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SOME RESULTS ON GENERALIZED $(k, \mu)-$CONTACT METRIC MANIFOLDS

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Abstract. The object of the present paper is to study three-dimensional generalized $(k, \mu)-$contact metric manifolds with $\eta-$recurrent Ricci tensor and harmonic curvature tensor. $\phi-$Ricci symmetric generalized $(k, \mu)-$contact metric manifolds of dimension three are also considered. Each sections are followed by examples to illustrate the obtained results.

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1. Introduction

Considering k, μ as smooth functions T. Koufogiorgos and C. Tsichlias introduced the notion of generalized $(k, \mu)-$contact metric manifolds and gave several examples [12]. They also proved that such manifolds of dimension greater than three do not exist. $(k, \mu)-$contact metric manifolds with k and μ as functions have also been studied by several authors, viz, [5], [6], [7], [9], [10], [11],[12], [13]. However, in the present paper we study three-dimensional generalized $(k, \mu)-$contact metric manifolds with $\eta-$recurrent Ricci tensor and harmonic curvature tensor. $\phi-$Ricci symmetric three-dimensional generalized $(k, \mu)-$contact metric manifolds have also been considered. The present paper is organized as follows:

After the introduction and preliminaries, three-dimensional generalized $(k, \mu)-$contact metric manifolds with $\eta-$recurrent Ricci tensor have been studied in Section 3 and it is proved that a three-dimensional generalized $(k, \mu)-$contact metric manifold has $\eta-$recurrent Ricci tensor if and only if the manifold is generalized $N(k)-$contact. Section 4 deals with three-dimensional generalized $(k, \mu)-$contact metric manifold with harmonic curvature tensor. In this section we obtain that a three-dimensional generalized $(k,\mu)-$contact metric manifold with harmonic curvature tensor becomes $(k,\mu)-$contact. Section 5 is devoted to study $\phi-$Ricci symmetric generalized $(k,\mu)-$contact metric manifolds of dimension three. Here we prove that a three-dimensional generalized $(k,\mu)-$contact metric manifold is ϕ-Ricci symmetric if and only if μ is constant and as a corollary we prove that every three-dimensional $(k,\mu)-$contact metric manifold is $\phi-$Ricci symmetric. Every section contains illustrative examples which are related to the results obtained.

2. Preliminaries

Let M be a $(2n+1)-$dimensional $C^\infty-$differentiable manifold. The manifold is said to admit an almost contact metric structure (ϕ,ξ,η,g) if it satisfies the following relations:

$$\phi^2X = -X+\eta(X)\xi,\ \ \eta(\xi)=1,\ \ g(X,\xi)=\eta(X), \tag{1}$$

$$\phi\xi=0,\ \ \eta\phi=0,\ \ g(X,\phi Y)=-g(\phi X,Y),\ \ g(X,\phi X)=0, \tag{2}$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{3}$$

where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is an 1-form and g is a Riemannian metric on M. A manifold equipped with an almost contact metric structure is called an almost contact metric manifold. An almost contact metric manifold is called a contact metric manifold if it satisfies

$$g(X,\phi Y)=d\eta(X,Y).$$

Given a contact metric manifold $M(\phi,\xi,\eta,g)$, we consider a $(1,1)$ tensor field h defined by $h=\frac{1}{2}L_\xi\phi$, where L denotes the Lie differentiation. h is a symmetric operator and satisfies $h\phi=-\phi h$. If λ is an eigenvalue of h with eigenvector X, then $-\lambda$ is also an eigenvalue of h with eigenvector ϕX. Again, we have $\mathrm{tr}h=\mathrm{tr}\phi h=0$, and $h\xi=0$. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds [3]:

$$\nabla_X\xi=-\phi X-\phi hX,\ \ (\nabla_X\eta)Y=g(X+hX,\phi Y). \tag{4}$$

The vector field ξ is a Killing vector field with respect to g if and only if $h=0$. A contact metric manifold $M(\phi,\xi,\eta,g)$ for which ξ is a Killing vector is said to be a $K-$contact manifold. A $K-$contact structure on M gives rise to an almost complex structure on the product $M\times\mathbb{R}$. If this almost complex structure is integrable, the contact metric manifold is said to be Sasakian. Equivalently, a contact metric manifold is said to be Sasakian if and only if

$$R(X,Y)\xi=\eta(Y)X-\eta(X)Y$$

holds for all X, Y, where R denotes the Riemannian curvature tensor of the manifold M. The $(k,\mu)-$nullity distribution of a contact metric manifold $M(\phi,\xi,\eta,g)$ is a distribution [3]

$$\begin{aligned} N(k,\mu) &: p \to N_p(k,\mu) \\ &= \{Z \in T_p(M) : R(X,Y)Z \\ &\quad = k(g(Y,Z)X - g(X,Z)Y) + \mu(g(Y,Z)hX - g(X,Z)hY)\}, \end{aligned} \quad (5)$$

for any $X, Y \in T_pM$. Hence, if the characteristic vector field ξ belongs to the $(k,\mu)-$nullity distribution, we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \quad (6)$$

A contact metric manifold with ξ belonging to $(k,\mu)-$nullity distribution is called a $(k,\mu)-$contact metric manifold. If $k = 1$, $\mu = 0$, then the manifold becomes Sasakian [3]. In particular, if $\mu = 0$, then the notion of $(k,\mu)-$nullity distribution reduces to $k-$nullity distribution introduced by S. Tanno [18]. A contact metric manifold with ξ belonging to $k-$nullity distribution is known as $N(k)-$contact metric manifold.

In a $(2n+1)-$dimensional $(k,\mu)-$contact metric manifold we have the following [3]:

$$h^2 = (k-1)\phi^2, \;\; k \leq 1. \quad (7)$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX). \quad (8)$$

$$Q\phi - \phi Q = 2(2(n-1) + \mu)h\phi. \quad (9)$$

Lemma 2.1.[2] *A contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ with $R(X,Y)\xi = 0$, for all vector fields X, Y on the manifold and $n > 1$, is locally isometric to the Riemannian product $E^{n+1} \times S^n(4)$, and for $n = 1$ the manifold is flat.*

Lemma 2.2.[16] *Let M^{2n+1} be a contact metric manifold with harmonic curvature tensor and ξ belonging to the $(k,\mu)-$nullity distribution. Then M is either*

(i) *an Einstein Sasakian manifold, or,*

(ii) *an $\eta-$Einstein manifold, or,*

(iii) *locally isometric to the Riemannian product $E^{n+1} \times S^n(4)$ including a flat contact metric structure for $n = 1$.*

A generalized $(k,\mu)-$contact metric manifold $M^3(\phi,\xi,\eta,g)$ is a $(k,\mu)-$contact metric manifold in which k and μ are smooth functions on M^3. A generalized $(k,\mu)-$contact metric manifold does not exist for dimension greater than three [4], [9], [12]. In a generalized $(k,\mu)-$contact metric manifold we have the following [4], [9], [10], [12], [13]:

For a contact metric manifold $M^3(\phi, \xi, \eta, g)$ with $\xi \in N(k, \mu)$, where k and μ are functions, the Ricci operator Q is given by

$$QX = \frac{1}{2}(r - 2k)X + \frac{1}{2}(6k - r)\eta(X)\xi + \mu hX, \tag{10}$$

where

$$r = 2(k - \mu). \tag{11}$$

r denotes the scalar curvature of the manifold. Using (10) and (11) we can write the Ricci tensor of the manifold as

$$S(X, Y) = -\mu g(X, Y) + \mu g(hX, Y) + (2k + \mu)\eta(X)\eta(Y). \tag{12}$$

$$S(X, \xi) = 2k\eta(X). \tag{13}$$

Also

$$\begin{aligned}(\nabla_X h)Y &= [(1 - k)g(X, \phi Y) + g(X, h\phi Y)]\xi \\ &+ \eta(Y)h(\phi X + \phi hX) - \mu\eta(X)\phi hY.\end{aligned} \tag{14}$$

In addition with all the above formulas on a $(k, \mu)-$contact metric manifold, $\xi k = 0$, $\xi r = 0$, and $h\text{grad}\mu = \text{grad}k$.

If μ vanishes identically for a generalized $(k, \mu)-$contact metric manifold, then we call the manifold as generalized $N(k)-$contact metric manifold.

3. Generalized $(k, \mu)-$contact metric manifolds of dimension three with $\eta-$recurrent Ricci tensor

Definition 3.1. *The Ricci tensor of a three-dimensional generalized $(k, \mu)-$contact metric manifold M^3 is called $\eta-$recurrent if there exists an $1-$form A such that*

$$(\nabla_Z S)(\phi X, \phi Y) = A(Z)S(\phi X, \phi Y), \tag{15}$$

where A is defined by $g(Z, \rho) = A(Z)$, ρ is a unit vector field and X, Y, Z are arbitrary differentiable vector fields on the manifold.

If the $1-$form vanishes identically on the manifold, then the Ricci tensor is called $\eta-$parallel. The notion of $\eta-$parallel Ricci tensor was introduced by M. Kon [14] in the context of Sasakian manifold. From the definition, it follows that if the Ricci tensor is $\eta-$parallel, then it is $\eta-$recurrent with $A(Z) = 0$, but the converse is not true, in general. From (12), using (4) and (14) we get

$$\begin{aligned}(\nabla_Z S)(X,Y) &= (Z\mu)[g(hX,Y)-g(X,Y)]+(2(Zk)+(Z\mu))\eta(X)\eta(Y)\\ &+ (2k+\mu)[g(Z,\phi X)\eta(Y)+g(hZ,\phi X)\eta(Y)\\ &+ g(Z,\phi Y)\eta(X)+g(hZ,\phi Y)\eta(X)]\\ &+ \mu(1-k)g(Z,\phi Y)\eta(X)+\mu^2 g(hX,\phi Y)\eta(Z)\\ &+ \mu(1-k)g(Z,\phi X)\eta(Y)+g(hZ,\phi X)\eta(Y)\\ &+ \mu g(\phi Z,hY)\eta(X). \qquad (16)\end{aligned}$$

From (16) we have

$$(\nabla_Z S)(\phi X,\phi Y)=(Z\mu)(g(h\phi X,\phi Y)-g(\phi X,\phi Y))+\mu^2 g(h\phi X,\phi^2 Y)\eta(Z). \qquad (17)$$

Let the Ricci tensor of M^3 is $\eta-$recurrent. Then by (12), (15) and (17), we get

$$\begin{aligned}&(Z\mu)(g(h\phi X,\phi Y)-g(\phi X,\phi Y))+\mu^2 g(h\phi X,\phi^2 Y)\eta(Z)\\ =\ &A(Z)[-\mu g(\phi X,\phi Y)+\mu g(h\phi X,\phi Y)]. \qquad (18)\end{aligned}$$

In the preliminary section we have mentioned that for a generalized $(k,\mu)-$contact metric manifold of dimension three $\xi k=\xi r=0$. Hence, in view of (11), $\xi\mu=0$. In the equation (18), taking $Z=\xi\neq\rho$, we get

$$\mu[A(\xi)g(\phi X-h\phi X,\phi Y)+\mu g(h\phi X,\phi^2 Y)]=0.$$

The above equation yields $\mu=0$, because $A(\xi)g(\phi X-h\phi X,\phi Y)+\mu g(h\phi X,\phi^2 Y)$ is not zero for all values of X,Y. Hence, the manifold is generalized $N(k)-$contact metric manifold.

Conversely, suppose that the manifold is generalized $N(k)-$contact metric manifold. Then (12) yields

$$S(X,Y)=2k\eta(X)\eta(Y).$$

Therefore,

$$(\nabla_W S)(X,Y)=2k[(\nabla_W\eta)(X)\eta(Y)+\eta(X)(\nabla_W\eta)(Y)].$$

From the above equation we obtain

$$(\nabla_W S)(\phi X,\phi Y)=0.$$

Consequently, the Ricci tensor of the manifold is $\eta-$parallel and hence $\eta-$recurrent. Now, we are in a position to state the following:

Theorem 3.1. *A three-dimensional generalized $(k, \mu)-$contact metric manifold has $\eta-$recurrent Ricci tensor if and only if the manifold is generalized $N(k)-$contact.*

Example 3.1. In the paper [3] the authors gave examples of $(k, \mu)-$contact metric manifolds. Let us consider one of these examples in the following:

Consider $M = \{(x, y, z) \in R^3, (x, y, z) \neq (0, 0, 0)\}$, where (x, y, z) are the standard coordinates in R^3. Let M be generated by three linearly independent vector fields e_1, e_2 and e_3 satisfying

$$[e_2, e_3] = 2e_1, \qquad [e_3, e_1] = c_2 e_2, \qquad [e_1, e_2] = c_3 e_3, \tag{19}$$

where c_2, c_3 are smooth functions. Let $\{\omega_i\}$ be the dual $1-$form to the vector field $\{e_i\}$. Using (19) we get

$$d\omega(e_1, e_2) = -dw_1(e_3, e_2) = 1 \quad \text{and} \quad dw_1(e_i, e_j) = 0$$

for others i, j. We take $e_1 = \xi$. Define the Riemannian metric by $g(e_i, e_j) = \delta_{ij}$. Let $\phi e_3 = -e_2, \phi e_2 = e_3$. For g as an associated metric, we have $\phi^2 = -I + \omega_1 \otimes e_1$. Hence $M(\phi, e_1, \omega_1, g)$ is a contact metric manifold. By Koszul formula we can calculate the following:

$$\begin{array}{lll} \nabla_{e_1} e_1 = 0, & \nabla_{e_2} e_2 = 0, & \nabla_{e_3} e_3 = 0, \\ \nabla_{e_1} e_2 = \frac{1}{2}(c_2 + c_3 - 2)e_3, & \nabla_{e_2} e_1 = \frac{1}{2}(c_3 - c_2 - 2)e_3, & \nabla_{e_1} e_3 = -\frac{1}{2}e_2, \\ \nabla_{e_3} e_1 = \frac{1}{2}(2 + c_2 - c_3)e_2. & & . \end{array}$$

The non-vanishing components of the curvature tensor of the manifold can be calculated as

$$R(e_2, e_1)e_1 = [1 - \frac{(c_3 - c_2)^2}{4}]e_2 + (2 - c_2 - c_3)he_2,$$

$$R(e_3, e_1)e_1 = [1 - \frac{(c_3 - c_2)^2}{4}]e_3 + (2 - c_2 - c_3)he_3.$$

Here $k = 1 - \frac{(c_3 - c_2)^2}{4}, \quad \mu = 2 - c_2 - c_3$. In this example, if we choose $c_2 = c_3 = 1$ everywhere on the manifold, then $k = 1$ and $\mu = 0$. Hence, it is generalized $N(k)-$contact. From the components of the curvature tensor it follows that the non-vanishing component of the Ricci tensor is

$$\begin{aligned} S(e_1, e_1) &= g(R(e_1, e_1)e_1, e_1) + g(R(e_2, e_1)e_1, e_2) + g(R(e_3, e_1)e_1, e_3) \\ &= 2. \end{aligned}$$

From above, it easily follows that the manifold has η-parallel and hence recurrent Ricci tensor. Thus, we verify Theorem 3.1 by the above example.

4. Three-dimensional generalized $(k, \mu)-$contact metric manifolds with harmonic curvature tensor

Definition 4.1. *If the divergence of the Riemannian curvature tensor of a $(k, \mu)-$contact metric manifold is equal to zero, then this curvature tensor is called harmonic.*

A Riemannian manifold has harmonic curvature tensor if and only if the Ricci operator Q satisfies $(\nabla_X Q)Y - (\nabla_Y Q)X = 0$. $(k, \mu)-$contact metric manifolds with harmonic curvature tensor was studied by B. J. Papantoniou [16]. In this section we study three-dimensional generalized $(k, \mu)-$contact metric manifold with $\text{div}R = 0$.

Let us consider a three-dimensional generalized $(k, \mu)-$contact metric manifold with $\text{div}R = 0$. It is well known that

$$(\nabla_Z S)(X, Y) - (\nabla_X S)(Z, Y) = (\text{div}R)(X, Z)Y.$$

Hence, from (16) and the above equation we get

$$\begin{aligned}
& (Z\mu)[g(hX, Y) - g(X, Y)] + (2(Zk) + (Z\mu))\eta(X)\eta(Y) \\
+ \quad & (2k + \mu)[g(Z, \phi X)\eta(Y) + g(hZ, \phi X)\eta(Y) \\
+ \quad & g(Z, \phi Y)\eta(X) + g(hZ, \phi Y)\eta(X)] \\
+ \quad & \mu(1 - k)g(Z, \phi Y)\eta(X) + \mu^2 g(hX, \phi Y)\eta(Z) \\
+ \quad & \mu(1 - k)g(Z, \phi X)\eta(Y) + g(hZ, \phi X)\eta(Y) \\
+ \quad & \mu g(\phi Z, hY)\eta(X) \\
= \quad & (X\mu)[g(hZ, Y) - g(Z, Y)] + (2(Xk) + (X\mu))\eta(Z)\eta(Y) \\
+ \quad & (2k + \mu)[g(X, \phi Z)\eta(Y) + g(hX, \phi Z)\eta(Y) \\
+ \quad & g(X, \phi Y)\eta(Z) + g(hX, \phi Y)\eta(Z)] \\
+ \quad & \mu(1 - k)g(X, \phi Y)\eta(Z) + \mu^2 g(hZ, \phi Y)\eta(X) \\
+ \quad & \mu(1 - k)g(X, \phi Z)\eta(Y) + g(hX, \phi Z)\eta(Y) \\
+ \quad & \mu g(\phi X, hY)\eta(Z) \qquad (20)
\end{aligned}$$

In (20), putting $X = Y = \xi$, we get $Zk = 0$. Hence, k is constant. Again, $\text{div}R = 0$ implies r is constant. So, from (11), it follows that μ is constant.

Now, we are in a position to state the following:

Theorem 4.1. *A three-dimensional generalized $(k, \mu)-$contact metric manifold with harmonic curvature tensor reduces to $(k, \mu)-$contact metric manifold.*

By Theorem 4.1 and Lemma 2.2 of the preliminary section, we obtain the following:

Corollary 4.1. *Let $M^3(\phi, \xi, \eta, g)$ be a generalized $(k, \mu)-$contact metric manifold with harmonic curvature tensor. Then it is either*

(i) *an Einstein Sasakian manifold, or,*
(ii) *an $\eta-$Einstein manifold, or,*
(iii) *locally isometric to a flat contact structure.*

Example 4.1. Consider the manifold given in Example 3.1. The non-vanishing components of the curvature tensor of the manifold are

$$R(e_2,e_1)e_1=[1-\frac{(c_3-c_2)^2}{4}]e_2+(2-c_2-c_3)he_2,$$

$$R(e_3,e_1)e_1=[1-\frac{(c_3-c_2)^2}{4}]e_3+(2-c_2-c_3)he_3.$$

It is easy to verify that for this manifold

$$k=1-\frac{(c_3-c_2)^2}{4},\qquad \mu=2-c_2-c_3.$$

To make divergence of the curvature tensor of the manifold is equal to zero, we have to choose c_2, c_3 as constants. Then k, μ are constants, and Example 4.1 agrees with Theorem 4.1.

5. Locally $\phi-$Ricci symmetric three-dimensional generalized $(k,\mu)-$contact metric manifolds

Definition 5.1. *A three-dimensional generalized $(k,\mu)-$contact metric manifold is called $\phi-$Ricci symmetric if the Ricci operator Q satisfies*

$$\phi^2(\nabla_X Q)Y=0,$$

for any differentiable vector fields X, Y on the manifold. If X, Y are orthogonal to ξ, the manifold is called locally $\phi-$Ricci symmetric.

The notion of $\phi-$Ricci symmetric manifolds was introduced by U. C. De and A. Sarkar [8] in the context of Sasakian geometry. In this connection, it should be mentioned that the notion of $\phi-$symmetry was introduced by T. Takahashi [17] as a generalization of local symmetry. Till today symmetry of manifolds has been weakened by several authors in several ways. However, in this section we study locally $\phi-$Ricci symmetric generalized $(k,\mu)-$contact metric manifolds of dimension three.

By virtue of (10), and (11) we get

$$\begin{aligned}(\nabla_W Q)X &= -d\mu(W)X+d\mu(W)hX+\mu(\nabla_W h)X\\ &+ (2k+\mu)((\nabla_W\eta)(X)\xi+\eta(X)(\nabla_W\xi))\\ &+ (2dk(W)+d\mu(W))\eta(X)\xi. \qquad (21)\end{aligned}$$

By (4) and (14), the above equation yields

$$\begin{aligned}(\nabla_W Q)X &= -d\mu(W)X + d\mu(W)hX\\ &+ \mu((1-k)g(W,\phi X) + g(W,h\phi X))\xi\\ &+ \mu\eta(X)h(\phi W + \phi hW) - \mu^2\eta(W)\phi hX\\ &+ (2k+\mu)(g(W+hW,\phi X)\xi + \eta(X)(-\phi W - \phi hW))\\ &+ (2dk(W) + d\mu(W))\eta(X)\xi. \end{aligned} \tag{22}$$

From the equation (22) we obtain

$$\begin{aligned}\phi^2(\nabla_W Q)X &= -d\mu(W)\phi^2 X + d\mu(W)\phi^2(hX)\\ &+ \mu\eta(X)\phi^2(h(\phi W + \phi hW)) - \mu^2\eta(W)\phi^2(\phi hX)\\ &- (2k+\mu)\eta(X)\phi^2(\phi W + \phi hW). \end{aligned} \tag{23}$$

For X, Y, W orthogonal to ξ, the above equation gives

$$\phi^2(\nabla_W Q)X = d\mu(W)X - d\mu(W)(hX). \tag{24}$$

Now, suppose that the manifold is locally $\phi-$Ricci symmetric. Then we obtain from the above equation

$$d\mu(W)X - d\mu(W)(hX) = 0. \tag{25}$$

Taking inner product of (25) with Y we get

$$d\mu(W)g(X,Y) + d\mu(W)g(hX,Y) = 0. \tag{26}$$

In (26), putting $X = Y = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i, $i = 1,2,3$, we get

$$d\mu(W) = 0.$$

Consequently, μ is constant.

Conversely, suppose that μ is constant. Then from (24)

$$\phi^2(\nabla_W Q)X = 0,$$

where W, X are orthogonal to ξ. Therefore, the manifold is locally $\phi-$Ricci symmetric. Thus, we have the following:

Theorem 5.1. *A three-dimensional generalized $(k,\mu)-$contact metric manifold is locally $\phi-$Ricci symmetric if and only if μ is constant.*

For a $(k,\mu)-$contact metric manifold μ is always constant. So, we have the following:

Corollary 5.1. *A three-dimensional $(k,\mu)-$contact metric manifold is always locally $\phi-$Ricci symmetric.*

Example 5.1. In Example 4.1, choosing $c_2 = 0$ and $c_3 = 1$ we get $\mu = 1$. Hence by Theorem 5.1 the manifold is locally $\phi-$Ricci symmetric.

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