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SUBORDINATION RESULTS FOR FUNCTIONS OF COMPLEX ORDER DEFINED BY CONVOLUTION

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Abstract. In this paper, we drive several interesting subordination results for functions of complex order defined by convolution.

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1. Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\phi \in A$ be given by

$$\phi(z) = z + \sum_{k=2}^{\infty} c_k z^k. \tag{1.2}$$

Definition 1 *(Hadamard product or convolution). Given two functions f and ϕ in the class A, where $f(z)$ is given by* (1.1) *and $\phi(z)$ is given by* (1.2) *the Hadamard product (or convolution) $f* \phi$ of f and ϕ is defined (as usual) by*

$$(f * \phi)(z) = z + \sum_{k=2}^{\infty} a_k c_k z^k = (\phi * f)(z). \tag{1.3}$$

We also denote by K the class of functions $f(z) \in A$ that are convex in $\mathbb{U}$.

A function $f(z) \in A$ is said to be in the class of starlike functions of complex order b, denoted by $S(b)$ if

$$Re\left\{1 + \frac{1}{b}(\frac{zf'(z)}{f(z)} - 1)\right\} > 0 \quad (b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\};\ z \in \mathbb{U}). \tag{1.4}$$

A function $f(z) \in A$ is said to be in the class of convex functions of complex order b, denoted by $C(b)$ if

$$Re\left\{1+\frac{1}{b}\frac{zf''(z)}{f'(z)}\right\}>0 \quad (b\in\mathbb{C}^*;\ z\in\mathbb{U}\ ). \tag{1.5}$$

The class $S(b)$ was introduced and studied by Nasr and Aouf [12] and the class $C(b)$ was introduced and studied by Nasr and Aouf [11] and Waitrowski [16].

A function $f(z) \in A$ is said to be in $S^{\eta}(\gamma) = S((1-\gamma)\cos\eta\ e^{-i\eta})$, the class of $\eta-$spirallike functions of order γ if

$$Re\left\{e^{i\eta}\frac{zf'(z)}{f(z)}\right\}>\gamma\cos\eta \quad (|\eta|<\frac{\pi}{2};0\le\gamma<1). \tag{1.6}$$

A function $f(z) \in A$ is said to be in $C^{\eta}(\gamma) = C\ ((1-\gamma)\cos\eta\ e^{-i\eta})$, the class of $\eta-$Robertson functions of order γ if

$$Re\left\{e^{i\eta}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\gamma\cos\eta \quad (|\eta|<\frac{\pi}{2};0\le\gamma<1). \tag{1.7}$$

It follows from (1.6) and (1.7) that

$$f(z)\in C^{\eta}(\gamma)\Leftrightarrow zf'(z)\in S^{\eta}(\gamma).$$

The class $S^{\eta}(\gamma)$ was introduced and studied by Libera [8] and the class $C^{\eta}(\gamma)$ was introduced and studied by Chichra [4].

For $0 \le \lambda \le 1$, $b \in \mathbb{C}^*$, *we denote by* $M(f, g, b, \lambda)$ *the subclass of* A *consisting of functions* $f(z)$ *of the form* (1.1), *functions* $g(z)$ *given by*

$$g(z)=z+\sum_{k=2}^{\infty}b_kz^k, \tag{1.8}$$

and satisfying the analytic criterion:

$$Re\left\{1+\frac{1}{b}\left(\frac{z(f*g)'(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}-1\right)\right\}>0. \tag{1.9}$$

We note that for suitable choices of g, b and λ , we obtain the following subclasses studied by various authors.

(*i*) $M(f, \frac{z}{(1-z)}, 1-\alpha, 0) = S^*(\alpha)$ $(0 \le \alpha \le 1)$ (see Robertson [13]);
(*ii*) $M(f, \frac{z}{(1-z)^2}, 1-\alpha, 0) = C(\alpha)$ $(0 \le \alpha \le 1)$ (see Robertson [13]);
(*iii*) $M(f, \frac{z}{(1-z)}, b, 0) = S(b)$ $(b \in \mathbb{C}^*)$ (see Nasr and Aouf [12]);

(*iv*) $M(f, \frac{z}{(1-z)^2}, b, 0) = C(b)$ $(b \in \mathbb{C}^*)$ (see Waitrowski [16], Nasr and Aouf [11]);

(*v*) $M(f, \frac{z}{(1-z)}, (1-\gamma)\cos\eta e^{-i\eta}, 0) = S^{\eta}(\gamma)$ $(|\eta| < \frac{\pi}{2}, 0 \le \gamma < 1)$ (see Libera [8]);

(*vi*) $M(f, \frac{z}{(1-z)^2}, (1-\gamma)\cos\eta e^{-i\eta}, 0) = C^{\eta}(\gamma)$ $(|\eta| < \frac{\pi}{2}, 0 \le \gamma < 1)$ (see Chichra [4]);

(*vii*) $M(f, z + \sum\limits_{k=2}^{\infty} \Gamma_k(\alpha_1) z^k, (1-\gamma)\cos\eta e^{-i\eta}, \lambda) = R_s^q(\eta, \gamma, \lambda)$ $(|\eta| < \frac{\pi}{2}, 0 \le \lambda \le 1, 0 \le \gamma < 1)$ (see Murugusundaramoorthy and Magesh [10]), where

$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1} \ldots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \ldots (\beta_s)_{k-1} (1)_{k-1}}, \tag{1.10}$$

for $\alpha_i > 0$, $i = 1, ..., q$; $\beta_j > 0$, $j = 1, ..., s$; $q \le s+1$; $q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N} = \{1, 2, ...\}$.

Also we note that:

(*i*) $M(f, g, b, 0) = M(f, g, b)$

$$= \left\{ f \in A : Re \left[1 + \frac{1}{b} \left(\frac{z(f*g)'(z)}{(f*g)(z)} - 1 \right) \right] > 0, b \in \mathbb{C}^* \right\};$$

(*ii*) $M(f, z + \sum\limits_{k=2}^{\infty} \Gamma_k(\alpha_1) z^k, b, \lambda) = M_{q,s}(\alpha_1, b, \lambda)$

$$= \left\{ f \in A : Re \left[1 + \frac{1}{b} \left(\frac{z \left(H_{q,s}(\alpha_1, \beta_1) f(z) \right)'}{(1-\lambda) H_{q,s}(\alpha_1, \beta_1) f(z) + \lambda z \left(H_{q,s}(\alpha_1, \beta_1) f(z) \right)'} - 1 \right) \right] > 0 \right\},$$

$$(0 \le \lambda \le 1, b \in \mathbb{C}^*, z \in \mathbb{U} \text{ and } \Gamma_k(\alpha_1) \text{ is defined by } (1.10)),$$

and the operator $H_{q,s}(\alpha_1, \beta_1)$ was introduced and studied by Dziok and Srivastava (see [5] and [6]), which is a generalization of many other linear operators considered earlier;

(*iii*) $M(f, z + \sum\limits_{k=2}^{\infty} \left[\frac{\ell+1+\mu(k-1)}{\ell+1} \right]^m z^k, b, \lambda) = M(m, \mu, \ell, b, \lambda)$

$$= \left\{ f \in A : Re \left[1 + \frac{1}{b} \left(\frac{z \left(I^m(\mu, \ell) f(z) \right)'}{(1-\lambda) I^m(\mu, \ell) f(z) + \lambda z \left(I^m(\mu, \ell) f(z) \right)'} - 1 \right) \right] > 0 \right\},$$

where $0 \le \lambda \le 1, b \in \mathbb{C}^*$, $m \in \mathbb{N}_0$, $\mu, \ell \ge 0, z \in \mathbb{U}$ and the operator $I^m(\mu, \ell)$ was defined by Cătaş et al. [3], which is a generalization of many other linear operators considered earlier;

(iv) $M(f,g,(1-\gamma)\cos\eta e^{-i\eta},\lambda)=M(f,g,\lambda,\gamma,\eta)$

$$=\left\{f\in A: Re\left[e^{i\eta}\frac{z\,(f*g)'(z)}{(1-\lambda)\,(f*g)(z)+\lambda z\,(f*g)'(z)}\right]>\gamma\cos\eta\right\},$$

where $|\eta|<\frac{\pi}{2}, 0\le\lambda\le1, 0\le\gamma<1$;

(v) $M(f,z+\sum\limits_{k=2}^{\infty}\left[\frac{\ell+1+\mu(k-1)}{\ell+1}\right]^m z^k,(1-\gamma)\cos\eta e^{-i\eta},\lambda)=M(m,\mu,\ell,\lambda,\gamma,\eta)$

$$=\left\{f\in A: Re\left[e^{i\eta}\frac{z\left(I^m(\mu,\ell)\,f(z)\right)'}{(1-\lambda)I^m(\mu,\ell)\,f(z)+\lambda z\left(I^m(\mu,\ell)\,f(z)\right)'}\right]>\gamma\cos\eta\right\}.$$

where $|\eta|<\frac{\pi}{2}, 0\le\lambda\le1, 0\le\gamma<1$.

Definition 2 *(Subordination principle). For two functions f and ϕ, analytic in U, we say that the function $f(z)$ is subordinate to $\phi(z)$ in U, written $f(z)\prec\phi(z)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0)=0$ and $|w(z)|<1$, such that $f(z)=\phi\,(w(z))$. Indeed it is known that*

$$f(z)\prec\phi(z)\Rightarrow f(0)=\phi(0)\text{ and }\ f(\mathbb{U})\subset\phi(\mathbb{U}\ ).$$

Furthermore, if the function ϕ is univalent in $\mathbb{U}$, then we have the following equivalence (see [2] *and* [9] *):*

$$f(z)\prec\phi(z)\Leftrightarrow f(0)=\phi(0)\text{ and }\ f(\mathbb{U})\subset\phi(\mathbb{U}\ ). \tag{1.11}$$

Definition 3 *(Subordinating factor sequence)* [17]. *A sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever f of the form* (1.1) *is analytic, univalent and convex in $\mathbb{U}$, we have*

$$\sum_{k=2}^{\infty}a_kc_kz^k\prec f(z)\ \ (a_1=1;z\in\mathbb{U}\ ). \tag{1.12}$$

2. Main Result

Unless otherwise mentioned, we assume throughout this section that $|\eta|<\frac{\pi}{2}$, $0\le\lambda\le1$, $0\le\gamma<1$, $b\in\mathbb{C}^*$, $z\in\mathbb{U}$ and $g(z)$ given by (1.8).

To prove our main result we need the following lemmas.

Lemma 1 [17]. *The sequence* $\{c_k\}_{k=1}^{\infty}$ *is a subordinating factor sequence if and only if*

$$Re\left\{1+2\sum_{k=1}^{\infty}c_k z^k\right\}>0. \tag{2.1}$$

Now, we prove the following Lemma which gives a sufficient condition for functions belonging to the class $M(f,g,b,\lambda)$:

Lemma 2. *A function* $f(z)$ *of the form* (1.1) *is said to be in the class* $M(f,g,b,\lambda)$ *if*

$$\sum_{k=2}^{\infty}\left\{(1-\lambda)(k-1)+|b|\left[1+\lambda(k-1)\right]\right\}b_k|a_k|\leq|b|, \tag{2.2}$$

where $b_{k+1}\geq b_k>0$ $(k\geq 2)$.

Proof. Assume that, the inequality (2.2) holds true. Then it suffices to show that

$$\left|\frac{z\ (f*g)'(z)}{(1-\lambda)(f*g)(z)+\lambda z\ (f*g)'(z)}-1\right|\leq|b|.$$

We have

$$\left|\frac{z\ (f*g)'(z)}{(1-\lambda)(f*g)(z)+\lambda z\ (f*g)'(z)}-1\right|$$

$$\leq\frac{\sum\limits_{k=2}^{\infty}(1-\lambda)(k-1)b_k|a_k|\left|z^{k-1}\right|}{1-\sum\limits_{k=2}^{\infty}\left[1+\lambda(k-1)\right]b_k|a_k|\left|z^{k-1}\right|}$$

$$\leq\frac{\sum\limits_{k=2}^{\infty}(1-\lambda)(k-1)b_k|a_k|}{1-\sum\limits_{k=2}^{\infty}\left[1+\lambda(k-1)\right]b_k|a_k|}\leq|b|.$$

This completes the proof of Lemma 2

Let $M^{*}(f,g,b,\lambda)$ denote the class of $f(z)\in A$ whose coefficients satisfy the condition (2.2). We note that $M^{*}(f,g,b,\lambda)\subseteq M(f,g,b,\lambda)$.

Employing the technique used earlier by Attiya [1] and Srivastava and Attiya [15], we prove:

Thereom 1. *Let $f(z) \in M^*(f,g,b,\lambda)$. Then*

$$\frac{[1-\lambda+|b|(1+\lambda)]b_2}{2\{|b|+[1-\lambda+|b|(1+\lambda)]b_2\}}(f*h)(z) \prec h(z) \tag{2.3}$$

$$(\ b_{k+1} \geq b_k > 0\ (k \geq 2)),$$

for every function $h \in K$, and

$$Re\{f(z)\} > -\frac{\{|b|+[1-\lambda+|b|(1+\lambda)]b_2\}}{[1-\lambda+|b|(1+\lambda)]b_2}. \tag{2.4}$$

The constant factor $\frac{[1-\lambda+|b|(1+\lambda)]b_2}{2\{|b|+[1-\lambda+|b|(1+\lambda)]b_2\}}$ in the subordination result (2.3) *can not be replaced by a larger one.*

Proof. Let $f(z) \in M^*(f,g,b,\lambda)$ and suppose that $h(z) = z + \sum\limits_{k=2}^{\infty} c_k z^k$, then

$$\frac{[1-\lambda+|b|(1+\lambda)]b_2}{2\{|b|+[1-\lambda+|b|(1+\lambda)]b_2\}}(f*h)(z)$$

$$= \frac{[1-\lambda+|b|(1+\lambda)]b_2}{2\{|b|+[1-\lambda+|b|(1+\lambda)]b_2\}}\left(z+\sum_{k=2}^{\infty} c_k a_k z^k\right). \tag{2.5}$$

Thus, by using Definition 3, the subordination result holds true if

$$\left\{\frac{[1-\lambda+|b|(1+\lambda)]b_2}{2\{|b|+[1-\lambda+|b|(1+\lambda)]b_2\}}a_k\right\}_{k=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalent to the following inequality:

$$Re\left\{1+\sum_{k=1}^{\infty}\frac{[1-\lambda+|b|(1+\lambda)]b_2}{\{|b|+[1-\lambda+|b|(1+\lambda)]b_2\}}a_k z^k\right\} > 0. \tag{2.6}$$

Now, since

$$\Psi(k) = \{(1-\lambda)(k-1)+|b|[1+\lambda(k-1)]\}b_k$$

is an increasing function of k $(k \geq 2)$, we have

$$Re\left\{1+\frac{[1-\lambda+|b|(1+\lambda)]b_2}{\{|b|+[1-\lambda+|b|(1+\lambda)]b_2\}}\sum_{k=1}^{\infty} a_k z^k\right\}$$

$$= Re\left\{1+\frac{[1-\lambda+|b|(1+\lambda)]\,b_2}{\{|b|+[1-\lambda+|b|(1+\lambda)]\,b_2\}}z+\frac{\sum\limits_{k=2}^{\infty}[1-\lambda+|b|(1+\lambda)]\,b_2a_kz^k}{\{|b|+[1-\lambda+|b|(1+\lambda)]\,b_2\}}\right\}$$

$$\geq \quad 1-\frac{[1-\lambda+|b|(1+\lambda)]\,b_2}{\{|b|+[1-\lambda+|b|(1+\lambda)]\,b_2\}}r$$
$$-\frac{1}{\{|b|+[1-\lambda+|b|(1+\lambda)]\,b_2\}}\sum_{k=2}^{\infty}[1-\lambda+|b|(1+\lambda)]\,b_2a_kr^k$$

$$\geq \quad 1-\frac{[1-\lambda+|b|(1+\lambda)]\,b_2}{\{|b|+[1-\lambda+|b|(1+\lambda)]\,b_2\}}r$$
$$-\frac{1}{[|b|+(1-\lambda+|b|(1+\lambda))\,b_2]}\sum_{k=2}^{\infty}\{(1-\lambda)(k-1)+|b|[1+\lambda(k-1)]\}\,b_k\,|a_k|\,r^k$$

$$\geq 1-\frac{[1-\lambda+|b|(1+\lambda)]\,b_2}{\{|b|+[1-\lambda+|b|(1+\lambda)]\,b_2\}}r-\frac{|b|}{\{|b|+[1-\lambda+|b|(1+\lambda)]\,b_2\}}r$$

$$\geq 1-\frac{[1-\lambda+|b|(1+\lambda)]\,b_2}{\{|b|+[1-\lambda+|b|(1+\lambda)]\,b_2\}}-\frac{|b|}{\{|b|+[1-\lambda+|b|(1+\lambda)]\,b_2\}}>0\,(|z|=r<1)\,,$$

where we have also made use of assertion (2.2) of Lemma 2. Thus (2.6) holds true in $\mathbb{U}$. This proves the inequality (2.3). The inequality (2.4) follows from (2.4) by taking the convex function

$$h(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k\in K. \tag{2.7}$$

To prove the sharpness of the constant

$$\frac{[1-\lambda+|b|(1+\lambda)]\,b_2}{2\,\{|b|+[1-\lambda+|b|(1+\lambda)]\,b_2\}},$$

we consider the function $f_0(z)\in M^*(f,g,b,\lambda)$ given by

$$f_0(z)=z-\frac{|b|}{[1-\lambda+|b|(1+\lambda)]\,b_2}z^2.$$

Thus from (2.4), we have

$$\frac{[1-\lambda+|b|(1+\lambda)]b_2}{2\{|b|+[1-\lambda+|b|(1+\lambda)]b_2\}}f_0(z)\prec\frac{z}{1-z}.$$

It is easily verified that

$$\min_{|z|\le r}\left\{Re\left(\frac{[1-\lambda+|b|(1+\lambda)]b_2}{2\{|b|+[1-\lambda+|b|(1+\lambda)]b_2\}}f_0(z)\right)\right\}=-\frac{1}{2}. \tag{2.8}$$

This show that the constant $\frac{[1-\lambda+|b|(1+\lambda)]b_2}{2\{|b|+[1-\lambda+|b|(1+\lambda)]b_2\}}$ is the best possible. This completes the proof of Theorem 1.

Remark 1.

(*i*) Taking $g(z)=\frac{z}{1-z}$, $b=1-\alpha$ $(0\le\alpha\le1)$ and $\lambda=0$ in Theorem 1, we obtain the result obtained by Frasin [7, Corollary 2.3];

(*ii*) Taking $g(z)=\frac{z}{1-z}$, $b=1$ and $\lambda=0$ in Theorem 1, we obtain the result obtained by Singh [14, Corollary 2.2];

(*iii*) Taking $g(z)=\frac{z}{(1-z)^2}$, $b=1-\alpha$ $(0\le\alpha\le1)$ and $\lambda=0$ in Theorem 1, we obtain the result obtained by Frasin [7, Corollary 2.6];

(*iv*) Taking $g(z)=\frac{z}{(1-z)^2}$, $b=1$ and $\lambda=0$ in Theorem 1, we obtain the result obtained by Frasin [7, Corollary 2.7];

(*v*) Taking $g(z)=\frac{z}{1-z}$, $b=\cos\eta e^{-i\eta}$ $\left(|\eta|<\frac{\pi}{2}\right)$ and $\lambda=0$ in Theorem 1, we obtain the result obtained by Singh [14];

(*vi*) Taking $g(z)=z+\sum\limits_{k=2}^{\infty}\Gamma_k(\alpha_1)z^k$, where $\Gamma_k(\alpha_1)$ given by (1.10) and $b=(1-\gamma)\cos\eta e^{-i\eta}$ $\left(|\eta|<\frac{\pi}{2},\ 0\le\gamma<1\right)$ in Theorem 1, we obtain the result obtained by Murugusundaramoorthy and Magesh [10].

Also, we establish subordination results for the associated subclasses, $M^*(f,g,b)$, $M^*_{q,s}(\alpha_1,b,\lambda)$, $M^*(m,\mu,\ell,b,\lambda)$, $M^*(f,g,\lambda,\gamma,\eta)$ and $M^*(m,\mu,\ell,\lambda,\gamma,\eta)$, whose coefficients satisfy the condition (2.2) in the special cases as mentioned in the introduction.

By taking $\lambda=0$ in Lemma 2 and Theorem 1, we have:

Corollary 1. *Let the function $f(z)$ defined by* (1.1) *be in the class $M^*(f,g,b)$ and satisfy the condition*

$$\sum_{k=2}^{\infty}(k-1+|b|)\,b_k\,|a_k|\le|b|. \tag{2.9}$$

Then for every function $h\in K$, we have

$$\frac{(1+|b|)\, b_2}{2\left[|b|+(1+|b|)\, b_2\right]}\,(f*h)\,(z) \prec h(z), \tag{2.10}$$

and

$$Re\,\{f(z)\} > -\frac{\left[|b|+(1+|b|)\, b_2\right]}{(1+|b|)\, b_2}. \tag{2.11}$$

The constant factor $\frac{(1+|b|)b_2}{2[|b|+(1+|b|)b_2]}$ *in* (2.10) *can not be replaced by a larger one.*

By taking $b_k = \Gamma_k(\alpha_1)$, where $\Gamma_k(\alpha_1)$ defined by (1.10), in Lemma 2 and Theorem 1, we have:

Corollary 2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $M^*_{q,s}\,(\alpha_1, b, \lambda)$ *and satisfy the condition*

$$\sum_{k=2}^{\infty}\left\{(1-\lambda)\,(k-1)+|b|\left[1+\lambda\,(k-1)\right]\right\}\Gamma_k(\alpha_1)\,|a_k| \leq |b|\,. \tag{2.12}$$

Then for every function $h \in K$*, we have*

$$\frac{\left[1-\lambda+|b|\,(1+\lambda)\right]\Gamma_2(\alpha_1)}{2\left\{|b|+\left[1-\lambda+|b|\,(1+\lambda)\right]\Gamma_2(\alpha_1)\right\}}\,(f*h)\,(z) \prec h(z), \tag{2.13}$$

and

$$Re\,\{f(z)\} > -\frac{\left\{|b|+\left[1-\lambda+|b|\,(1+\lambda)\right]\Gamma_2(\alpha_1)\right\}}{\left[1-\lambda+|b|\,(1+\lambda)\right]\Gamma_2(\alpha_1)}. \tag{2.14}$$

The constant factor $\frac{[1-\lambda+|b|(1+\lambda)]\Gamma_2(\alpha_1)}{2\{|b|+[1-\lambda+|b|(1+\lambda)]\Gamma_2(\alpha_1)\}}$ *in* (2.13) *can not be replaced by a larger one.*

By taking $b_k = \left(\frac{\ell+1+\mu(k-1)}{\ell+1}\right)^m$ $(m \in \mathbb{N}_0,\ \mu, \ell \geq 0)$ in Lemma 2 and Theorem 1, we have:

Corollary 3. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $M^*(m, \mu, \ell, b, \lambda)$ *and satisfy the condition*

$$\sum_{k=2}^{\infty}\left\{(1-\lambda)\,(k-1)+|b|\left[1+\lambda\,(k-1)\right]\right\}\left[\frac{\ell+1+\mu(k-1)}{\ell+1}\right]^m |a_k| \leq |b|\,. \tag{2.15}$$

Then for every function $h \in K$*, we have*

$$\frac{[1-\lambda+|b|(1+\lambda)][\ell+1+\mu]^m}{2\{(\ell+1)^m|b|+[1-\lambda+|b|(1+\lambda)][\ell+1+\mu]^m\}}(f*h)(z)\prec h(z), \qquad (2.16)$$

and

$$Re\{f(z)\}>-\frac{\{(\ell+1)^m|b|+[1-\lambda+|b|(1+\lambda)][\ell+1+\mu]^m\}}{[1-\lambda+|b|(1+\lambda)][\ell+1+\mu]^m}. \qquad (2.17)$$

The constant factor $\frac{[1-\lambda+|b|(1+\lambda)][\ell+1+\mu]^m}{2\{(\ell+1)^m|b|+[1-\lambda+|b|(1+\lambda)][\ell+1+\mu]^m\}}$ *in* (2.16) *can not be replaced by a larger one.*

By taking $b=(1-\gamma)\cos\eta e^{-i\eta}\left(|\eta|<\frac{\pi}{2},0\leq\gamma<1\right)$ in Lemma 2 and Theorem 1, we have:

Corollary 4. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $M^*(f,g,\lambda,\gamma,\eta)$ *and satisfy the condition*

$$\sum_{k=2}^{\infty}\{(1-\lambda)(k-1)\sec\eta+(1-\gamma)[1+\lambda(k-1)]\}b_k|a_k|\leq 1-\gamma. \qquad (2.18)$$

Then for every function $h\in K$*, we have*

$$\frac{[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)]b_2}{2\{1-\gamma+[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)]b_2\}}(f*h)(z)\prec h(z), \qquad (2.19)$$

and

$$Re\{f(z)\}>-\frac{\{1-\gamma+[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)]b_2\}}{[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)]b_2}. \qquad (2.20)$$

The constant factor $\frac{[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)]b_2}{2\{1-\gamma)+[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)]b_2\}}$ *in* (2.19) *can not be replaced by a larger one.*

By taking $b_k=\left(\frac{\ell+1+\mu(k-1)}{\ell+1}\right)^m$ ($m\in\mathbb{N}_0,\ \mu,\ell\geq 0$) and $b=(1-\gamma)\cos\eta e^{-i\eta}$ ($|\eta|<\frac{\pi}{2},\ 0\leq\gamma<1$) in Lemma 2 and Theorem 1, we have:

Corollary 5. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $M^*(m,\mu,\ell,\lambda,\gamma,\eta)$ *and satisfy the condition*

$$\sum_{k=2}^{\infty}\{(1-\lambda)(k-1)\sec\eta+(1-\gamma)[1+\lambda(k-1)]\}\left[\frac{\ell+1+\mu(k-1)}{\ell+1}\right]^m|a_k|\leq 1-\gamma \qquad (2.21)$$

Then for every function $h \in K$, we have

$$\frac{[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)][\ell+1+\mu]^m}{2\{(1-\gamma)(\ell+1)^m+[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)][\ell+1+\mu]^m\}}(f*h)(z)\prec h(z) \tag{2.22}$$

and

$$Re\{f(z)\}>-\frac{\{(1-\gamma)(\ell+1)^m+[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)][\ell+1+\mu]^m\}}{[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)][\ell+1+\mu]^m}. \tag{2.23}$$

The constant factor

$$\frac{[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)][\ell+1+\mu]^m}{2\{(1-\gamma)(\ell+1)^m+[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)][\ell+1+\mu]^m\}}$$

in (2.22) *can not be replaced by a larger one.*

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