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f-ASYMPTOTICALLY LACUNARY EQUIVALENT SEQUENCES

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Abstract. This paper presents introduce some new notions, f-asymptotically equivalent of multiple L, strong f-asymptotically equivalent of multiple L, and strong f-asymptotically lacunary equivalent of multiple L which is a natural combination of the definition for asymptotically equivalent, Statistically limit, Lacunary sequence, and Modulus function. We study some connections between the asymptotically equivalent sequences and $f-$asymptotically equivalent sequences.

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1. Introduction

Let s, ℓ_∞, c denote the spaces of all real sequences, bounded, and convergent sequences,respectively. Any subspace of s is called a sequence space.
Following Freedman et al.[4], we call the sequence $\theta = (k_r)$ lacunary if it is an increasing sequence of integers such that $k_0 = 0, h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$ .The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = k_r/k_{r-1}$ .These notations will be used troughout the paper. The sequence space of lacunary strongly convergent sequences N_θ was defined by Freedman et al.[4], as follows:

$$N_\theta = \{x = (x_i) \in s : h_r^{-1} \sum_{i \in I_r} |x_i - s| = 0 \text{ for some } s\}.$$

The notion of modulus function was introduced by Nakano [11]. We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$such that (i) $f(x) = 0$ if and only if $x = 0$,(ii) $f(x + y) \leq f(x) + f(y)$ for $x, y \geq 0$, (iii) f is increasing and (iv) f is continuous from the right at 0. Hence f must be continuous everywhere on $[0, \infty)$. Connor [2], Kolk [8], Maddox [9], Öztürk and Bilgin [12], Pehlivan and Fisher [15], Ruckle [16] and others used a modulus function to construct sequence spaces.

Marouf presented definitions for asymptotically equivalent sequences and asymptotic regular matrices in [10]. Patterson extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices in[13]

Recently , the concept of asymptotically equivalent was generalized by Patterson and Savas [14], Savas and Basarir[17],and Savas and Patterson[18]. This paper presents introduce some new notions, f-asymptotically equivalent of multiple L, strong f-asymptotically equivalent of multiple L, and strong f-asymptotically lacunary equivalent of multiple L which is a natural combination of the definition for asymptotically equivalent, Statistically limit, Lacunary sequence, and Modulus function. In addition to these definitions, natural inclusion theorems shall also be presented.

2. Definitions and Notations

Now we recall some definitions of sequence spaces (see [3], [5], [9], [10], [13], and [14]).

Definition 1 *A sequence* $[x]$ *is statistically convergent to* L *if*
$\lim_n \frac{1}{n}\{\text{the number of } k \leq n : |x_k - L| \geq \varepsilon\} = 0$ *for every* $\varepsilon > 0$,
(denoted by $st - limx = L$*).*

Definition 2 *A sequence* $[x]$ *is strongly(Cesaro) summable to* L *if* $\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0$ *,(denoted by* $w - limx = L$*).*

Definition 3 *Let* f *be any modulus; the sequence* $[x]$ *is strongly (Cesaro) summable to* L *with respect to a modulus if*

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} f(|x_k - L|) = 0$$

denoted by $w_f - limx = L$.

Definition 4 *Two nonnegative sequences* $[x]$*and* $[y]$*are said to be asymptotically equivalent if* $\lim_k \frac{x_k}{y_k} = 1$,*(denoted by* $x \backsim y$*).*

Definition 5 *Two nonnegative sequences* $[x]$ *and* $[y]$ *are said to be asymptotically statistical equivalent of multiple* L *provided that for every* $\varepsilon > 0$,
$\lim_n \frac{1}{n}\left\{\text{the number of } k \leq n : \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\} = 0$,*(denoted by* $x \overset{S}{\backsim} y$*) and simply asymptotically statistical equivalent, if* $L = 1$.

Definition 6 *Two nonnegative sequences* $[x]$ *and* $[y]$ *are said to be strong asymptotically equivalent of multiple L provided that*

$\lim_n \frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_k}{y_k}-L\right| = 0$*(denoted by* $x \overset{w}{\backsim} y$*) and simply strong asymptotically equivalent, if* $L=1$.

Definition 7 *Let* θ *be a lacunary sequence; the two nonnegative sequences* $[x]$ *and* $[y]$ *are said to be asymptotically lacunary statistical equivalent of multiple L provided that for every* $\varepsilon>0$,

$\lim_r \frac{1}{h_r}\left\{\text{the number of } k\in I_r : \left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\} = 0$,*(denoted by* $x \overset{S_\theta}{\backsim} y$*) and simply asymptotically lacunary statistical equivalent, if* $L=1$.

Definition 8 *Let* θ *be a lacunary sequence; the two nonnegative sequences* $[x]$ *and* $[y]$ *are said to be strong asymptotically lacunary equivalent of multiple L provided that* $\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_k}{y_k}-L\right| = 0$ *(denoted by* $x \overset{N_\theta}{\backsim} y$*) and simply strong asymptotically lacunary equivalent, if* $L=1$.

Following these results we shall now introduce some new notions, f-asymptotically equivalent of multiple L, strong f-asymptotically equivalent of multiple L, and strong f-asymptotically lacunary equivalent of multiple L. Then we use these definitions to prove strong f -asymptotically equivalent and strong f-asymptotically lacunary equivalent analogues of Connor's results in [1] and Fridy and Orhan's results in [5,6] .

Definition 9 *Let* f *be any modulus; the two nonnegative sequences* $[x]$ *and* $[y]$*are said to be* f*-asymptotically equivalent of multiple L provided that,*

$lim_k f(\left|\frac{x_k}{y_k}-L\right|) = 0$ *(denoted by* $x \overset{f}{\backsim} y$*)and simply strong* $f-$ *asymptotically equivalent, if* $L=1$.

Since f *is continuous and* $f(x)=0$ *if and only if* $x=0$*, then*

$\lim_k f(\left|\frac{x_k}{y_k}-1\right|) = f(lim_k\left|\frac{x_k}{y_k}-1\right|) = 0$ *if and only if* $\lim_k(\frac{x_k}{y_k}-1)=0$ *i.e.*$\lim_k \frac{x_k}{y_k} = 1$. *Therefore* $x \backsim y \iff x \overset{f}{\backsim} y$.

Definition 10 *Let* f *be any modulus; the two nonnegative sequences* $[x]$ *and* $[y]$ *are said to be strong* f*-asymptotically equivalent of multiple L provided that,*

$\lim_n \frac{1}{n}\sum_{k=1}^{n} f(\left|\frac{x_k}{y_k}-L\right|) = 0$ *(denoted by* $x \overset{w_f}{\backsim} y$*) and simply strong* $f-$*asymptotically equivalent, if* $L=1$.

Definition 11 *Let* f *be any modulus and* θ *be a lacunary sequence; the two nonnegative sequences* $[x]$ *and* $[y]$ *are said to be strong* f*-asymptotically lacunary equivalent* *of multiple* L *provided that*

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} f(\left|\frac{x_k}{y_k} - L\right|) = 0$$

denoted by $x \overset{N_{\theta,f}}{\backsim} y$ *and simply strong* f*-asymptotically lacunary equivalent, if* $L = 1$.

3. Main Theorems

We start this section with the following Theorem to show that the relation between f-asymptotically equivalence and strong f-asymptotically equivalence.

Theorem 12 *Let* f *be any modulus then*
(i) if $x \overset{w}{\backsim} y$ *then* $x \overset{w_f}{\backsim} y$*,and*
(ii) if $lim_{t \longrightarrow \infty} \frac{f(t)}{t} = \beta > 0$*, then*$x \overset{w}{\backsim} y \iff x \overset{w_f}{\backsim} y$.

Proof: Part (i): Let $x \overset{w}{\backsim} y$ and $\varepsilon > 0$. We choose $0 < \delta < 1$such that $f(u) < \varepsilon$ for every u with $0 \leq u \leq \delta$. We can write

$$\tfrac{1}{n} \sum_{k=1}^{n} f(\left|\tfrac{x_k}{y_k} - L\right|) = \tfrac{1}{n} \sum_1 f(\left|\tfrac{x_k}{y_k} - L\right|) + \tfrac{1}{n} \sum_2 f(\left|\tfrac{x_k}{y_k} - L\right|)$$

where the first summation is over$\left|\frac{x_k}{y_k} - L\right| \leq \delta$ and the second summation over $\left|\frac{x_k}{y_k} - L\right| > \delta$.By definition of f, we have

$$\tfrac{1}{n} \sum_{k=1}^{n} f(\left|\tfrac{x_k}{y_k} - L\right|) \leq \varepsilon + 2f(1)\delta^{-1} \tfrac{1}{n} \sum_{k=1}^{n} \left|\tfrac{x_k}{y_k} - L\right|$$

and the result follows on applying the operator $lim_{\varepsilon \longrightarrow 0}\ lim_{n \longrightarrow \infty}$
Therefore $x \overset{w_f}{\backsim} y$.
Part (ii): If $lim_{t \longrightarrow \infty} \frac{f(t)}{t} = \beta > 0$, then $f(t) \geq \beta t$ for all $t > 0$.Let $x \overset{w_f}{\backsim} y$,clearly

$$\tfrac{1}{n} \sum_{k=1}^{n} f(\left|\tfrac{x_k}{y_k} - L\right|) \geq \tfrac{1}{n} \sum_{k=1}^{n} \beta \left|\tfrac{x_k}{y_k} - L\right| = \beta\ \tfrac{1}{n} \sum_{k=1}^{n} \left|\tfrac{x_k}{y_k} - L\right|,$$

therefore $x \overset{w}{\backsim} y$. By using (i) the proof is complete.
In the following theorem we study the relationship between the asymptotically statistical equivalence and the strong $f-$asymptotically equivalence.

Theorem 13 *Let* f *be any modulus then*
(i) if $x \overset{w_f}{\backsim} y$ *then* $x \overset{S}{\backsim} y$*, and*
(ii) if f *is bounded then* $x \overset{w_f}{\backsim} y \iff x \overset{S}{\backsim} y$.

Proof: Part (i): Take $\varepsilon > 0$ and let $\sum_1$ denote the sum over $k \leq n$ with $\left|\frac{x_k}{y_k} - L\right| \geq \varepsilon$ .Then

$$\frac{1}{n}\sum_{k=1}^{n} f(\left|\frac{x_k}{y_k} - L\right|) \geq \frac{1}{n}\sum_1 f(\left|\frac{x_k}{y_k} - L\right|)$$
$$\geq f(\varepsilon)\frac{1}{n}\left\{\text{the number of } k \leq n : \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\},$$

from which the result follows.

Part (ii): Suppose that f is bounded and $x \overset{S}{\backsim} y$. We split the sum for $k \leq n$ into sums over $\left|\frac{x_k}{y_k} - L\right| \geq \varepsilon$ and $\left|\frac{x_k}{y_k} - L\right| < \varepsilon$.Then

$$\frac{1}{n}\sum_{k=1}^{n} f(\left|\frac{x_k}{y_k} - L\right|) \leq \sup f(t)\frac{1}{n}\left\{\text{the number of } k \leq n : \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\} + f(\varepsilon)$$

and the result follows on applying the operator $lim_{\varepsilon \longrightarrow 0}\ lim_{n \longrightarrow \infty}$.

The next theorem shows the relationship between the strong $f-$ asymptotically equivalence and the strong $f-$asymptotically lacunary equivalence.

Theorem 14 *Let f be a any modulus then*

(i) if $\liminf_r q_r > 1$ *then* $x \overset{w_f}{\backsim} y$ *implies* $x \overset{N_{\theta,f}}{\backsim} y$,

(ii) if $\limsup_r q_r < \infty$ *then* $x \overset{N_{\theta,f}}{\backsim} y$ *implies* $x \overset{w_f}{\backsim} y$

(iii) if $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$,*then* $x \overset{w_f}{\backsim} y \iff x \overset{N_{\theta,f}}{\backsim} y$.

Proof: Part (i): Let $x \overset{w_f}{\backsim} y$ and $\liminf_r q_r > 1$ .There exist $\delta > 0$ such that $q_r = (k_r/k_{r-1}) \geq 1 + \delta$ for sufficiently large r. We have,for sufficiently large r, that $(h_r/k_r) \geq \delta/(1+\delta)$. Then

$$\frac{1}{k_{r-1}}\sum_{k=1}^{k_r} f(\left|\frac{x_k}{y_k} - L\right|) \geq \frac{1}{k_{r-1}}\sum_{k \in I_r} f(\left|\frac{x_k}{y_k} - L\right|)$$
$$= (h_r/k_r)\frac{1}{h_r}\sum_{k \in I_r} f(\left|\frac{x_k}{y_k} - L\right|)$$
$$\geq [\delta/(1+\delta)]\frac{1}{h_r}\sum_{k \in I_r} f(\left|\frac{x_k}{y_k} - L\right|)$$

which yields that $x \overset{N_{\theta,f}}{\backsim} y$.

Part (ii): If $\limsup_r q_r < \infty$ then there exists $K > 0$ such that $q_r < K$ for every r.

Now suppose that $x \overset{N_{\theta,f}}{\backsim} y$ and $\varepsilon > 0$. There exists m_0 such that for every $m \geq m_0$,

$$H_m = h_m^{-1}\sum_{k \in I_m} f(\left|\frac{x_k}{y_k} - L\right|) < \varepsilon$$

We can also find $R > 0$ such that $H_m \leq R$ for all m. Let n be any integer with $k_r \geq n > k_{r-1}$.Now write

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_k}{y_k}-L\right|\right) \le \frac{1}{k_{r-1}}\sum_{k=1}^{k_r} f\left(\left|\frac{x_k}{y_k}-L\right|\right)$$

$$= \frac{1}{k_{r-1}}\left(\sum_{m=1}^{m_0}+\sum_{m=m_0+1}^{k_r}\right)\sum_{k\in I_m} f\left(\left|\frac{x_k}{y_k}-L\right|\right)$$

$$= \frac{1}{k_{r-1}}\sum_{m=1}^{m_0}\sum_{k\in I_m} f\left(\left|\frac{x_k}{y_k}-L\right|\right)+\frac{1}{k_{r-1}}\sum_{m=m_0+1}^{k_r}\sum_{k\in I_m} f\left(\left|\frac{x_k}{y_k}-L\right|\right)$$

$$\le \frac{1}{k_{r-1}}\sum_{m=1}^{m_0} h_m h_m^{-1}\sum_{k\in I_m} f\left(\left|\frac{x_k}{y_k}-L\right|\right)+\varepsilon(k_r-k_{m_0})\frac{1}{k_{r-1}}$$

$$\le \frac{k_{m_0}}{k_{r-1}}\sup_{1\le k\le m_0} H_k+\varepsilon K$$

$$< R\frac{k_{m_0}}{k_{r-1}}+\varepsilon K$$

from which we deduce that $x \overset{w_f}{\backsim} y$.

Part (iii): This immediately follows from (i) and (ii).

Theorem 15 *Let f be any modulus then*

(i) if $x \overset{N_\theta}{\backsim} y$ then $x \overset{N_{\theta,f}}{\backsim} y$, and

(ii) if $\lim_{t\longrightarrow\infty}\frac{f(t)}{t}=\beta>0$, then $x \overset{N_\theta}{\backsim} y \iff x \overset{N_{\theta,f}}{\backsim} y$

ProofThe proof of Theorem 3.4. is very similar to the Theorem 3.1. Then we omit it.

Finally we give relation between asymptotically lacunary statistical equivalence and strong f-asymptotically lacunary equivalence.Also we give relation between asymptotically lacunary statistical equivalence and strong f-asymptotically equivalence.

Theorem 16 *Let f be any modulus then*

(i) if $x \overset{N_{\theta,f}}{\backsim} y$ then $x \overset{S_\theta}{\backsim} y$,

(ii) if f is bounded then $x \overset{N_{\theta,f}}{\backsim} y \iff x \overset{S_\theta}{\backsim} y$, and

(iii) if f is bounded then $x \overset{S_\theta}{\backsim} y$ implies $x \overset{w_f}{\backsim} y$.

Proof: Part (i): Take $\varepsilon>0$ and let $\sum_1$ denote the sum over $k\le n$ with $\left|\frac{x_k}{y_k}-L\right|\ge\varepsilon$ .Then

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_k}{y_k}-L\right|\right) \ge \frac{1}{h_r}\sum_1 f\left(\left|\frac{x_k}{y_k}-L\right|\right)$$

$$\ge f(\varepsilon)\frac{1}{h_r}\left\{\text{the number of } k\in I_r : \left|\frac{x_k}{y_k}-L\right|\ge\varepsilon\right\},$$

from which the result follows.

Part (ii): Suppose that f is bounded and $x \overset{S_\theta}{\backsim} y$. Since f is bounded, there exists an integer T such that $|f(x)|\le T$ for all $x\ge 0$. We see that

$\frac{1}{h_r}\sum\limits_{k\in I_r} f(\left|\frac{x_k}{y_k}-L\right|) \leq T\frac{1}{h_r}\left\{\text{the number of } k\in I_r : \left|\frac{x_k}{y_k}-L\right| \geq \varepsilon\right\} + f(\varepsilon)$
and the result follows on applying the operator $lim_{\varepsilon\longrightarrow 0}\ lim_{r\longrightarrow\infty}$.
Part (iii):Let n be any integer with $n\in I_r$, then

$$\frac{1}{n}\sum_{k=1}^{n} f(\left|\frac{x_k}{y_k}-L\right|) = \frac{1}{n}\sum_{p=1}^{r-1}\sum_{k\in I_p} f(\left|\frac{x_k}{y_k}-L\right|) + \frac{1}{n}\sum_{k=1+k_{r-1}}^{n} f(\left|\frac{x_k}{y_k}-L\right|) \quad (1)$$

Consider the first term on the right in (1);

$$\frac{1}{n}\sum_{p=1}^{r-1}\sum_{k\in I_p} f(\left|\frac{x_k}{y_k}-L\right|) \leq \frac{1}{k_{r-1}}\sum_{p=1}^{r-1}\sum_{k\in I_p} f(\left|\frac{x_k}{y_k}-L\right|)$$
$$= \frac{1}{k_{r-1}}\sum_{p=1}^{r-1} h_p(\frac{1}{h_p}\sum_{k\in I_p} f(\left|\frac{x_k}{y_k}-L\right|))$$

Since f is bounded and $x \overset{S_\theta}{\backsim} y$, it follows (ii) that

$$\frac{1}{h_p}\sum_{k\in I_p} f(\left|\frac{x_k}{y_k}-L\right|) \to 0$$

Hence

$$\frac{1}{k_{r-1}}\sum_{p=1}^{r-1} h_p(\frac{1}{h_p}\sum_{k\in I_p} f(\left|\frac{x_k}{y_k}-L\right|)) \to 0. \quad (2)$$

Consider the second term on the right in (1); Since f is bounded,there exists an integer T such that $|f(x)| \leq T$ for all $x \geq 0$. We split the sum for $k_{r-1} < k \leq n$ into sums over $\left|\frac{x_k}{y_k}-L\right| \geq \varepsilon$ and $\left|\frac{x_k}{y_k}-L\right| < \varepsilon$.Therefore we have for every $\varepsilon > 0$, that

$$\frac{1}{n}\sum_{k=1+k_{r-1}}^{n} f(\left|\frac{x_k}{y_k}-L\right|) \leq T\frac{1}{h_r}\left\{\text{number of } k\in I_r : \left|\frac{x_k}{y_k}-L\right| \geq \varepsilon\right\} + f(\varepsilon) \quad (3)$$

Since $x \overset{S_\theta}{\backsim} y$, f is continuous from the right at 0, and ε is arbitrary, the expression on left side of (3) tends to zero as $r\to\infty$. Hence (1),(2) and (3) imply that $x \overset{w_f}{\backsim} y$.

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