1) - x_2(n)) &= \lambda \sum_{\mathbb{I}_\omega}[r_2(n) - f_2(n)e^{x_2(n)} - p_2(n) \\ & \quad +p_2(n)e^{x_1(n)-x_2(n)}] \\ \sum_{\mathbb{I}_\omega}(y_1(n+1) - y_1(n)) &= \lambda \sum_{\mathbb{I}_\omega}[r_3(n) - f_3(n)e^{y_1(n)} - \frac{h_1(n)e^{y_2(n)}}{e^{y_1(n)}+\beta_2(n)e^{y_2(n)}} \\ & \quad -\frac{g_2(n)e^{x_1(n-\tau)}}{e^{x_1(n-\tau)}+\beta_1(n)e^{y_1(n-\tau)}}] \\ \sum_{\mathbb{I}_\omega}(y_2(n+1) - y_2(n)) &= \lambda \sum_{\mathbb{I}_\omega}[r_4(n) - f_4(n)e^{y_2(n)} + \frac{h_2(n)e^{y_1(n-\tau)}}{e^{y_1(n-\tau)}+\beta_2(n)e^{y_2(n-\tau)}}]. \end{cases} \quad (7)$$

Therefore, it follows that

$$\begin{cases} 0 &= \omega\bar{r}_1 - \sum_{\mathbb{I}_\omega}[f_1(n)e^{x_1(n)} + p_1(n) - p_1(n)e^{x_2(n)-x_1(n)} + \frac{g_1(n)e^{y_1(n-\tau)}}{e^{x_1(n-\tau)}+\beta_1(n)e^{y_1(n-\tau)}}] \\ 0 &= \omega\bar{r}_2 - \sum_{\mathbb{I}_\omega}[f_2(n)e^{x_2(n)} - p_2(n) - p_2(n)e^{x_1(n)-x_2(n)}] \\ 0 &= \omega\bar{r}_3 - \sum_{\mathbb{I}_\omega}[f_3(n)e^{y_1(n)} + \frac{g_2(n)e^{x_1(n-\tau)}}{e^{x_1(n-\tau)}+\beta_1(n)e^{y_1(n-\tau)}} + \frac{h_1(n)e^{y_2(n)}}{e^{y_1(n)}+\beta_2(n)e^{y_2(n)}}] \\ 0 &= \omega\bar{r}_4 - \sum_{\mathbb{I}_\omega}[f_4(n)e^{y_2(n)} - \frac{h_2(n)e^{y_1(n-\tau)}}{e^{y_1(n-\tau)}+\beta_2(n)e^{y_2(n-\tau)}}], \end{cases} \quad (8)$$

since

$$\begin{array}{ll} \sum_{\mathbb{I}_\omega}(x_1(n+1) - x_1(n)) = 0, & \sum_{\mathbb{I}_\omega}(x_2(n+1) - x_2(n)) = 0, \\ \sum_{\mathbb{I}_\omega}(y_1(n+1) - y_1(n)) = 0, & \sum_{\mathbb{I}_\omega}(y_2(n+1) - y_2(n)) = 0, \end{array}$$

for $x_i(n), y_i(n) \in \Gamma$. Moreover, combine (7) with (8), we get

$$\begin{aligned}
\sum_{\mathbb{I}_\omega} |x_1(n+1) - x_1(n)| &\leq \lambda \sum_{\mathbb{I}_\omega} [r_1(n) + f_1(n)e^{x_1(n)} + p_1(n)e^{x_2(n)-x_1(n)} \\
&\quad + p_1(n) + \frac{g_1(n)e^{y_1(n-\tau)}}{e^{x_1(n-\tau)} + \beta_1(n)e^{y_1(n-\tau)}}] \\
&\leq 2\omega\bar{r}_1 + 2\sum_{\mathbb{I}_\omega} p_1(n)e^{x_2(n)}, &(9)\\
\sum_{\mathbb{I}_\omega} |x_2(n+1) - x_2(n)| &\leq \lambda \sum_{\mathbb{I}_\omega} [r_2(n) + f_2(n)e^{x_2(n)} + p_2(n) + p_2(n)e^{x_1(n)-x_2(n)}] \\
&\leq 2\omega\bar{r}_2, &(10)\\
\sum_{\mathbb{I}_\omega} |y_1(n+1) - y_1(n)| &\leq \lambda \sum_{\mathbb{I}_\omega} [r_3(n) + f_3(n)e^{y_1(n)} + \frac{h_1(n)e^{y_2(n)}}{e^{y_1(n)} + \beta_2(n)e^{y_2(n)}} \\
&\quad + \frac{g_2(n)e^{x_1(n-\tau)}}{e^{x_1(n-\tau)} + \beta_1(n)e^{y_1(n-\tau)}}] \\
&\leq 2\omega\bar{r}_3, &(11)\\
\sum_{\mathbb{I}_\omega} |y_2(n+1) - y_2(n)| &\leq \lambda \sum_{\mathbb{I}_\omega} [r_4(n) + f_4(n)e^{y_2(n)} + \frac{h_2(n)e^{y_1(n-\tau)}}{e^{y_1(n-\tau)} + \beta_2(n)e^{y_2(n-\tau)}}] \\
&\leq 2\omega\bar{r}_2 + 2\omega\bar{h}_2. &(12)
\end{aligned}$$

Because $\mathbf{x} \in \Gamma$, there exist $n_i, \kappa_i \in \mathbb{I}_\omega (i = 1, 2, 3, 4)$ such that $x_i(n_i) = \min_{n\in\mathbb{I}_\omega} x_i(n)$, $x_i(\kappa_i) = \max_{n\in\mathbb{I}_\omega} x_i(n)$, $y_i(n_{i+2}) = \min_{n\in\mathbb{I}_\omega} y_i(n)$ and $y_i(\kappa_{i+2}) = \max_{n\in\mathbb{I}_\omega} y_i(n) (i = 1, 2)$. Finally, we complete the estimation of solution with four steps.

Step one. From the second equation of (8), we obtain

$$\omega\bar{f}_2 e^{x_2(n_2)} \leq \sum_{\mathbb{I}_\omega} f_2(n)e^{x_2(n)} \leq \omega\bar{r}_2 - \omega\bar{p}_2,$$

which yields to $x_2(n_2) \leq \ln\frac{\bar{r}_2 - \bar{p}_2}{\bar{f}_2}$ according to assumption that $\bar{r}_2 - \bar{p}_2 > 0$. Furthermore, From [6], we know that

$$x_2(n) \leq x_2(n_2) + \sum_{\mathbb{I}_\omega} |x_2(n+1) - x_2(n)|,$$

and thus with (10) we have

$$x_2(n) \leq 2\omega\bar{r}_2 + \ln\frac{\bar{r}_2 - \bar{p}_2}{\bar{f}_2}. \tag{13}$$

On the other hand, the second equation of (8) also implies that

$$\omega \bar{f}_2 e^{x_2(\kappa_2)} \geq \sum_{\mathbb{I}_\omega} f_2(n) e^{x_2(n)} \geq \omega \bar{r}_2 - \omega \bar{p}_2,$$

which follows that $x_2(\kappa_2) \geq \ln \frac{\bar{r}_2 - \bar{p}_2}{\bar{f}_2}$. Moreover, from (10) that

$$x_2(n) \geq x_2(\kappa_2) - \sum_{\mathbb{I}_\omega} |x_2(n+1) - x_2(n)| \geq \ln \frac{\bar{r}_2 - \bar{p}_2}{\bar{f}_2} - 2\omega \bar{r}_2. \tag{14}$$

So

$$|x_2(n)| \leq C_2 := \max\{|\ln \frac{\bar{r}_2 - \bar{p}_2}{\bar{f}_2} \pm 2\omega \bar{r}_2|\}, \tag{15}$$

by (13) and (14).

Step two. From the first equation of (8), we get

$$\begin{aligned} \omega \bar{f}_1 e^{x_1(n_1)} &\leq \sum_{\mathbb{I}_\omega} f_1(n) e^{x_1(n)} \leq \omega \bar{r}_1 + \omega \bar{p}_1 + \sum_{\mathbb{I}_\omega} p_1(n) e^{x_2(n)} \\ &\leq \omega \bar{r}_1 + \omega \bar{p}_1 + \omega \bar{p}_1 e^{x_2(\kappa_2)} \leq \omega \bar{r}_1 + \omega \bar{p}_1 + \omega \bar{p}_1 e^{C_1}, \end{aligned}$$

which implies

$$x_1(n_1) \leq \ln((\bar{r}_1 + \bar{p}_1 + \bar{p}_1 e^{C_1}) \bar{f}_1^{-1}).$$

Thus

$$x_1(n) \leq x_1(n_1) + \sum_{\mathbb{I}_\omega} |x_1(n+1) - x_1(n)| \quad \leq 2\omega \bar{r}_1 + 2\bar{p}_1 \ln((\bar{r}_1 + \bar{p}_1 + \bar{p}_1 e^{C_1}) \bar{f}_1^{-1}) \tag{16}$$

according to (9). From the first equation of (8), we also obtain that

$$\omega \bar{f}_1 e^{x_1(\kappa_1)} \geq \sum_{\mathbb{I}_\omega} f_1(n) e^{x_1(n)} \geq \omega \bar{r}_1 - \omega \bar{g}_1,$$

which yields to

$$x_1(\kappa_1) \geq \ln \frac{\bar{r}_1 - \overline{(g_1/\beta_1)}}{\bar{f}_1} \tag{17}$$

according to assumption that $\bar{r}_1 - \overline{(g_1/\beta_1)} > 0$. Noticing that (9) gives

$$x_1(n) \geq x_1(\kappa_1) - \sum_{\mathbb{I}_\omega} |x_1(n+1) - x_1(n)| \geq \ln \frac{\bar{r}_1 - \overline{(g_1/\beta_1)}}{\bar{f}_1} - 2\omega \bar{r}_2 - 2\omega \bar{p}_1 e^{C_1},$$

and thus

$$|x_1(n)| \leq C_1 \quad := \quad \max\{|2\omega\bar{r}_1 + 2\bar{p}_1 \ln((\bar{r}_1 + \bar{p}_1 + \bar{p}_1 e^{C_1})\bar{f}_1^{-1})|, \\ |\ln \frac{\bar{r}_1 - \overline{(g_1/\beta_1)}}{\bar{f}_1} - 2\omega\bar{r}_2 - 2\omega\bar{p}_1 e^{C_1}|\}. \tag{18}$$

Step three. By the third equation of (8) we have

$$\omega\bar{f}_3 e^{y_1(n_3)} \leq \sum_{\mathbb{I}_\omega} f_3(n) e^{y_1(n)} \leq \omega\bar{r}_3,$$

which implies that $y_1(n_3) \leq \ln \frac{\bar{r}_3}{\bar{f}_3}$. Thus

$$y_1(n) \leq |y_1(n_3)| + \sum_{\mathbb{I}_\omega} |y_1(n+1) - y_1(n)| \leq 2\omega\bar{r}_3 + |\ln \frac{\bar{r}_3}{\bar{f}_3}| \tag{19}$$

from (11). Also from the third equation of (8), we obtain

$$\omega\bar{f}_3 e^{y_1(\kappa_3)} \geq \sum_{\mathbb{I}_\omega} f_3(n) e^{y_1(n)} \geq \omega(\bar{r}_3 + \bar{g}_2 + \overline{h_1/\beta_2}),$$

then

$$y_1(\kappa_3) \geq \ln \frac{\bar{r}_3 + \bar{g}_2 + \overline{h_1/\beta_2}}{\bar{f}_3}. \tag{20}$$

Moreover, it follows by (11) that

$$y_1(n) \geq y_1(\kappa_3) - \sum_{\mathbb{I}_\omega} |y_1(n+1) - y_1(n)| \geq \ln \frac{\bar{r}_3 + \bar{g}_2 + \overline{h_1/\beta_2}}{\bar{f}_3} - 2\omega\bar{r}_3. \tag{21}$$

Therefore,

$$|y_1(n)| \leq C_3 := \max\{2\omega\bar{r}_3 + |\ln \frac{\bar{r}_3}{\bar{f}_3}|, \ln \frac{\bar{r}_3 + \bar{g}_2 + \overline{h_1/\beta_2}}{\bar{f}_3} - 2\omega\bar{r}_3\}. \tag{22}$$

Step four. From the fourth equation of (8), we have

$$\omega\bar{f}_4 e^{y_2(n_4)} \leq \sum_{\mathbb{I}_\omega} f_4(n) e^{y_2(n)} \leq \omega\bar{r}_4 + \omega\bar{h}_2$$

which implies $y_2(n_4) \leq \ln \frac{\bar{r}_4 + \bar{h}_2}{\bar{f}_4}$. Thus

$$y_2(n) \leq y_2(n_4) + \sum_{\mathbb{I}_\omega} |y_2(n+1) - y_2(n)| \leq 2\omega\bar{r}_4 + \omega\bar{h}_2 + |\ln \frac{\bar{r}_4 + \bar{h}_2}{\bar{f}_4}| \tag{23}$$

from (12). Also from the fourth equation of (8) we obtain

$$\omega\bar{f}_4 e^{y_2(\kappa_4)} \geq \sum_{\mathbb{I}_\omega} f_4(n)e^{y_2(n)} \geq \omega\bar{r}_4,$$

and thus $y_2(\kappa_4) \geq \ln\frac{\bar{r}_4}{\bar{f}_4}$. Then according to (12), we have

$$y_2(n) \geq y_2(\kappa_4) - \sum_{\mathbb{I}_\omega}|y_2(n+1)-y_2(n)| \geq \ln\frac{\bar{r}_4}{\bar{f}_4} - 2\omega\bar{r}_4 - 2\omega\bar{h}_2, \tag{24}$$

which follows from (23) that

$$|y_2(n)| \leq C_4 := \max\{2\omega\bar{r}_4 + \omega\bar{h}_2 + |\ln\frac{\bar{r}_4+\bar{h}_2}{\bar{f}_4}|, \ln\frac{\bar{r}_4}{\bar{f}_4} - 2\omega\bar{r}_4 - 2\omega\bar{h}_2\}. \tag{25}$$

Combining (15), (18), (22) and (25), we see that

$$|x_1(n)| + |x_2(n)| + |y_1(n)| + |y_2(n)| \leq M_1 \tag{26}$$

with $M_1 = C_1 + C_2 + C_3 + C_4 + 1$. Obviously, M_1 is independent of $\lambda \in (0,1)$. □

Now consider the algebraic system

$$\begin{cases} 0 = \bar{r}_1 - \bar{f}_1 e^{v_1} + \mu(-\bar{p}_1 + \bar{p}_1 e^{v_2 - v_1}) - \frac{\mu}{\omega}\sum_{\mathbb{I}_\omega}\frac{g_1(n)e^{v_3}}{e^{v_1}+\beta_1(n)e^{v_3}} \\ 0 = \bar{r}_2 - \bar{f}_2 e^{v_2} + \mu(-\bar{p}_2 + \bar{p}_2 e^{v_1 - v_2}) \\ 0 = \bar{r}_3 - \bar{f}_3 e^{v_3} - \frac{\mu}{\omega}\sum_{\mathbb{I}_\omega}\frac{g_2(n)e^{v_1}}{e^{v_1}+\beta_1(n)e^{v_3}} - \frac{\mu}{\omega}\sum_{\mathbb{I}_\omega}\frac{h_1(n)e^{v_4}}{e^{v_3}+\beta_2(n)e^{v_4}} \\ 0 = \bar{r}_4 - \bar{f}_4 e^{v_4} + \frac{\mu}{\omega}\sum_{\mathbb{I}_\omega}\frac{h_2(n)e^{v_3}}{e^{v_3}+\beta_2(n)e^{v_4}}. \end{cases} \tag{27}$$

We have the following useful lemma.

Lemma 3 *Suppose* $\mu \in (0,1)$, $\bar{r}_1\beta^s - \bar{g}_1 > 0$ *and* $\frac{\bar{r}_2}{\bar{f}_4} - \frac{\bar{g}_2 C'}{\bar{f}_4\beta_1^s} - \frac{\bar{h}_1}{\bar{f}_4\beta_2^s} > 0$ *with* $C' = \max\{|\ln\frac{\bar{r}_1\beta_1^s - \bar{g}_1}{\bar{f}_1\beta_1^s}|, \ |\ln\frac{\bar{r}_1}{\bar{f}_1}|, |\ln\frac{\bar{r}_2}{\bar{f}_2}|\}$. *Then any solution of algebraic system (27) is bounded by some constant.*

Proof. When $v_2 \leq v_1$, from the first two equations of (27), we obtain

$$\begin{aligned} \bar{f}_1 e^{v_1} &= \bar{r}_1 + \mu(-\bar{p}_1 + \bar{p}_1 e^{v_2 - v_1}) - \frac{\mu e^{v_3}}{\omega}\sum_{\mathbb{I}_\omega}\frac{g_1(n)}{e^{v_1}+\beta_1(n)e^{v_3}} \leq \bar{r}_1, \\ \bar{f}_2 e^{v_2} &= \bar{r}_2 + \mu(-\bar{p}_2 + \bar{p}_2 e^{v_1 - v_2}) \geq \bar{r}_2, \end{aligned}$$

which implies that

$$\ln\frac{\bar{r}_2}{\bar{f}_2} \leq v_2 \leq v_1 \leq \ln\frac{\bar{r}_1}{\bar{f}_1}. \tag{28}$$

Analogously, when $v_1 < v_2$, we have

$$\begin{aligned}\overline{f}_1 e^{v_1} &= \bar{r}_1 + \mu(-\bar{p}_1 + \bar{p}_1 e^{v_2 - v_1}) - \frac{\mu e^{v_3}}{\omega}\sum_{\mathbb{I}_\omega}\frac{g_1(n)}{e^{v_1} + \beta_1(n)e^{v_3}} \geq \bar{r}_1 - \frac{\bar{g}_1}{\beta_1^s},\\ \overline{f}_2 e^{v_2} &= \bar{r}_2 + \mu(-\bar{p}_2 + \bar{p}_2 e^{v_1 - v_2}) \leq \bar{r}_2,\end{aligned}$$

and thus

$$\ln\frac{\bar{r}_1\beta_1^s - \bar{g}_1}{\overline{f}_1\beta_1^s} \leq v_1 < v_2 \leq \ln\frac{\overline{r}_2}{\overline{f}_2} \tag{29}$$

by the assumption that $\bar{r}_1\beta^s - g_1^M > 0$.

Then, by setting

$$C_5 := \max\{|\ln\frac{\bar{r}_1\beta_1^s - \bar{g}_1}{\overline{f}_1\beta_1^s}|, |\ln\frac{\overline{r}_1}{\overline{f}_1}|, |\ln\frac{\overline{r}_2}{\overline{f}_2}|\}$$

we have $|v_1| + |v_2| \leq 2C_5$.

On the other hand, with the similar method, from the last two equations of (27) we obtain that

$$\ln(\frac{\overline{r}_2}{\overline{f}_4} - \frac{\bar{g}_2 C_5}{\overline{f}_4\beta_1^s} - \frac{\bar{h}_1}{\overline{f}_4\beta_2^s}) \leq v_3 \leq \ln\frac{\overline{r}_3}{\overline{f}_3}, \tag{30}$$

$$\ln\frac{\overline{r}_4}{\overline{f}_4} \leq v_4 \leq \ln\frac{\overline{r}_4 + \bar{h}_2}{\overline{f}_4}, \tag{31}$$

it follows that $|v_3| + |v_4| \leq 2C_6$ with

$$C_6 := \max\{|\ln(\frac{\overline{r}_2}{\overline{f}_4} - \frac{\bar{g}_2 C_5}{\overline{f}_4\beta_1^s} - \frac{\bar{h}_1}{\overline{f}_4\beta_2^s})|, |\ln\frac{\overline{r}_3}{\overline{f}_3}|, |\ln\frac{\overline{r}_4 + \bar{h}_2}{\overline{f}_4}|\}.$$

Therefore, $\|\mathbf{v}\| \leq 2C_5 + 2C_6$. □

With the preparations above, we complete the proof of the main result as follows.

Proof of Theorem 1. Let $M_2 := 2C_5 + 2C_6 + 1$. From Lemma 3, any solution $\mathbf{v}$ of (27) satisfies that $\|\mathbf{v}\| < M_2$. Take $C := M_1 + M_2$, and define $\Omega = \{\mathbf{u} \in \Gamma : \|\mathbf{u}\| < C\}$. Due to Lemma 2 and 3, we know that condition *(c1)* in Lemma 1 is holding.

Let

$$\mathcal{L} : \mathrm{Dom}\mathcal{L} \cap \Gamma \to \Gamma$$

$$\mathcal{L}\mathbf{u}(n) = \begin{pmatrix} x_1(n+1) - x_1(n) \\ x_2(n+1) - x_2(n) \\ y_1(n+1) - y_1(n) \\ y_2(n+1) - y_2(n) \end{pmatrix}$$

where $\mathrm{Dom}\mathcal{L} = \{\mathbf{u} := (x_1(n), x_2(n), y_1(n), y_2(n))^T, x_i, y_i : \mathbb{Z} \mapsto \mathbb{R}\}$.

$$\mathcal{N} : \Gamma \to \Gamma$$
$$\mathcal{N}\mathbf{u}(n) = (N_1(n), N_2(n), N_3(n), N_4(n))^T,$$

$$\begin{aligned}
N_1(n) &= r_1(n) - p_1(n) - f_1(n)e^{x_1(n)} + p_1(n)e^{x_2(n)-x_1(n)} - \tfrac{g_1(n)e^{y_1(n-\tau)}}{e^{x_1(n-\tau)}+\beta_1(n)e^{y_1(n-\tau)}},\\
N_2(n) &= r_2(n) - f_2(n)e^{x_2(n)} - p_2(n) + p_2(n)e^{x_1(n)-x_2(n)},\\
N_3(n) &= r_3(n) - f_3(n)e^{y_1(n)} - \tfrac{h_1(n)e^{y_2(n)}}{e^{y_1(n)}+\beta_2(n)e^{y_2(n)}} - \tfrac{g_2(n)e^{x_1(n-\tau)}}{e^{x_1(n-\tau)}+\beta_1(n)e^{y_1(n-\tau)}},\\
N_4(n) &= r_4(n) - f_4(n)e^{y_2(n)} + \tfrac{h_2(n)e^{y_1(n-\tau)}}{e^{y_1(n-\tau)}+\beta_2(n)e^{y_2(n-\tau)}}.
\end{aligned}$$

With the definitions above, we obtain that $\mathcal{L}\mathbf{u} = \mathcal{N}\mathbf{u}$ for $\mathbf{u} \in \mathrm{Dom}\mathcal{L} \cap \Gamma$ with $\mathrm{Im}\mathcal{L} = \{\mathbf{u} \in \Gamma : \sum_{\mathbb{I}_\omega} x_i(n) = 0, \sum_{\mathbb{I}_\omega} y_i(n) = 0, n \in \mathbb{Z}, i = 1, 2\}$ and $\ker \mathcal{L} = \mathbb{R}^4$ which is closed in Γ, and thus $\dim(\ker \mathcal{L}) = \mathrm{codim}(\mathrm{Im}\mathcal{L}) = 4$. Therefore, $\mathcal{L}$ is a Fredholm mapping of index zero. Moreover, define two projectors P, Q such that $\mathrm{Im}P = \ker \mathcal{L}$ and $\mathrm{Im}\mathcal{L} = \ker Q = \mathrm{Im}(I - Q)$, where

$$P = Q : \Gamma \to \Gamma$$
$$P\mathbf{u} = Q\mathbf{u} = (\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2)^T$$

So $\Gamma = \ker \mathcal{L} \oplus \ker P = \ker \mathcal{L} \oplus \ker Q$. Now choose Ψ as the identity isomorphism of $\mathrm{Im}Q$ to $\ker P$. Hence, the generalized inverse (to $\mathcal{L}$) exists and is given by

$$\Phi : \mathrm{Im}\mathcal{L} \to \mathrm{Dom}\mathcal{L} \cap \ker P$$
$$\Phi\mathbf{u} = (x_1, x_2, y_1, y_2)^T,$$

where $x_i = \sum_{\mathbb{I}_\omega} x_i(n) - \frac{1}{\omega}\sum_{\mathbb{I}_\omega}(\omega - n)x_i(n)$,$y_i = \sum_{\mathbb{I}_\omega} y_i(n) - \frac{1}{\omega}\sum_{\mathbb{I}_\omega}(\omega - n)y_i(n)$ for $i = 1, 2$.

Thus

$$Q \circ \mathcal{N}\mathbf{u} = (\frac{1}{\omega}\sum_{\mathbb{I}_\omega} N_1(n), \frac{1}{\omega}\sum_{\mathbb{I}_\omega} N_2(n), \frac{1}{\omega}\sum_{\mathbb{I}_\omega} N_3(n), \frac{1}{\omega}\sum_{\mathbb{I}_\omega} N_4(n))^T.$$

Consequently, $Q \circ \mathcal{N}$ and $\Phi \circ (I - Q)$ are well defined. It is easy to check by the Lebesgue convergence theorem and the Arzela-Ascoli theorem, that $\Phi \circ (I - Q)(\overline{\Omega})$ is relatively compact for any open bounded set $\Omega \subset \Gamma$. Moreover, $Q \circ \mathcal{N}(\overline{\Omega})$ is bounded. Thus, $\mathcal{N}$ is $\mathcal{L}$-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset \Gamma$. When $\mathbf{u} \in \partial\Omega \cap \mathbb{R}^4$, that $\mathbf{u}$ is a constant vector in $\mathbb{R}^4$, and $Q \circ \mathcal{N}\mathbf{u} \neq 0$ since $Q \circ \mathcal{N}\mathbf{u} = 0$ is (27) with $\mu = 1$. The condition *(c2)* in Lemma 1 is also holding.

Finally, we claim that $\deg(\Psi \circ Q \circ \mathcal{N}, \Omega, O) \neq 0$, where $O := (0,0,0,0)^T$. In fact, consider the homotopy

$$H_\mu \mathbf{v} = \mu Q \circ \mathcal{N}\mathbf{v} + (1-\mu)G\mathbf{v}, \mu \in [0,1]$$

where $G\mathbf{v} = (\overline{r}_1 - \overline{f}_1 e^{v_1}, \overline{r}_2 - \overline{f}_2 e^{v_2}, \overline{r}_3 - \overline{f}_3 e^{v_3}, \overline{r}_4 - \overline{f}_4 e^{v_4})^T$.

When $\mathbf{v} \in \Omega \cap \ker \mathcal{L} = \Omega \cap \mathbb{R}^4$, that $\mathbf{v}$ is a constant vector with $\|\mathbf{v}\| = C$. From Lemma 3, we get that $H_\mu \mathbf{v} \neq O$ on $\partial\Omega \cap \ker \mathcal{L}$. Since $\mathrm{Im}Q = \ker \mathcal{L}$ and $(v_1^*, v_2^*, v_3^*, v_4^*)^T \in \Omega \cap \ker \mathcal{L}$ is the unique solution of the algebraic equations $G\mathbf{v} = (0,0,0,0)^T$, by the homotopy invariance of Brouwer degree, we get that

$$\deg(\Psi \circ Q \circ \mathcal{N}, \partial\Omega \cap \ker \mathcal{L}, O) = \mathrm{sign}(-\overline{f}_1 \overline{f}_2 \overline{f}_3 \overline{f}_4 e^{v_1^* + v_2^* + v_3^* + v_4^*}) \neq 0.$$

Therefore, all the conditions of the Lemma 1 are fulfilled. So, difference system (5) has at least one ω-periodic solution lying in $\mathrm{Dom}\mathcal{L} \cap \overline{\Omega}$ which follows the result for system (2). □

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Zhi-jie Nan
Department of Mathematics, Physics and Information Engineering,
Jiaxing University, Jiaxing 314001, Zhejiang, P. R. China
email: *nanzj@mail.zjxu.edu.cn*

Wei-jun Chen
Department of Basic Course,
Jiaxing Vocational and Technical College, Jiaxing 314001, Zhejiang, P. R. China
email: *cwj0617@163.com*

Ming-xing Luo
School of Information Science and Technology,Information Security and National Computing Grid Laboratory
Southwest Jiaotong University, Chengdu 610031, SiChuan, P. R. China
email: *luomxgg@yahoo.com.cn*

Lin Li
Department of Mathematics, Physics and Information Engineering,
Jiaxing University,Jiaxing 314001, Zhejiang, P. R. China
email: *matlinl@yahoo.com.cn*