Acta Universitatis Apulensis
ISSN: 1582-5329

No. 30/2012
pp. 337-354

RULED SURFACES AND DUAL SPHERICAL CURVES

Yusuf Yayli and Semra Saracoglu

Abstract. In this paper, using curves in E(3) with tangent, normal, binormal and darboux lines, we studied their dual spherical indicatrices. In addition, the dual angles and lengths of pitch of the closed ruled surfaces are given. We showed that tangent and binormal indicatrice curves are involutes of darboux indicatrice curves. Depending on some differences that found, important results are presented about these dual spherical indicatrice curves.

2000 *Mathematics Subject Classification*: 53A04, 53A05, 53A17

1. Introduction

Dual numbers were ıntroduced in the 19th century by Clifford and their applications to rigid body kinematics was generalized by Study in their principle of transference [16]. As it is known, the analytical tools in the study of 3-dimensional kinematics and differential geometry of ruled surfaces are based on dual vector calculus.

Important contributions to the curvature theory, frame approaches have been made by R.L. Bishop, Mc Carthy, Bottema, Blaschke, Hacısalihoglu H., Rashad A. Abdel-Baky...etc. In the study of P. Azariadis and N. Aspragathos, an alternative representational model for 3 dimensional geometric entities is expressed, which is based on dual unit vectors. Dual points are used to describe curves and surfaces in E(3) as well as geometric invariant properties such as normal vectors or curvature vectors.

In the first part of this study we briefly give basic concepts for the reader who isn't familiar with darboux, blaschke, frenet frames...etc. Necessary mathematical formulations and conventions are presented. Next part of this study includes dual spherical motions of curves with its tangent, normal, binormal and darboux lines. When we give representation of geometric entities using different indicatrices, we use the 5th part of P. Azariadis. S. G. Papageorgiou and N.A. Aspragathos expressed similarly that the 3D surface is given by unique parametric equation. A general spatial displacement of an object is equivalent to a rotation around a line and a translation along the same line. The name of motion is screw displacements around an axis by a dual angle. At this time, Rashad A.Abdel-Baky' s article shows us that we can study a ruled surface as a curve on the dual unit sphere by using Blaschke approach.

In addition, we present the kinematic interpretation of dual representations by using relations between Blaschke, Darboux, Frenet Frames. Some theorems and results are given for the special cases which the line is being the principal normal and binormal of the base curve α. By analyzing Yaylı's, Gürsoy's, Helmutth's, Keles and Karadag's, Abdel Baky's articles, these theorems and results are developed. Fenchel showed that the unit spherical closed curve is the principal normal indicatrice of a closed space curve. In additon, the dual angle length of pitch of the closed ruled surface are computed. Tangent and binormal indicatrice curves are involutes of darboux indicatrice curve.

2. Basic Concepts

Now we give basic concepts on classical differential geometry of space curves. We define dual numbers, Frenet frame, Blaschke frame and involute benefiting from the references [1, 5, 8, 15, 16]

2.1. Dual Numbers

If p and p*are real numbers, the form

$$\widetilde{P} = p + \varepsilon p^\star \tag{1}$$

is called a dual number.ε is the dual unit with the properties $\varepsilon^2 = 0$, $\varepsilon \neq 0$, $\varepsilon.1 = 1.\varepsilon = \varepsilon$. We can easily say that dual numbers forms a ring over the real number field. We show the set of all dual numbers with ID. Such as, two dual numbers $\widetilde{P}$ and $\widetilde{Q}$ where

$$\begin{aligned} \widetilde{P} &= p + \varepsilon p^\star, & (2)\\ \widetilde{Q} &= q + \varepsilon q^\star & (3) \end{aligned}$$

are added componentwise

$$\widetilde{P} + \widetilde{Q} = (p + q) + \varepsilon(p^\star + q^\star) \tag{4}$$

and they are multiplied by

$$\widetilde{P}.\widetilde{Q} = p.q + \varepsilon(p^\star.q + p.q^\star). \tag{5}$$

For the equality of A and B , we have

$$\widetilde{P} = \widetilde{Q} \Leftrightarrow p = q \tag{6}$$

and

$$p^{\star} = q^{\star}. \tag{7}$$

ID^3 is a set that is a module over the ring ID. It is called dual space. Dual vectors are the elements of this dual space. So we can write a dual vector as

$$\overrightarrow{P} = p + \varepsilon p^{\star} \tag{8}$$

where $p^{\star} = q \Lambda p$ and $p, q, p^{\star}$ are real vectors in $\mathbb{R}^3$.

2.2. Frenet Frame [15]

We assume that the curve α is parametrized by arclength. Then, $\alpha'(s)$ is the unit tangent vector to the curve, which we denote by $T(s)$ Since t has constant length, $T'(s)$ will be orthogonal to $T(s)$. If $T'(s) \neq 0$ then we define principal normal

$$N(s) = \frac{T'(s)}{T'(s)} \tag{9}$$

vectorand the curvature

$$k_1(s) = \left\| T'(s) \right\|$$

So far, we have

$$T'(s) = k_1(s).T(s) \tag{10}$$

If $k_1(s) = 0$, the principal normal vector is not defined. If $k_1(s) \neq 0$ then the binormal vector b(s) is given by

$$B(s) = T(s) \times N(s)$$

Then $\{T(s), N(s), B(s)\}$ form a right -handed orthonormal basis for IR^3. In summary Frenet formulas can be given as

$$T'(s) = k_1(s).N(s) \tag{11}$$
$$N'(s) = -k_1(s).T(s) + k_2(s).B(s) \tag{12}$$
$$B'(s) = -k_2(s).N(s). \tag{13}$$

2.3. Blaschke Frame [13]

Let $M(s,u)$ be the ruled surface and $A(s)$ be the dual spherical curve in $\mathbb{D}^3$;

$$A(s) = a(s) + \varepsilon a^{\star}(s) \tag{14}$$

We now define an orthonormal moving frame along this dual curve as follows:

$$A_1 = A(s), \tag{15}$$

$$A_2 = \frac{A_1'}{\|A_1'\|}, \tag{16}$$

$$A_3 = A_1 \times A_2. \tag{17}$$

From now on we consider the case without $a(s) = \textit{constant vector}$ and $a^{\star}(s) = 0$. In the case $a(t) = \textit{constant vector}$ the ruled surface $M(s,u)$ is a cylinder and in the case $a^{\star}(s) = 0$ the ruled surface M(t,u) is a cone. The frame $\{A_1, A_2, A_3\}$ is called Blaschke frame.

2.4. Involute

The orbit that is the perpendicular to the tangents of a curve is involute of this curve.

2.5. Summary Representation

In addition, we can give a summary representation of this study as follows:

2.5.1. Spherical Indicatrice

Let α be a curve in E^3,

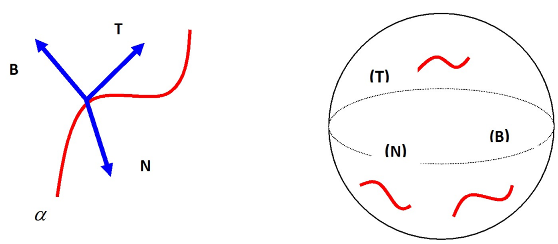

Figure 1: Indicatrices on S^2 unit sphere.

2.5.2. Ruled Surface and Dual Spherical Curves

Let α be base curve and $\overrightarrow{X}$ is director vector of α.

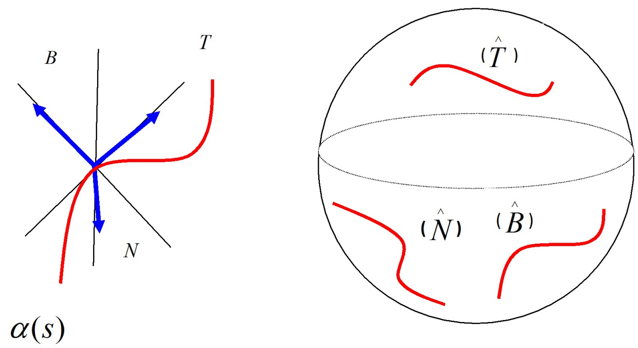

Figure 2: Indicatrices on unit dual sphere.

Ruled Surfaces

$$\begin{aligned}
\Phi_{\widehat{T}}(s,v) &= \alpha(s)+v.T\\
\Phi_{\widehat{N}}(s,v) &= \alpha(s)+v.N\\
\Phi_{\widehat{B}}(s,v) &= \alpha(s)+v.B
\end{aligned}$$

Dual Curves

$$\begin{aligned}
\widehat{T} &= T+\varepsilon\alpha\wedge T\\
\widehat{N} &= N+\varepsilon\alpha\wedge N\\
\widehat{B} &= B+\varepsilon\alpha\wedge B
\end{aligned}$$

Let β be the other curve and $\{T,N,B\}$ is Frenet Frame of α.

Ruled Surfaces

$$\begin{aligned}
\Phi_{\overline{T}}(s,v) &= \beta(s)+v.T\\
\Phi_{\overline{N}}(s,v) &= \beta(s)+v.N\\
\Phi_{\overline{B}}(s,v) &= \beta(s)+v.B
\end{aligned}$$

Dual Curves

$$\begin{aligned}
\overline{T} &= T+\varepsilon\beta\wedge T\\
\overline{N} &= N+\varepsilon\beta\wedge N\\
\overline{B} &= B+\varepsilon\beta\wedge B
\end{aligned}$$

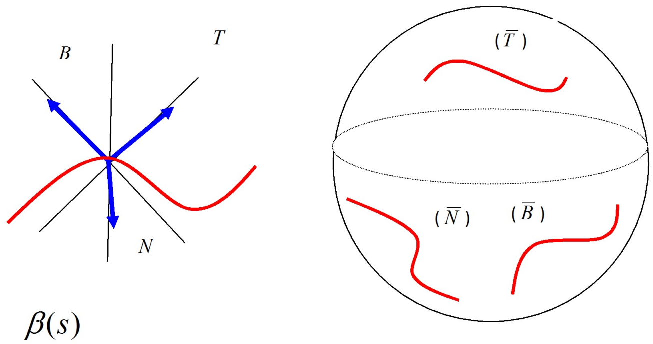

Figure 3: Indicatrices on unit dual sphere.

3. Indicatrices and Dual Spherical Motion

3.1. First Kind of Indicatrice Curve

Let

$$\begin{aligned} \alpha : I &\to E^3 \\ s &\mapsto \alpha(s) \end{aligned} \tag{18}$$

be unit speed curve and $\{T, N, B\}$Frenet frame of α. T, N, B are the unit tangent, principal, normal and binormal vectors respectively. With the assistance of α, we define a dual curve in ID^3. So, let us have a closed spherical dual curve $\widehat{\alpha}$ of class C^1 on a unit dual sphere S^{1}in ID^3. The curve α describes a closed dual spherical motion.If

$$\begin{aligned} \widehat{\alpha} : S^1 &\to ID^3 \\ s &\mapsto \alpha(s) + \oint \alpha \wedge T dt = \oint (T + \varepsilon(\alpha \wedge T)) dt \end{aligned} \tag{19}$$

then the curve α can be written as :

$$\widehat{\alpha} = \alpha(s) + \varepsilon \alpha^{\star}(s). \tag{20}$$

Therefore, when we have

$$\alpha : I \to E^3$$

$$s \mapsto \alpha(s)$$

the Peano directiion of α and the projection of P_α can be given by **[16]**

$$P_\alpha = \oint_{S^1} \alpha \wedge T dt \tag{21}$$

Hence, F_α is a projection area of closed curve α on the plane that its normal is N.

$$F_\alpha = \langle P_\alpha, N \rangle . \tag{22}$$

We now define an orthonormal moving frame along dual curve as follows in ID^3; The tangent indicatrice of $\widehat{\alpha}$ is

$$\widehat{T} = \frac{d\widehat{\alpha}}{ds} \tag{23}$$

$$\widehat{T} = T + \varepsilon\alpha \wedge T, \qquad \left\| \widehat{T} \right\| = 1. \tag{24}$$

The principal normal indicatrice of $\widehat{\alpha}$ is

$$\widehat{N} = \frac{\dfrac{d\widehat{T}}{ds}}{\left\| \dfrac{d\widehat{T}}{ds} \right\|}, \tag{25}$$

$$\widehat{N} = N + \varepsilon\alpha \wedge N \tag{26}$$

The binormal indicatrice of $\widehat{\alpha}$ is

$$\begin{aligned} \widehat{B} &= \widehat{T} \times \widehat{N} \\ &= B + \varepsilon\alpha \wedge B \end{aligned} \tag{27}$$

Subsequently, we can write matrix form as:

$$\begin{bmatrix} \widehat{T}' \\ \widehat{N}' \\ \widehat{B}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 + \varepsilon \\ 0 & -k_2 - \varepsilon & 0 \end{bmatrix} . \begin{bmatrix} \widehat{T} \\ \widehat{N} \\ \widehat{B} \end{bmatrix} \tag{28}$$

On the other hand, as [14], we can define the Darboux screw. It can be seen that the real part of $\widehat{W}$ is the darboux vector of $\{T, N, B\}$ Frenet motion.

Theorem 1 *Darboux screw is given by*

$$\widehat{W} = (k_2 + \varepsilon).\widehat{T}(t) + k_1.\widehat{B}(t). \tag{29}$$

Proof. To proof this theorem, we have to show the truth of these following equations:

$$\begin{aligned} \widehat{W} \wedge \widehat{T} &= \widehat{T}' \\ \widehat{W} \wedge \widehat{N} &= \widehat{N}' \\ \widehat{W} \wedge \widehat{B} &= \widehat{B}'. \end{aligned} \tag{30}$$

Theorem 2 *$\widehat{T}, \widehat{B}$ dual spherical curves on dual sphere are involutes of $\widehat{C}$ where $\widehat{C} = \dfrac{\widehat{W}}{\left\|\widehat{W}\right\|}$.*

Proof. According to the theorem.1, If

$$\widehat{W} = (k_2 + \varepsilon).\widehat{T}(t) + k_1.\widehat{B}(t), \tag{31}$$

then

$$\widehat{C} = \frac{\widehat{W}}{\left\|\widehat{W}\right\|} = (\sqrt{k_1^2 + k_2^2} + \varepsilon \frac{k_2}{\sqrt{k_1^2 + k_2^2}}).\widehat{W} \tag{32}$$

and

$$\begin{aligned} \left\langle \frac{d\widehat{C}}{dS}, \frac{d\widehat{T}}{ds} \right\rangle &= 0, \\ \left\langle \frac{d\widehat{C}}{ds}, \frac{d\widehat{B}}{ds} \right\rangle &= 0. \end{aligned} \tag{33}$$

Hence, in dual sphere T and B are involutes of $\widehat{C}$ ($\widehat{C}$ is fixed pole curve). Privately, if we take the real part of the following equations,

$$\begin{aligned} \widehat{T} &= T + \varepsilon\alpha \wedge T \\ \widehat{N} &= N + \varepsilon\alpha \wedge N \\ \widehat{B} &= B + \varepsilon\alpha \wedge B \end{aligned} \tag{34}$$

We can easily say that $\widehat{W}$ is the darboux vector of the motion of $\left\{\widehat{T}, \widehat{N}, \widehat{B}\right\}$. If T,N,B are spherical indicatrices, then

$$\begin{aligned} \left\langle \frac{d\widehat{C}}{dS}, \frac{d\widehat{T}}{ds} \right\rangle &= 0, \\ \left\langle \frac{d\widehat{C}}{ds}, \frac{d\widehat{B}}{ds} \right\rangle &= 0. \end{aligned} \tag{35}$$

Result. α is base curve on E^3, T and B are tangent, binormal indicatrices respectively and the fixed centrode is the curve C descibed by c=c(s). From this, it follows that the indicatrices T and B are spherical involutes of C.

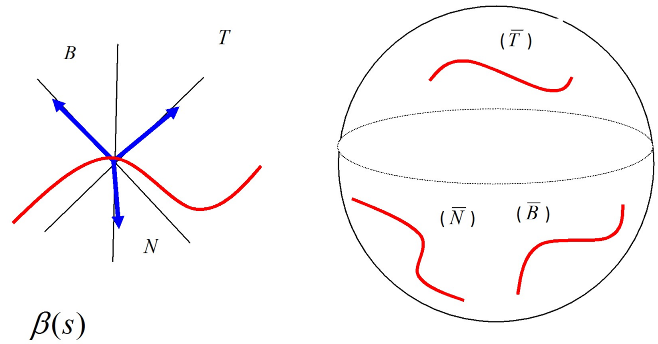

Figure 4: T and B are spherical linvolutes of C. [3]

If we take $A_1 = T$, $A_2 = N$, $A_3 = B$ then the Blaschke's inviriants of the dual curve $\widehat{T}(t)$ is given by

$$\begin{aligned} P &= p + \varepsilon p^{\star} = \left\| T' \right\| = k_1 \\ Q &= q + \varepsilon q^{\star} = \frac{\det \left\| \widehat{T}, \widehat{T}', \widehat{T}" \right\|}{P^2} \end{aligned} \tag{36}$$

where

$$k_1 = P \tag{37}$$

$$k_2 + \varepsilon = Q \tag{38}$$

On the other hand, it can be easily seen that there is a ruled surface corresponding to $\widehat{T}(s)$ on ID^3

$$\widehat{\alpha}(s) = \widehat{T}(s) = T + \varepsilon(\alpha \wedge T) \tag{39}$$

where

$$\begin{aligned} \widehat{T}(s) &= \alpha(s), \\ \alpha^{\star}(s) &= \varepsilon(\widehat{\alpha} \wedge T) \end{aligned} \tag{40}$$

then

$$\begin{aligned} \Phi_{\widehat{T}}(s,v) &= \alpha \wedge \alpha^{\star} + v.T(s) \\ &= T \wedge (\alpha \wedge T) + v.T \\ &= \alpha(s) - \langle T, \alpha \rangle .T + v.T \\ &= \alpha(s) + (v - \langle T, \alpha \rangle).T \\ &= \alpha(s) + v.T \end{aligned} \tag{41}$$

Smiliarly, if $\widehat{N}(s), \widehat{B}(s)$ are normal and binormal indicatrices of $\widehat{\alpha}(s)$ respectively, then we can give the ruled surfaces corresponding to the curve $\widehat{N}(s), \widehat{B}(s)$. In this case, the equations can be given as:

$$\begin{aligned} \widehat{N} &= N(s) + \varepsilon \alpha \wedge N \\ \widehat{N} &= \frac{\dfrac{d\widehat{T}}{ds}}{\left\| \dfrac{d\widehat{T}}{ds} \right\|} \\ \Phi_{\widehat{N}}(s,v) &= \alpha(s) + v.N \\ \widehat{B} &= B(s) + \varepsilon \alpha \wedge B = \widehat{T} \wedge \widehat{N} \\ \Phi_{\widehat{B}}(s,v) &= \alpha(s) + v.B \end{aligned} \tag{42}$$

Let $\left\{ O, \widehat{T}, \widehat{N}, \widehat{B} \right\}$ and $\{O, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ be two orthonormal coordinate systems of S moving unit dual sphere and S' fixed unit dual sphere with the same 'O' origin, and let represent one parameter dual spherical motion (dual rotation) between S, S' dual spheres with S/S$'$. During the spherical motion S/S' on the α base curve $\alpha : I \to E^3, \overrightarrow{X}$ is director vector and s is the arc parameter of α the base curve of $\Phi(s,v)$ closed surface. In this case, Steiner vector is

$$\begin{aligned} \widehat{D} &= \oint \widehat{W}.ds; \\ \widehat{D} &= \oint (k_2 + \varepsilon)\widehat{T} + \oint k_1 \widehat{B}. \end{aligned} \tag{43}$$

If $\overrightarrow{X}$ director vector is

$$\widehat{X} = (x_1 + \varepsilon x_1^{\star}).\widehat{T} + (x_2 + \varepsilon x_2^{\star})\widehat{N} + (x_1 + \varepsilon x_1^{\star})\widehat{B} \tag{44}$$

where

$$\begin{aligned} \left\| \widehat{X} \right\| &= 1, \\ \widehat{x}_1 &= x_1 + \varepsilon x_1^{\star}, \widehat{x}_2 = x_2 + \varepsilon x_2^{\star}, \widehat{x}_3 = x_3 + \varepsilon x_3^{\star} \end{aligned} \tag{45}$$

then the dual angle of pitch of the closed ruled surface generated by a director vector X of $\widehat{\alpha}(s)$ closed curve is

$$\wedge_{\widehat{X}} = -\left\langle \widehat{D}, \widehat{X} \right\rangle. \tag{46}$$

Thus, with the aid of

$$\widehat{X} = \widehat{x}_1.\widehat{T} + \widehat{x}_2.\widehat{N} + \widehat{x}_3.\widehat{B} \tag{47}$$

$$\widehat{D} = \oint (k_2 + \varepsilon) ds \widehat{T} + \oint k_1.ds \widehat{B} \tag{48}$$

the dual angle of pitch of the indicatrices are obtained as:

$$\wedge_{\widehat{T}} = \oint (k_2 + \varepsilon) ds \tag{49}$$

$$\wedge_{\widehat{N}} = 0 \tag{50}$$

$$\wedge_{\widehat{B}} = \oint k_1.ds. \tag{51}$$

Thus,

$$\begin{aligned} \wedge_{\widehat{X}} &= -[\widehat{x}_1. \oint (k_2 + \varepsilon) ds + \widehat{x}_3 \oint k_1 ds] \qquad (52) \\ &= -[\widehat{x}_1. \oint k_2 ds + \varepsilon \widehat{x}_1. \oint ds + \widehat{x}_3 \oint k_1 ds] \\ &= -[\widehat{x}_1. \oint k_2 ds + \widehat{x}_3 \oint k_1 ds + \varepsilon \widehat{x}_1. \oint ds] \\ &= -[(x_1 + \varepsilon x_1^\star). \oint k_2 ds + (x_3 + \varepsilon x_3^\star). \oint k_1 ds + \varepsilon (x_1 + \varepsilon x_1^\star). \oint ds] \\ &= -\{[x_1. \oint k_2 ds + x_3. \oint k_1 ds] + \varepsilon [x_1^\star. \oint k_2 ds + x_3^\star \oint k_1 ds + x_1. \oint ds]\} \end{aligned}$$

In this case, if we compose dual angle of pitch of the closed ruled surface with dual and real parts,

$$\begin{aligned} \wedge_{\widehat{X}} &= -\left\langle \widehat{D}, \widehat{X} \right\rangle \qquad (53) \\ &= -\langle d, x \rangle - \varepsilon (\langle d^\star, x \rangle + \langle d, x^\star \rangle) \end{aligned}$$

can be seen. Thus the relation between $\lambda_{\widehat{X}}$ angle of pitch of the closed ruled surface that corresponds to $\widehat{X}$ and $L_{\widehat{X}}$the length of pitch is:

$$\wedge_{\widehat{X}} = \lambda_{\widehat{X}} - \varepsilon . L_{\widehat{X}}. \tag{54}$$

Then, the real and dual part can be designed as:

$$\lambda_{\widehat{X}} = -\langle d, x\rangle = -[x_1.\oint k_2 ds + x_3.\oint k_1 ds] \tag{55}$$

$$L_{\widehat{X}} = [x_1^{\star}.\oint k_2 ds + x_3^{\star}\oint k_1 ds + x_1.\oint ds] = (\langle d^{\star}, x\rangle + \langle d, x^{\star}\rangle) \tag{56}$$

SPECIAL CASES

1. The case $\widehat{X} = \widehat{T}$;
In this case; $\widehat{x}_1 = 1, \widehat{x}_2 = \widehat{x}_3 = 0$. Thus,

$$\begin{aligned} \wedge_{\widehat{T}} &= -\left\langle \widehat{D}, \widehat{T}\right\rangle \\ &= -\oint (k_2 + \varepsilon) ds \\ &= -\oint k_2 ds - \varepsilon \oint ds = \lambda_{\widehat{T}} - \varepsilon L_{\widehat{T}}. \end{aligned} \tag{57}$$

Here $\lambda_{\widehat{T}}$ is dual angle of pitch of the closed ruled surface and $L_{\widehat{T}}$ is dual length of pitch of closed ruled surface that drawn by $\widehat{T}$ during $\alpha(s)$.

2. The case $\widehat{X} = \widehat{N}$;
In this case; $\widehat{x}_2 = 1, \widehat{x}_1 = \widehat{x}_3 = 0$. Thus,

$$\wedge_{\widehat{N}} = 0 \tag{58}$$

3. The case $\widehat{X} = \widehat{B}$;
In this case; $\widehat{x}_3 = 1, \widehat{x}_1 = \widehat{x}_2 = 0$. Thus,

$$\wedge_{\widehat{B}} = -\left\langle \widehat{D}, \widehat{B}\right\rangle = -\oint k_1 ds = -L_{\widehat{B}} \tag{59}$$

3.2. Second Kind of Indicatrice Curve

Let $\beta(s)$ be a curve and let its parameter be the same as $\alpha(s)$.

$$\beta : I \to E^3 \tag{60}$$

$$s \mapsto \beta(s)$$

Then we get ruled surfaces that produced by $\{\overline{T}, \overline{N}, \overline{B}\}$frame as:

$$\begin{aligned} \Phi(s, v) &: I \times \mathbb{R} \to E^3 \\ \Phi(s, v) &= \beta(s) + v.\overline{X}(s). \end{aligned} \tag{61}$$

Smiliarly, dual indicatrice curves can be given as:

$$\begin{aligned}
\overline{T} &= T + \varepsilon\beta \wedge T && (62)\\
\overline{N} &= N + \varepsilon\beta \wedge N && (63)\\
\overline{B} &= B + \varepsilon\beta \wedge B && (64)
\end{aligned}$$

Thus, we can write the ruled surfaces for $\overline{T}, \overline{N}, \overline{B}$ as follows:

$$\begin{aligned}
\Phi_{\overline{T}}(s,v) &= \beta(s) + vT && (65)\\
\Phi_{\overline{N}}(s,v) &= \beta(s) + vN\\
\Phi_{\overline{B}}(s,v) &= \beta(s) + vB.
\end{aligned}$$

Accordingly, we get $\overline{T}$ as:

$$\begin{aligned}
\overline{T} &= T + \varepsilon\beta \wedge T && (66)\\
\overline{T}' &= T' + \varepsilon(\beta' \wedge T + \beta \wedge T)\\
&= T' + \varepsilon(\beta' \wedge T + \beta \wedge (k_1 N))\\
&= T' + \varepsilon\left[(\lambda_1 T + \lambda_2 N + \lambda_3 B) \wedge T + k_1\beta \wedge N\right]\\
&= k_1(N + \varepsilon\beta \wedge N) - \varepsilon\lambda_2 B + \varepsilon\lambda_3 N\\
&= k_1.\overline{N} + \varepsilon(\lambda_3 N + \varepsilon\lambda_3\beta \wedge N) - \varepsilon\lambda_2 B\\
&= (k_1 + \varepsilon\lambda_3).\overline{N} - \varepsilon\lambda_2\overline{B}
\end{aligned}$$

Then we get $\overline{N}$ as:

$$\begin{aligned}
\overline{N} &= N + \varepsilon\beta \wedge N && (67)\\
\overline{N}' &= N' + \varepsilon(\beta' \wedge N + \beta \wedge N')\\
&= (-k_1 T + k_2 B) + \varepsilon[(\lambda_1 T + \lambda_2 N + \lambda_3 B) \wedge N + \beta \wedge (-k_1 T + k_2 B)]\\
&= (-k_1 T + k_2 B) + \varepsilon(-\lambda_3 T + \lambda_1 B - k_1\beta \wedge T + k_2\beta \wedge B)\\
&= (-k_1\overline{T} + k_2\overline{B}) - \varepsilon\lambda_3\overline{T} + \varepsilon\lambda_1\overline{B})\\
&= (-k_1 - \varepsilon\lambda_3).\overline{T} + (k_2 + \varepsilon\lambda_1).\overline{B}
\end{aligned}$$

Smiliarly, we get $\overline{B}$ as:

$$\begin{aligned}
\overline{B} &= B + \varepsilon\beta \wedge B && (68)\\
\overline{B}' &= B' + \varepsilon(\beta' \wedge B + \beta \wedge B') = -k_2 N + \varepsilon\left[(\lambda_1 T + \lambda_2 N + \lambda_3 B) \wedge B + \beta \wedge (-k_2 N)\right]\\
&= -k_2 N + \varepsilon\left[(\lambda_2 T - \lambda_1 N - k_2\beta \wedge N)\right] = (-k_2 - \varepsilon\lambda_1)\overline{N} + \varepsilon\lambda_2\overline{T}.
\end{aligned}$$

Consequently, the matrix form can be given as:

$$\begin{bmatrix} \overline{T}' \\ \overline{N}' \\ \overline{B}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 + \varepsilon\lambda_3 & -\varepsilon\lambda_2 \\ -k_1 - \varepsilon\lambda_3 & 0 & k_2 + \varepsilon\lambda_1 \\ \varepsilon\lambda_2 & -k_2 - \varepsilon\lambda_1 & 0 \end{bmatrix} . \begin{bmatrix} \overline{T} \\ \overline{N} \\ \overline{B} \end{bmatrix} . \tag{69}$$

We get Darboux screw as:

$$\overline{W} = (k_2 + \varepsilon\lambda_1).\overline{T} + \varepsilon\lambda_2.\overline{N} + (k_1 + \varepsilon\lambda_3).\overline{B} \tag{70}$$

Specially, for $\lambda_1 = 1, \lambda_2 = \lambda_3 = 0$; Darboux screw for Frenet.

$$\overline{W} = (k_2 + \varepsilon)\overline{N} + k_1\overline{B}, \tag{71}$$

can be found. Thus,

$$\left\langle \overline{W}, \overline{W} \right\rangle = (k_2 + \varepsilon\lambda_1)^2 + (k_1 + \varepsilon\lambda_3)^2 = k_1^2 + k_2^2 + 2\varepsilon(\lambda_1 k_2 + \lambda_3 k_1) = x + 2\varepsilon x^\star \tag{72}$$

Then, the norm of $\overline{W}$ Darboux screw is

$$\begin{aligned} \sqrt{x + \varepsilon x^\star} &= \sqrt{x} + \varepsilon \frac{x^\star}{\sqrt{x}} \\ \|\overline{W}\| &= \sqrt{k_1^2 + k_2^2} + \varepsilon \frac{\lambda_1 k_2 + \lambda_3 k_1}{\sqrt{k_1^2 + k_2^2}} = \Psi + \varepsilon \Psi^\star \end{aligned} \tag{73}$$

So,

$$\Psi^\star = \frac{\lambda_1 k_2 + \lambda_3 k_1}{\sqrt{k_1^2 + k_2^2}}. \tag{74}$$

Now, we present the dual angles of pitch and Steiner vector of spherical motion. Steiner vector of this motion is,

$$\overline{D} = \oint (k_2 + \varepsilon\lambda_1) ds\overline{T} + \varepsilon \oint \lambda_2 ds\overline{N} + \oint (k_1 + \varepsilon\lambda_3) ds\overline{B}. \tag{75}$$

Theorem 3 *$\overline{T}, \overline{B}$ dual spherical curves on dual sphere aren't involutes of $\overline{C}$ where $\overline{C} = \dfrac{\overline{W}}{\|\overline{W}\|}$.*

Proof. We can write the matrix form as:

$$\begin{bmatrix} \overline{T}' \\ \overline{N}' \\ \overline{B}' \end{bmatrix} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} \overline{T} \\ \overline{N} \\ \overline{B} \end{bmatrix} \tag{76}$$

where

$$\begin{aligned} a &= k_2 + \varepsilon.\lambda_1 \\ b &= \varepsilon.\lambda_2 \\ c &= k_1 + \varepsilon.\lambda_3 \end{aligned} \tag{77}$$

Thus, we get $\overline{C}$ as follows:

$$\begin{aligned} \overline{C} &= \frac{\overline{W}}{\|\overline{W}\|} \\ &= (\frac{a}{\sqrt{a^2+b^2+c^2}}).\overline{T} + (\frac{b}{\sqrt{a^2+b^2+c^2}}).\overline{N} + (\frac{c}{\sqrt{a^2+b^2+c^2}}).\overline{B} \end{aligned} \tag{78}$$

Then,

$$\overline{C}' = (\frac{a}{\sqrt{a^2+b^2+c^2}})'.\overline{T} + (\frac{b}{\sqrt{a^2+b^2+c^2}})'.\overline{N} + (\frac{c}{\sqrt{a^2+b^2+c^2}})'.\overline{B}$$

This shows that

$$\begin{aligned} \left\langle \frac{d\overline{C}}{dS}, \frac{d\overline{T}}{ds} \right\rangle &\neq 0, \\ \left\langle \frac{d\overline{C}}{ds}, \frac{d\overline{B}}{ds} \right\rangle &\neq 0. \end{aligned} \tag{79}$$

From this, we can easily say that $\overline{T}, \overline{B}$ dual spherical curves on dual sphere aren't involutes of $\overline{W}$ where $\overline{C} = \dfrac{\overline{W}}{\|\overline{W}\|}$. But if we get special values as $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$, these $\overline{T}, \overline{B}$ dual spherical curves on dual sphere can be involutes of $\overline{W}$ where $\overline{C} = \dfrac{\overline{W}}{\|\overline{W}\|}$.

Theorem 4 *On S/S' unit dual spherical motion with one parameter, let $\wedge_X$ be the angel of pitch of closed ruled surface that is drawn by $\overrightarrow{X} = \overrightarrow{x} + \varepsilon \overrightarrow{x^\star}$ unit dual vector. $\wedge_X$ is denoted by:*

$$\wedge_{\overline{X}} = -\left\langle \overline{D}, \overline{X} \right\rangle. \tag{80}$$

Here, dual Steiner vector of dual spherical motion is

$$\overline{D} = \overrightarrow{d} + \varepsilon \overrightarrow{d^\star} = \oint (k_2 + \varepsilon\lambda_1) ds \overline{T} + \varepsilon \oint \lambda_2 ds \overline{N} + \oint (k_1 + \varepsilon\lambda_3) ds \overline{B}, \tag{81}$$

where

$$\overline{x}_1 = x_1 + \varepsilon x_1^\star,\ \overline{x}_2 = x_2 + \varepsilon x_2^\star,\ \overline{x}_3 = x_3 + \varepsilon x_3^\star \tag{82}$$

the elements of ID and $\overrightarrow{X}$ *vector is*

$$\overline{X} = \overline{x}_1 \overline{T} + \overline{x}_2 \overline{N} + \overline{x}_3 \overline{B}. \tag{83}$$

Thus,

$$\begin{aligned}
\wedge_{\overline{X}} &= -[\overline{x}_1 \oint (k_2 + \varepsilon\lambda_1) ds + \varepsilon \overline{x}_2 \oint \lambda_2 ds + \overline{x}_3 \oint (k_1 + \varepsilon\lambda_3) ds] \qquad (84)\\
&= -[\overline{x}_1 \oint k_2 ds + \varepsilon \overline{x}_1 \oint \lambda_1 ds + \varepsilon \overline{x}_2 \oint \lambda_2 ds + \overline{x}_3 \oint k_1 ds + \varepsilon \overline{x}_3 \oint \lambda_3 ds]\\
&= -[\overline{x}_1 \oint k_2 ds + \varepsilon \overline{x}_1 \oint \lambda_1 ds + \varepsilon \overline{x}_2 \oint \lambda_2 ds + \overline{x}_3 \oint k_1 ds + \varepsilon \overline{x}_3 \oint \lambda_3 ds]\\
&= -[(x_1 + \varepsilon x_1^\star) \oint k_2 ds + \varepsilon (x_1 + \varepsilon x_1^\star) \oint \lambda_1 ds + \varepsilon (x_2 + \varepsilon x_2^\star) \oint \lambda_2 ds +\\
&\quad (x_3 + \varepsilon x_3^\star) \oint k_1 ds + \varepsilon (x_3 + \varepsilon x_3^\star) \oint \lambda_3 ds]\\
&= -[x_1 \oint k_2 ds + \varepsilon x_1^\star \oint k_2 ds + \varepsilon x_1 \oint \lambda_1 ds + \varepsilon x_2 \oint \lambda_2 ds\\
&\quad + x_3 \oint k_1 ds + \varepsilon x_3^\star \oint k_1 ds + \varepsilon x_3 \oint \lambda_3 ds]\\
&= -\{x_1 \oint k_2 ds + x_3 \oint k_1 ds + \varepsilon [x_1^\star \oint k_2 ds + x_1 \oint \lambda_1 ds + x_2 \oint \lambda_2 ds\\
&\quad + x_3^\star \oint k_1 ds + x_3 \oint \lambda_3 ds]\}
\end{aligned}$$

Depending on this, we get

$$\wedge_{\overline{X}} = -\left\langle \overline{D}, \overline{X} \right\rangle = -\langle d, x\rangle - \varepsilon(\langle d^\star, x\rangle + \langle d, x^\star\rangle) \tag{85}$$

and then we can give the relation between $\lambda_{\overline{X}}$ *and* $L_{\overline{X}}$ *as follows:*

$$\wedge_{\overline{X}} = \lambda_{\overline{X}} - \varepsilon L_{\overline{X}}. \tag{86}$$

Accordingly to this, we can give

$$\begin{aligned}
\lambda_{\overline{X}} &= -\langle d, x\rangle = -[x_1 \oint k_2 ds + x_3 \oint k_1 ds] \qquad (87)\\
L_{\overline{X}} &= \langle d^\star, x\rangle + \langle d, x^\star\rangle = [x_1^\star \oint k_2 ds + x_1 \oint \lambda_1 ds + x_2 \oint \lambda_2 ds \qquad (88)\\
&\quad + x_3^\star \oint k_1 ds + x_3 \oint \lambda_3 ds]
\end{aligned}$$

where

$$\overline{X} = (\overline{x}_1, \overline{x}_2, \overline{x}_3),\ \overline{X^\star} = (\overline{x_1^\star}, \overline{x_2^\star}, \overline{x_3^\star}) \tag{89}$$

SPECIAL CASES

1. The case $\overline{X} = \overline{T}$;
In this case; $\overline{x}_1 = 1, \overline{x}_2 = \overline{x_3} = 0$. Thus,

$$\wedge_{\overline{T}} = -\left\langle \overline{D}, \overline{T} \right\rangle = -\oint (k_2 + \varepsilon\lambda_1) ds = -\oint k_2 ds - \varepsilon \oint \lambda_1 ds = L_{\overline{T}} + \varepsilon\lambda_{\overline{T}}. \tag{90}$$

2. The case $\overline{X} = \overline{N}$;
In this case; $\overline{x}_2 = 1, \overline{x}_1 = \overline{x_3} = 0$. Thus,

$$\wedge_{\overline{N}} = 0 \tag{91}$$

3. The case $\overline{X} = \overline{B}$;
In this case; $\overline{x_3} = 1, \overline{x}_1 = \overline{x}_2 = 0$. Thus,

$$\wedge_{\overline{B}} = -\left\langle \overline{D}, \overline{B} \right\rangle = -\oint k_1 ds - \varepsilon \oint \lambda_3 ds = \lambda_{\overline{B}} - \varepsilon L_{\overline{B}}. \tag{92}$$

4. Conclusions

In this study, at first we get Frenet motion on Euclide space, by taking two different curves.Then, investigating this motion on dual sphere, we give the generalization of Frenet motion on dual sphere. At this time, angles and lengths of pitch of closed ruled surfaces are presented. Some differences are found . Depending on this, important results are given about these dual spherical indicatrice curves.

References

[1] Bottema, O., Roth, B., *Theoretical Kinematics*. North-Holland Publishing Company, New York, (1979).

[2] Kılıç, E., Keles, S., *On Holditch's theorem and polar inertia momentum*, Commun. Fac. Sci. Univ. Ank. Ser. A, 43, 1-2, (1994), 41–47.

[3] Fenchel W., *On the Differential Geometry of Closed Space Curves*, Bull. Amer. Math. Soc., 57, (1951), 44–54.

[4] Gursoy, O., *The dual angle of pitch of a closed ruled surface*, Mech. Mach. Theory, 25, (1990), 131–140.

[5] Gugenheimer, H. W., *Differential Geometry*, McGraw-Hill, New York, (1956), pp. 162–169.

[6] Hacısalihoglu, H. H., *On the pitch of a closed ruled surface*, Mech. Mach. Theory, 7, (1972), 291–305.

[7] Hacısalihoglu, H. H., Abdel Baky, R. A. (1997) *Holditch's theorem for oneparameter closed motions*. Mech. Mach. Theory 32: 235–239

[8] Lipschutz M.M., *Diferential Geometry*, New York, USA, McGraw-Hill, (1969).

[9] McCarthy, J. Micheal, *An Introduction to Theoretical Kinematics*, MIT Press, Cambridge, (1990).

[10] Niku, Saed B.Niku, *Introduction to Robotics*, Prentice Hall Press, New Jersey, (2001).

[11] Phillip Azariadis, Nikos Aspragathos, *Computer graphics representation and transformation of geometric entities using dual unit vectors and line transformations Computers*, Graphics, 25, (2001), 195–209.

[12] Rashad A. Abdel-Baky, *One-Parameter Closed Dual Spherical Motions and Holditch's Theorem*, Sitzungsber. Abt. II, 214, (2005), 27–41.

[13] Rashad A. Abdel-Baky, *On the Blaschke Approach of Ruled Surface*, Tamkang Journal of Math., (2003), 34(2), 107–116.

[14] Schaaf, A., Ravani, B., *Geometric Continuity of Ruled Surfaces*, Computer Aided Geometric Design, 15, (1998), 289–310.

[15] Shifrin, T., *Differential Geometry: A first Course in Curves and Surfaces(Preliminary Version)*, University of Georgia, (2010).

[16] Study, E., *Geometry der Dynamen*, Lepzig, (1901).

[17] Veldkamp, G. R., *On the use of dual numbers, vectors, and matrices in instantaneous spatial kinematics*, Mech. Mach. Theory, 11, (1976), 141–156

[18] Yaylı Y., *On the Motion of Frenet Vectors and Spacelike Ruled Surfaces in the Minkowski 3-Space*, Mathematical and Computational Applications, 5, (2000), 49–55.

[19] Weiner, J. L, *Closed Curves of Constant Torsion-II*, Proc. Amer. Math. Soc., 67, 2, (1977), 306–308.

Yusuf YAYLI
Faculty of Science, Department of Mathematics
Ankara University
Ankara, TURKEY
email: *yayli@science.ankara.edu.tr*

Semra SARACOGLU
Faculty of Science and Arts, Department of Mathematics
Siirt University
Siirt, TURKEY
email: *semrasaracoglu65@gmail.com*