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CERTAIN MULTIPLE INTEGRAL RELATIONS INVOLVING GENERALIZED MELLIN-BARNES TYPE OF CONTOUR INTEGRAL

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Abstract. A remarkably large number of integrals involving a product of the variety of special functions have been investigated by many authors. Motivated by these avenues of works,very recently,Agarwal (Tamsui Oxf. J. Inf. Math. Sci.,27(4) (2011), 449-462) gave two interesting unified integral relations involving the product of the multivariable polynomial and two generalized Mellin-Barnes type of contour integrals. In the present sequel to the aforementioned investigations and some of the earlier works listed in the reference, we present two generalized integral relations involving the product of the two $\overline{H}$-functions due to Inayat-Hussain. Next, by suitably specializing the function f in the first integral relation, we have evaluated a multiple integral which is new and quite general in nature.Moreover, results for some particular values of the parameters are also pointed out.

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1. Introduction and preliminaries

The study of $\overline{H}$-functions has a very long history (see, *e.g.*, [9, 10]) and now stands on fairly firm footing through the research contributions of various authors (see, *e.g.*,[1,2,3,4,5,6,7,8,9,10] and [12]).The $\overline{H}$-functions are important special functions and their closely related ones are widely used in physics and engineering; therefore, they are of interest to physicists and engineers as well as mathematicians. So it looks natural that many research works on the further investigations of the $\overline{H}$-functions have recently come up. Numerous integral formulas involving a variety of special functions have been developed by many authors (see, *e.g.*, [9, 10]; for a very recent work, see also [1]).

Also many integral formulas associated with the $\overline{H}$-functions of several kinds have been presented (see, *e.g.*, [3, 1-7]; see also [2]).Those integrals involving $\overline{H}$-functions are not only of great interest to the pure mathematics, but they are often of extreme importance in many branches of theoretical and applied physics and engineering. Several methods for evaluating infinite integrals involving $\overline{H}$-functions have been known (see, *e.g.*, [6] and [8]). However, these methods usually work on a case-by-case basis.

Motivated by these avenues of applications, very recently,Agarwal [1] gave two interesting unified integral relations involving the product of the multivariable polynomial and two generalized Mellin-Barnes type of contour integrals. In the present sequel to the aforementioned investigations and some of the earlier works listed in the reference, we present two generalized integral relations involving the product of the two $\overline{H}$-functions due to Inayat-Hussain. Next, by suitably specializing the function f in the first integral relation, we have evaluated a multiple integral which is new and quite general in nature.Some interesting special cases and (potential) usefulness of our main results are also considered and remarked, respectively.

For our purpose, we begin by recalling some known functions and earlier works.
The $\overline{\mathrm{H}}$-function is defined and represented in the following manner (see, *e.g.*,[7]).

$$\overline{H}_{p,q}^{m,n}[z] = \overline{H}_{p,q}^{m,n}\left[z\left|{}^{(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}}_{(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}}\right.\right] = \frac{1}{2\pi i}\int_L z^{\xi}\overline{\phi}(\xi)d\xi \ (z \neq 0) \tag{1}$$

where

$$\overline{\phi}(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)} \tag{2}$$

It may be noted that the $\overline{\phi}(\xi)$ contains fractional powers of some of the gamma functions. Here z may be real or complex but is not equal to zero or an empty product is interpreted as unity; m, n, p, q are integers such that $1 \leq m \leq q, 0 \leq n \leq p$; $\alpha_j > 0(j = 1, ..., p), \beta_j > 0(j = 1, ..., q)$ and $a_j(j = 1, ..., p)$ and $b_j(j = 1, ..., q)$ are complex numbers. The exponents

$A_j(j=1,...,n) and B_j(j=m+1,...,q)$ take on non-integer values.

The nature of contour L, sufficient conditions of convergence of defining integral (1) and other details about the $\overline{\text{H}}$-function can be seen in the papers (see, *e.g.*,[7] and [8]).

The behavior of the $\overline{\text{H}}$-function for small values of $|z|$ follows easily from a result given by Rathie (see, *e.g.*,[12]):

$\overline{H}_{p,q}^{m,n}[z] = o\left(|z|^{\alpha}\right)$; where

$$\alpha = \underbrace{\min}_{1\leq j\leq m} \text{Re}\left(\frac{b_j}{\beta_j}\right), \ |z| \to 0 \tag{3}$$

The following series representation for the $\overline{\text{H}}$-function given by Saxena et al. (see, *e.g.*,[14]) will be required later on:

$$\overline{H}_{p,q}^{m,n}\left[z\left|{}^{(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}}_{(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}}\right.\right] = \sum_{k=0}^{\infty}\sum_{h=1}^{p}\overline{f}(\zeta)z^{\zeta} \tag{4}$$

where

$$\overline{f}(\zeta) = \frac{\prod\limits_{\substack{j=1\\ j\neq h}}^{m}\Gamma(b_j-\beta_j\zeta)\prod\limits_{j=1}^{n}\{\Gamma(1-a_j+\alpha_j\zeta)\}^{A_j}}{\prod\limits_{j=m+1}^{q}\{\Gamma(1-b_j+\beta_j\zeta)\}^{B_j}\prod\limits_{j=n+1}^{p}\Gamma(a_j-\alpha_j\zeta)}\frac{(-1)^k}{k!\beta_h}, \tag{5}$$

$$\zeta = \frac{b_h+k}{\beta_h} \tag{6}$$

$$\mu_1 = \sum_{j=1}^{m}|\beta_j| + \sum_{j=1}^{n}|\alpha_j A_j| - \sum_{j=m+1}^{q}|\beta_j B_j| - \sum_{j=n+1}^{p}|\alpha_j| > 0, \tag{7}$$

$0 < |z| < \infty.$

The multivariables general class of polynomials $S_{n_1,\cdots,n_r}^{m_1,\cdots,m_r}[x_1,\cdots,x_r]$defined and represented as follows(see, *e.g.*,[16]):

$$S_{n_1,\cdots,n_r}^{m_1,\cdots,m_r}[x_1,\cdots,x_r]=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\left\{\frac{(-n_i)_{m_ik_i}}{k_i!}A(k_1,\cdots,k_r)x_i^{k_i}\right\} \quad (8)$$

where $n_i, m_i = 1,\cdots; m_i \neq 0, \forall\, i \in 1,\cdots,r$; the coefficients $A(k_1,\cdots,k_r)$ $(k_i \geq 0)$ are arbitrary constant ,real or complex. The general class of polynomials (8) is capable of reducing to a number of familiar multivariable polynomials by suitably specializing the arbitrary coefficients $A(k_1,\cdots,k_r)$ $(k_i \geq 0)$.

In the sequel, we also required the following well known formula(see, *e.g.*,[13]):

$$\int_0^{\pi/2} cos(2u\theta)\,(sin\theta)^{\nu}\,d\theta=\frac{\Gamma(\frac{1}{2}-u)\Gamma(\frac{1}{2}+u)\Gamma(\nu+1)}{2^{\nu+1}\Gamma(\frac{\nu}{2}+u+1)\Gamma(\frac{\nu}{2}-u+1)} \quad (\Re(\nu)>0) \quad (9)$$

2.DOUBLE INTEGRAL RELATIONS

In this section, we established two integrals relations involving the products of the general class of polynomials of multivariables $S_{n_1,\cdots,n_r}^{m_1,\cdots,m_r}[x_1,\cdots,x_r]$ and two generalized Mellin Barnes type of contour integrals.

First Integral Relation

$$\int_0^{\infty}\int_0^{\infty} cos\left(2u\,tan^{-1}\frac{y}{x}\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right)^{\nu}\overline{H}_{p_1,q_1}^{m_1,n_1}\left[a\frac{y^{\sigma}}{(x^2+y^2)^{\sigma/2}}\Big|_{(b_j,\beta_j)_{1,m_1},(b_j,\beta_j;B_j)_{m_1+1,q_1}}^{(a_j,\alpha_j;A_j)_{1,n_1},(a_j,\alpha_j)_{n_1+1,p_1}}\right]$$

$$\overline{H}_{p_2,q_2}^{m_2,n_2}\left[by^{\omega}(x^2+y^2)^{\rho-\omega/2}\Big|_{(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;D_j)_{m_2+1,q_2}}^{(c_j,\gamma_j;C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}}\right] f(x^2+y^2)dx\,dy$$

$$=\frac{\Gamma(\frac{1}{2}\pm u)}{4\sqrt{\pi}}\sum_{t_1=0}^{\infty}\sum_{h_1=1}^{m_1}a^{\zeta}\overline{f}(\zeta)\int_0^{\infty}\overline{H}_{p_2+2,q_2+2}^{m_2,n_2+2}\left[bz^{\rho}\Big|_{Q}^{P}\right]f(z)dz \quad (10)$$

where $\overline{f}(\zeta)$ is given by (5) and

$$P=\left(\frac{1}{2}-\frac{\nu}{2}-\frac{\sigma\zeta}{2},\frac{\omega}{2};1\right),\left(-\frac{\nu}{2}-\frac{\sigma\zeta}{2},\frac{\omega}{2};1\right),(c_j,\gamma_j;C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}$$

$$Q=(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;D_j)_{m_2+1,q_2},\left(-\frac{\nu}{2}-\frac{\sigma\zeta}{2}+u,\frac{\omega}{2};1\right),\left(-\frac{\nu}{2}-\frac{\sigma\zeta}{2}-u,\frac{\omega}{2};1\right) \tag{11}$$

Second Integral Relation

$$\int_0^\infty\int_0^\infty cos\left(2u\,tan^{-1}\frac{y}{x}\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right)^\nu \overline{H}^{m_1,n_1}_{p_1,q_1}\left[a\frac{y^\sigma}{(x^2+y^2)^{\sigma/2}}\Big|^{(a_j,\alpha_j;A_j)_{1,n_1},(a_j,\alpha_j)_{n_1+1,p_1}}_{(b_j,\beta_j)_{1,m_1},(b_j,\beta_j;B_j)_{m_1+1,q_1}}\right]$$

$$\overline{H}^{m_2,n_2}_{p_2,q_2}\left[by^\omega(x^2+y^2)^{\rho-\omega/2}\Big|^{(c_j,\gamma_j;C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}}_{(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;D_j)_{m_2+1,q_2}}\right]$$

$$S^{m_1,\cdots,m_r}_{n_1,\cdots,n_r}\left[c_1\frac{y^{\eta_1}}{(x^2+y^2)^{\eta_1/2}},\cdots,c_r\frac{y^{\eta_r}}{(x^2+y^2)^{\eta_r/2}}\right]f(x^2+y^2)dx\,dy$$

$$=\frac{\Gamma(\frac{1}{2}\pm u)}{4\sqrt{\pi}}\sum_{t_1=0}^{\infty}\sum_{h_1=1}^{m_1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\left\{\frac{(-n_i)_{m_ik_i}}{k_i!}A(k_1,\cdots,k_r)c_i^{k_i}\right\} \tag{12}$$

where $\overline{f}(\zeta)$ is given by (5) and

$$P'=\left(\frac{1}{2}-\frac{\nu}{2}-\frac{\sigma\zeta}{2}-\sum_{i=1}^{r}\frac{\eta_ik_i}{2},\frac{\omega}{2};1\right),\left(-\frac{\nu}{2}-\frac{\sigma\zeta}{2}-\sum_{i=1}^{r}\frac{\eta_ik_i}{2},\frac{\omega}{2};1\right),$$
$$(c_j,\gamma_j;C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}$$

$$Q'=(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;D_j)_{m_2+1,q_2},\left(-\frac{\nu}{2}-\frac{\sigma\zeta}{2}-\sum_{i=1}^{r}\frac{\eta_ik_i}{2}+u,\frac{\omega}{2};1\right),$$
$$\left(-\frac{\nu}{2}-\frac{\sigma\zeta}{2}-\sum_{i=1}^{r}\frac{\eta_ik_i}{2}-u,\frac{\omega}{2};1\right) \tag{13}$$

Above integral relations are valid provided that $\Re(\nu, \sigma, \omega, \rho, \eta_i) > 0, \Re(1 + \nu + \sigma(b_j/\beta_j) + \omega(d_j/\delta_j)) > 0. i = 1, \cdots, r; u = 0, 1, ...$ and

$$|arg(a)| < \frac{1}{2}\Omega\pi; |arg(b)| < \frac{1}{2}\Omega'\pi \tag{14}$$

where

$$\Omega = \sum_{j=1}^{m_1} \beta_j + \sum_{j=1}^{n_1} A_j\alpha_j - \sum_{j=m_1+1}^{q_1} B_j\beta_j - \sum_{j=n_1+1}^{p_1} \alpha_j \tag{15}$$

and

$$\Omega' = \sum_{j=1}^{m_2} \delta_j + \sum_{j=1}^{n_2} C_j\gamma_j - \sum_{j=m_2+1}^{q_2} D_j\delta_j - \sum_{j=n_2+1}^{p_2} \gamma_j \tag{16}$$

To establish the integral relations (10) and (12), we required the following interesting basic integral formulas.

3.BASIC INTEGRALS

In this section, we established two basic integrals involving the products of the general class of polynomials of multivariables (8) and two generalized Mellin Barnes type of contour integrals.

First Basic Integral

$$\int_0^{\pi/2} cos(2u\theta)(sin\theta)^\nu \overline{H}_{p_1,q_1}^{m_1,n_1} \left[a(sin\theta)^\sigma \Big|_{(b_j,\beta_j)_{1,m_1},(b_j,\beta_j;B_j)_{m_1+1,q_1}}^{(a_j,\alpha_j;A_j)_{1,n_1},(a_j,\alpha_j)_{n_1+1,p_1}} \right]$$

$$\overline{H}_{p_2,q_2}^{m_2,n_2} \left[bz^\rho(sin\theta)^\omega \Big|_{(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;D_j)_{m_2+1,q_2}}^{(c_j,\gamma_j;C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}} \right] d\theta$$

$$= \frac{\Gamma(\frac{1}{2} \pm u)}{2\sqrt{\pi}} \sum_{t_1=0}^{\infty} \sum_{h_1=1}^{m_1} a^\zeta \overline{f}(\zeta) \overline{H}_{p_2+2,q_2+2}^{m_2,n_2+2} \left[bz^\rho \Big|_{Q}^{P} \right] \tag{17}$$

where $\overline{f}(\zeta)$ and P,Q are given by equation (5) and (11), respectively.

Second Basic Integral

$$\int_0^{\pi/2} cos(2u\theta)(sin\theta)^{\nu}\overline{H}^{m_1,n_1}_{p_1,q_1}\left[a(sin\theta)^{\sigma}\Big|^{(a_j,\alpha_j;A_j)_{1,n_1},(a_j,\alpha_j)_{n_1+1,p_1}}_{(b_j,\beta_j)_{1,m_1},(b_j,\beta_j;B_j)_{m_1+1,q_1}}\right]$$

$$\overline{H}^{m_2,n_2}_{p_2,q_2}\left[bz^{\rho}(sin\theta)^{\omega}\Big|^{(c_j,\gamma_j;C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}}_{(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;D_j)_{m_2+1,q_2}}\right] S^{m_1,\cdots,m_r}_{n_1,\cdots,n_r}\left[c_1(sin\theta)^{\eta_1},\cdots,c_r(sin\theta)^{\eta_r}\right]d\theta$$

$$= \frac{\Gamma(\frac{1}{2}\pm u)}{2\sqrt{\pi}}\sum_{t_1=0}^{\infty}\sum_{h_1=1}^{m_1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\left\{\frac{(-n_i)_{m_ik_i}}{k_i!}A(k_1,\cdots,k_r)c_i^{k_i}\right\}$$

$$a^{\zeta}\overline{f}(\zeta)\overline{H}^{m_2,n_2+2}_{p_2+2,q_2+2}\left[bz^{\rho}\Big|^{P'}_{Q'}\right] \quad (18)$$

where $\overline{f}(\zeta)$ and P', Q' are given by (5) and (13), respectively. and provided that the condition of validity of (17) and (18) are easily follow from those given in (10) and (12),respectively.

Proof: For convenience, let the left-hand side of (17) be denoted by $\mathcal{I}$. Applying (4) and (1) to (17) and changing the order of integrations and summations , we find

$$\mathcal{I} = \sum_{t_1=0}^{\infty}\sum_{h_1=1}^{m_1}\overline{f}(\zeta)a^{\zeta}\frac{1}{2\pi\, i}\int_L b^{\xi}z^{\rho\xi}\overline{\Phi}(\xi)\left\{\int_0^{\pi/2} cos(2u\theta)(sin\theta)^{\nu+\sigma\zeta+\omega\xi}d\theta\right\}d\xi \quad (19)$$

Now we can apply (9) , (19) becomes:

$$\mathcal{I} = \sum_{t_1=0}^{\infty}\sum_{h_1=1}^{m_1}\overline{f}(\zeta)a^{\zeta}\frac{1}{2\pi\, i}\int_L b^{\xi}z^{\rho\xi}\overline{\Phi}(\xi)\frac{\Gamma(\frac{1}{2}\pm u)\Gamma(\frac{1+\nu+\sigma\zeta+\omega\xi}{2})\Gamma(\frac{2+\nu+\sigma\zeta+\omega\xi}{2})}{2\sqrt{\pi}\Gamma(\frac{2+\nu+\sigma\zeta+\omega\xi\pm u}{2})}d\xi \quad (20)$$

This, in accordance with definition (1) , gives the required result (17). The proof of (18) is same on parallel lines.

Derivation of the Double Integral Relations (10) and (12): To prove the (10), we start with (17),by replacing $z\, by\, r^2$ and multiply both sides by $rf(r^2)$,then integrate the resulting equation with respect to r over the semi-infinite ray$(0,\infty)$, we find:

$$\int_0^\infty rf(r^2)\int_0^{\pi/2} cos(2u\theta)(sin\theta)^\nu \overline{H}_{p_1,q_1}^{m_1,n_1}\left[a(sin\theta)^\sigma\Big|_{(b_j,\beta_j)_{1,m_1},(b_j,\beta_j;B_j)_{m_1+1,q_1}}^{(a_j,\alpha_j;A_j)_{1,n_1},(a_j,\alpha_j)_{n_1+1,p_1}}\right]$$

$$\overline{H}_{p_2,q_2}^{m_2,n_2}\left[br^{2\rho}(sin\theta)^\omega\Big|_{(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;D_j)_{m_2+1,q_2}}^{(c_j,\gamma_j;C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}}\right]d\theta dr$$

$$=\int_0^\infty rf(r^2)\frac{\Gamma(\frac{1}{2}\pm u)}{2\sqrt{\pi}}\sum_{t_1=0}^{\infty}\sum_{h_1=1}^{m_1} a^\zeta \overline{f}(\zeta)\overline{H}_{p_2+2,q_2+2}^{m_2,n_2+2}\left[br^{2\rho}\Big|_{Q}^{P}\right]dr \quad (21)$$

By changing the polar coordinates occurring in the L.H.S of (21) in to the cartesian coordinates by means of the substitutions $x=rcos\theta, y=rsin\theta, r^2=x^2+y^2\, and\, \theta=tan^{-1}(\frac{y}{x})$ and putting $r^2=z$ in the R.H.S. of (21), we easily get the (10) after a little simplifications.

The proof of (12) is same on parallel lines.

Remark 1. If we reduce the $\overline{H}$-function to the familiar Fox H-function(see, for example [17] and set $H_{p_1,q_1}^{m_1,n_1}=1$, in (10) and give some suitable parametric replacement in the resulting identity, we can arrive at the known results due to Prasad and Ram [11] and Shrivastava [15]

4.APPLICATIONS

In this section, we established very interesting multiple integral relation by choosing suitably specializing the function f in the(10) and (12) , a large number of interesting double integrals can be easily evaluated. Here, we shall ,present only one integral which is new and quite general in nature.

If we set $f(z)=z^{\mu-1}H_{P,Q}^{M,N}\left[wz\Big|_{(f_j,F_j)_{1,Q}}^{(e_j,E_j)_{1,P}}\right]$ in (10) and making use of the known result given by Gupta and Soni(see, *e.g.*, [8]) , we obtain:

$$\int_0^\infty \int_0^\infty cos\left(2utan^{-1}\frac{y}{x}\right)(y)^\nu(x^2+y^2)^{\mu-\nu/2-1}H_{P,Q}^{M,N}\left[w(x^2+y^2)\Big|_{(f_j,F_j)_{1,Q}}^{(e_j,E_j)_{1,P}}\right]$$

$$\overline{H}_{p_1,q_1}^{m_1,n_1}\left[a\frac{y^\sigma}{(x^2+y^2)^{\sigma/2}}\Big|_{(b_j,\beta_j)_{1,m_1},(b_j,\beta_j;B_j)_{m_1+1,q_1}}^{(a_j,\alpha_j;A_j)_{1,n_1},(a_j,\alpha_j)_{n_1+1,p_1}}\right]$$

$$\overline{H}_{p_2,q_2}^{m_2,n_2}\left[by^\omega(x^2+y^2)^{\rho-\omega/2}\Big|_{(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;D_j)_{m_2+1,q_2}}^{(c_j,\gamma_j;C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}}\right]f(x^2+y^2)dx\,dy$$

$$= w^{-\mu}\frac{\Gamma(\frac{1}{2}\pm u)}{4\sqrt{\pi}}\sum_{t_1=0}^{\infty}\sum_{h_1=1}^{m_1}a^\zeta\overline{f}(\zeta)\overline{H}_{p_2+Q+2,q_2+P+2}^{m_2+N,n_2+M+2}\left[bw^{-\rho}\Big|_{Q^*}^{P^*}\right] \quad (22)$$

where

$$P^* = \left(\frac{1}{2}-\frac{\nu}{2}-\frac{\sigma\zeta}{2},\frac{\omega}{2};1\right),\left(-\frac{\nu}{2}-\frac{\sigma\zeta}{2},\frac{\omega}{2};1\right),(c_j,\gamma_j;C_j)_{1,n_2},$$
$$(1-f_j-F_j\mu,F_j\rho;1)_{1,M},(c_j,\gamma_j)_{n_2+1,p_2},(1-f_j-F_j\mu,F_j\rho)_{M+1,Q}$$

$$Q^* = (d_j,\delta_j)_{1,m_2},(1-e_j-E_j\mu,E_j\rho)_{1,N},(d_j,\delta_j;D_j)_{m_2+1,q_2},$$
$$(1-e_j-E_j\mu,E_j\rho;1)_{N+1,P},\left(-\frac{\nu}{2}-\frac{\sigma\zeta}{2}+u,\frac{\omega}{2};1\right),\left(-\frac{\nu}{2}-\frac{\sigma\zeta}{2}-u,\frac{\omega}{2};1\right) \quad (23)$$

provided that the following conditions are satisfied: $\rho>0, |arga|<\frac{1}{2}\Omega\pi, \Omega$ is given by (15) and $|argw|<\frac{1}{2}\Omega''\pi$

where

$$\Omega'' = \sum_{j=1}^{N}E_j - \sum_{j=N+1}^{P}E_j + \sum_{j=1}^{M}F_j - \sum_{j=M+1}^{Q}F_j > 0.$$

5.CONCLUDING REMARKS

The $\overline{H}$-functions is potentially useful in engineering and applied sciences. Here, we now, briefly consider some consequences of the results derived in the previous sections. Following Gupta and Soni (see, *e.g.*, [8]), when $m = 1, n = p, q = q + 1$ and $m = 1, n = p = 0, q = 2$ then $\overline{H}$-functions would reduce immediately to the extensively investigated generalized Wright hypergeometric function ${}_p\overline{\Psi}_q(\text{z})$ and generalized Wright Bessel functions $\overline{J}_\lambda^{\nu,\mu}(\text{z})$. Therefore, integral relations and basic integral formulas yield the various integral formulas for the generalized special functions. Furthermore, as an immediate consequence of the definition (1),we have the following special cases of the $\overline{H}$-functions:

$$
{}_p\overline{\Psi}_q\left[\begin{matrix}(a_j,\alpha_j;A_j);\\(b_j,\beta_j;B_j);\end{matrix}\ z\right] = \overline{H}^{1,p}_{p,q+1}\left[-z\Big|^{(1-a_j,\alpha_j;A_j)}_{(0,1),(1-b_j,\beta_j;B_j)}\right] \tag{24}
$$

and

$$
\overline{J}_\lambda^{\nu,\mu}(z) = \overline{H}^{1,0}_{0,2}\left[-z\Big|^{-}_{(0,1),(-\lambda,\nu;\mu)}\right] \tag{25}
$$

Therefore, if we give some suitable parametric replacement in the results drive in this paper then all results yield the various integral formulas for the generalized special functions.

The $\overline{H}$- functions defined by (1) , possess the advantage that a number of generalized special function happen to be the particular cases of these functions. Therefore, we conclude this paper with the remark that, the results deduced above are significant and can lead to yield numerous other integral formulas involving various special functions by the suitable specializations of arbitrary parameters in the integral relations. The results thus derived in this paper are general in character and likely to find certain applications in the theory of special functions.

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