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RADIUS PROBLEMS FOR CERTAIN SUBCLASSES OF MEROMORPHIC FUNCTIONS

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Abstract. For analytic functions $f(z)$ in the punctured open unit disk $\mathbb{D}$, subclasses $\mathfrak{M}(\beta_1, \beta_2, \beta_3; \lambda)$, $\Sigma_\theta^*(\alpha)$ and $\Sigma_\theta(\alpha, \beta)$ are introduced. The object of the present paper is to discuss some interesting properties of functions $f(z)$ associated with classes $\mathfrak{M}(\beta_1, \beta_2, \beta_3; \lambda)$. Radius problems for the classes $\Sigma_\theta^*(\alpha)$ and $\Sigma_\theta(\alpha, \beta)$ are also obtained.

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1. Introduction and definitions

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the punctured unit disk $\mathbb{D} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. We also write by $\mathbb{U}$ the open unit disk.

A function $f(z)$ in Σ is said to be meromorphically starlike of order α if and only if

$$\mathfrak{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \mathbb{U}), \tag{2}$$

for some $\alpha (0 \leq \alpha < 1)$. We denote by $\Sigma^*(\alpha)$ the class of all meromorphically starlike functions of order α.

The class $\Sigma^*(\alpha)$ and various other subclasses of Σ have been studied rather extensively by Nehari and Netanyahu [7], Clunie [3], Pommerenke [8, 9], Miller [6], Royster [10], Cho et al.[2], Aouf [1] and others.

A function $f(z)$ in Σ is said to be in the class $\Sigma(\alpha, \beta)$ if and only if

$$\mathfrak{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \beta\left|\frac{zf'(z)}{f(z)}+1\right|+\alpha \qquad (z\in\mathbb{U}), \tag{3}$$

for some $\alpha(0\le\alpha<1)$ and $\beta\ge 0$. Clearly, $\Sigma(\alpha,0)=\Sigma^*(\alpha)$.

Let $\mathfrak{M}(\beta_1,\beta_2,\beta_3;\lambda)$ be the subclass of Σ consisting of all functions $f(z)$ satisfying

$$\left|\beta_1(z^2f'(z)+1)+\beta_2 z(z^2f'(z))'+\beta_3 z^2(z^2f'(z))''\right|\le\lambda \qquad (z\in\mathbb{U}) \tag{4}$$

for some complex numbers β_1, β_2 and β_3, and for some real $\lambda>0$.

Example 1.1 Let us consider the function $f_\gamma(z)$ given by

$$f_\gamma(z)=\frac{1-z+z(1+z)^\gamma}{z},(\gamma\in\mathbb{R}).$$

Then we have

$$\begin{aligned}&\left|\beta_1(z^2f_\gamma'(z)+1)+\beta_2 z(z^2f_\gamma'(z))'+\beta_3 z^2(z^2f_\gamma'(z))''\right|\\ =&\left|\sum_{n=1}^{\infty}n\gamma n(\beta_1+(n+1)\beta_2+n(n+1)\beta_3)z^{n+1}\right|,\end{aligned}$$

where

$$\gamma n=\frac{\gamma(\gamma-1)(\gamma-2)...(\gamma-n+1)}{n!}.$$

Therefore, if $\gamma=1$, then

$$\left|\beta_1(z^2f_1'(z)+1)+\beta_2 z(z^2f_1'(z))'+\beta_3 z^2(z^2f_1'(z))''\right|=\left|(\beta_1+2\beta_2+2\beta_3)z^2\right|\le|\beta_1|+2|\beta_2|+2|\beta_3|.$$

This implies that $f_1(z)\in\mathfrak{M}(\beta_1,\beta_2,\beta_3;\lambda)$ for $\lambda\ge|\beta_1|+2|\beta_2|+2|\beta_3|$. If $\gamma=2$, then

$$\begin{aligned}&\left|\beta_1(z^2f_2'(z)+1)+\beta_2 z(z^2f_2'(z))'+\beta_3 z^2(z^2f_2'(z))''\right|\\ =&\left|2(\beta_1+2\beta_2+2\beta_3)z^2+2(\beta_1+3\beta_2+6\beta_3)z^3\right|\le 4|\beta_1|+10|\beta_2|+16|\beta_3|.\end{aligned}$$

Therefore, $f_2(z)\in\mathfrak{M}(\beta_1,\beta_2,\beta_3;\lambda)$ for $\lambda\ge 4|\beta_1|+10|\beta_2|+16|\beta_3|$. Further, if $\gamma=3$, then we have

$$\begin{aligned}&\left|\beta_1(z^2f_3'(z)+1)+\beta_2 z(z^2f_3'(z))'+\beta_3 z^2(z^2f_3'(z))''\right|\\ =&\left|3(\beta_1+2\beta_2+2\beta_3)z^2+6(\beta_1+3\beta_2+6\beta_3)z^3+3(\beta_1+4\beta_2+12\beta_3)z^4\right|\\ \le& 12|\beta_1|+36|\beta_2|+90|\beta_3|.\end{aligned}$$

Thus, $f_3(z)\in\mathfrak{M}(\beta_1,\beta_2,\beta_3;\lambda)$ for $\lambda\ge 12|\beta_1|+36|\beta_2|+90|\beta_3|$.

Now, let Σ_θ denote the subclass of Σ consisting of functions $f(z)$ with

$$a_n = |a_n| e^{i(n+1)\theta} \qquad (n = 1, 2, ...).$$

Also, we introduce the subclasses $\Sigma^*_\theta(\alpha)$ and $\Sigma_\theta(\alpha, \beta)$ of Σ_θ as follows:

$$\Sigma^*_\theta(\alpha) = \Sigma_\theta \cap \Sigma^*(\alpha) \;\; and \Sigma_\theta(\alpha, \beta) = \Sigma_\theta \cap \Sigma(\alpha, \beta).$$

Radius problems for some subclasses of analytic functions in the open unit disk $\mathbb{U}$ and also for some subclasses of meromorphic functions in the punctured unit disk $\mathbb{D}$ have been considered by several authors (see, for example, [4, 5, 11]). In this paper, we proved some properties of the subclass $\mathfrak{M}(\beta_1, \beta_2, \beta_3; \lambda)$ of meromorphic functions. Radius problems for the classes $\Sigma^*_\theta(\alpha)$ and $\Sigma_\theta(\alpha, \beta)$ are also derived.

2. Properties of the class $\mathfrak{M}(\beta_1, \beta_2, \beta_3; \lambda)$

We first prove

Theorem 2.1 If $f(z) \in \Sigma$ satisfies

$$\sum_{n=1}^{\infty} n(|\beta_1| + (n+1)|\beta_2| + n(n+1)|\beta_3|)|a_n| \leq \lambda \tag{5}$$

for some complex numbers β_1, β_2 and β_3, and for some real $\lambda > 0$, then $f(z) \in \mathfrak{M}(\beta_1, \beta_2, \beta_3; \lambda)$.

Proof. We observe that

$$\begin{aligned} & \left|\beta_1(z^2 f'(z) + 1) + \beta_2 z(z^2 f'(z))' + \beta_3 z^2 (z^2 f'(z))''\right| \\ = & \left|\sum_{n=1}^{\infty} n(\beta_1 + (n+1)\beta_2 + n(n+1)\beta_3) a_n z^{n+1}\right| \\ \leq & \sum_{n=1}^{\infty} n(|\beta_1| + (n+1)|\beta_2| + n(n+1)|\beta_3|)|a_n| |z|^{n+1} \\ < & \sum_{n=1}^{\infty} n(|\beta_1| + (n+1)|\beta_2| + n(n+1)|\beta_3|)|a_n|. \end{aligned}$$

Therefore, if $f(z)$ satisfies the inequality (5), then $f(z) \in \mathfrak{M}(\beta_1, \beta_2, \beta_3; \lambda)$.

Next, we prove

Theorem 2.2 If $f(z) \in \mathfrak{M}(\beta_1, \beta_2, \beta_3; \lambda)$ with $\arg \beta_1 = \arg \beta_2 = \arg \beta_3 = \phi$ and $a_n = |a_n| e^{i((n+1)\theta - \phi)}$ $(n = 1, 2, ...)$, then we have

$$\sum_{n=1}^{\infty} n(|\beta_1| + (n+1)|\beta_2| + n(n+1)|\beta_3|)|a_n| \leq \lambda.$$

Proof. Let $f(z) \in \mathfrak{M}(\beta_1, \beta_2, \beta_3; \lambda)$ with $\arg \beta_1 = \arg \beta_2 = \arg \beta_3 = \phi$ and $a_n = |a_n| e^{i((n+1)\theta - \phi)}$ $(n = 1, 2, ...)$. Then we see that

$$\begin{aligned} & \left|\beta_1(z^2 f'(z) + 1) + \beta_2 z(z^2 f'(z))' + \beta_3 z^2 (z^2 f'(z))''\right| \\ = & \left|\sum_{n=1}^{\infty} n(\beta_1 + (n+1)\beta_2 + n(n+1)\beta_3) a_n z^{n+1}\right| \\ = & \left|\sum_{n=1}^{\infty} n(|\beta_1| + (n+1)|\beta_2| + n(n+1)|\beta_3|)|a_n| e^{i(n+1)\theta} z^{n+1}\right| \le \lambda \end{aligned}$$

for all $z \in \mathbb{U}$. Let us consider a point $z \in \mathbb{U}$ such that $z = |z| e^{-i\theta}$. Then we have

$$\sum_{n=1}^{\infty} n(|\beta_1| + (n+1)|\beta_2| + n(n+1)|\beta_3|)|a_n| |z|^{n+1} \le \lambda.$$

Letting $|z| \to 1^-$, we obtain

$$\sum_{n=1}^{\infty} n(|\beta_1| + (n+1)|\beta_2| + n(n+1)|\beta_3|)|a_n| \le \lambda.$$

Corollary 2.3 If $f(z) \in \mathfrak{M}(\beta_1, \beta_2, \beta_3; \lambda)$ with $\arg \beta_1 = \arg \beta_2 = \arg \beta_3 = \phi$ and $a_n = |a_n| e^{i((n+1)\theta - \phi)}$ $(n = 1, 2, ...)$, then we have

$$|a_n| \le \frac{\lambda}{n(|\beta_1| + (n+1)|\beta_2| + n(n+1)|\beta_3|)} \qquad (n = 1, 2, ...).$$

Example 2.4 Let us consider the function $f(z) \in \mathfrak{M}(\beta_1, \beta_2, \beta_3; \lambda)$ with $\arg \beta_1 = \arg \beta_2 = \arg \beta_3 = \phi$ and

$$a_n = \frac{\lambda e^{i((n+1)\theta - \phi)}}{n^2(n+1)(|\beta_1| + (n+1)|\beta_2| + n(n+1)|\beta_3|)}, \qquad (n = 1, 2, ...).$$

Then we see that

$$\begin{aligned} \sum_{n=1}^{\infty} n(|\beta_1| + (n+1)|\beta_2| + n(n+1)|\beta_3|)|a_n| &= \lambda \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \\ &= \lambda \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lambda. \end{aligned}$$

Corollary 2.5 If $f(z) \in \mathfrak{M}(\beta_1, \beta_2, \beta_3; \lambda)$ with $\arg\beta_1 = \arg\beta_2 = \arg\beta_3 = \phi$ and $a_n = |a_n| e^{i((n+1)\theta-\phi)}$ $(n = 1, 2, ...)$, then we have

$$\frac{1}{|z|} - \sum_{n=1}^{j} |a_n| |z|^n - A_j |z|^{j+1} \leq |f(z)| \leq \frac{1}{|z|} + \sum_{n=1}^{j} |a_n| |z|^n + A_j |z|^{j+1}$$

with

$$A_j = \frac{\lambda - \sum\limits_{n=1}^{j} n(|\beta_1| + (n+1)|\beta_2| + n(n+1)|\beta_3|)|a_n|}{(j+1)(|\beta_1| + (j+2)|\beta_2| + (j+1)(j+2)|\beta_3|)}$$

and

$$\frac{1}{|z|^2} - \sum_{n=1}^{j} n|a_n| |z|^n - B_j |z|^j \leq |f'(z)| \leq \frac{1}{|z|^2} + \sum_{n=1}^{j} n|a_n| |z|^n + B_j |z|^j$$

with

$$B_j = \frac{\lambda - \sum\limits_{n=1}^{j} n(|\beta_1| + (n+1)|\beta_2| + n(n+1)|\beta_3|)|a_n|}{(|\beta_1| + (j+2)|\beta_2| + (j+1)(j+2)|\beta_3|)}.$$

Proof. In view of Theorem 2.1, we know that

$$\begin{aligned} &\sum_{n=j+1}^{\infty} n(|\beta_1| + (n+1)|\beta_2| + n(n+1)|\beta_3|)|a_n| \\ \leq\ & \lambda - \sum_{n=1}^{j} n(|\beta_1| + (n+1)|\beta_2| + n(n+1)|\beta_3|)|a_n|. \end{aligned}$$

Further, we note that

$$\begin{aligned} &(j+1)(|\beta_1| + (j+2)|\beta_2| + (j+1)(j+2)|\beta_3|) \sum_{n=j+1}^{\infty} |a_n| \\ \leq\ & \sum_{n=j+1}^{\infty} n(|\beta_1| + (n+1)|\beta_2| + n(n+1)|\beta_3|)|a_n|. \end{aligned}$$

Therefore, we see that

$$\sum_{n=j+1}^{\infty} |a_n| \leq \frac{\lambda - \sum\limits_{n=1}^{j} n(|\beta_1| + (n+1)|\beta_2| + n(n+1)|\beta_3|)|a_n|}{(j+1)(|\beta_1| + (j+2)|\beta_2| + (j+1)(j+2)|\beta_3|)} = A_j.$$

Thus, we have

$$\begin{aligned} |f(z)| &\le \frac{1}{|z|} + \sum_{n=1}^{j} |a_n| \, |z|^n + \sum_{n=j+1}^{\infty} |a_n| \, |z|^n \\ &\le \frac{1}{|z|} + \sum_{n=1}^{j} |a_n| \, |z|^n + A_j \, |z|^{j+1} \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\ge \frac{1}{|z|} - \sum_{n=1}^{j} |a_n| \, |z|^n - \sum_{n=j+1}^{\infty} |a_n| \, |z|^n \\ &\ge \frac{1}{|z|} - \sum_{n=1}^{j} |a_n| \, |z|^n - A_j \, |z|^{j+1} \, |z|^{j+1} . \end{aligned}$$

Next, we observe that

$$\begin{aligned} & (|\beta_1| + (j+2) \, |\beta_2| + (j+1)(j+2) \, |\beta_3|) \sum_{n=j+1}^{\infty} n \, |a_n| \\ \le & \sum_{n=j+1}^{\infty} n(|\beta_1| + (n+1) \, |\beta_2| + n(n+1) \, |\beta_3|) \, |a_n| \\ \le & \ \lambda - \sum_{n=1}^{j} n(|\beta_1| + (n+1) \, |\beta_2| + n(n+1) \, |\beta_3|) \, |a_n| , \end{aligned}$$

that is, that

$$\sum_{n=j+1}^{\infty} n \, |a_n| \le \frac{\lambda - \sum\limits_{n=1}^{j} n(|\beta_1| + (n+1) \, |\beta_2| + n(n+1) \, |\beta_3|) \, |a_n|}{(|\beta_1| + (j+2) \, |\beta_2| + (j+1)(j+2) \, |\beta_3|)} = B_j .$$

Therefore, we obtain that

$$\begin{aligned} \left|f'(z)\right| &\le \frac{1}{|z|^2} + \sum_{n=1}^{j} n \, |a_n| \, |z|^{n-1} + \sum_{n=j+1}^{\infty} n \, |a_n| \, |z|^{n-1} \\ &\le \frac{1}{|z|^2} + \sum_{n=1}^{j} n \, |a_n| \, |z|^{n-1} + B_j \, |z|^j \end{aligned}$$

and

$$\begin{aligned}\left|f'(z)\right| &\geq \frac{1}{|z|^2} - \sum_{n=1}^{j} n\,|a_n|\,|z|^{n-1} - \sum_{n=j+1}^{\infty} n\,|a_n|\,|z|^{n-1} \\ &\geq \frac{1}{|z|^2} - \sum_{n=1}^{j} n\,|a_n|\,|z|^{n-1} - B_j\,|z|^j\,.\end{aligned}$$

3.Radius problems for the class $\Sigma^*_\theta(\alpha)$

To obtain the radius problem for the class $\Sigma^*_\theta(\alpha)$, we need that following lemma.
Lemma 3.1 *If* $f(z) \in \Sigma^*_\theta(\alpha)$, then

$$\sum_{n=1}^{\infty}(n+\alpha)\,|a_n| \leq 1-\alpha. \tag{6}$$

Proof. Let $f(z) \in \Sigma^*_\theta(\alpha)$. Then, we have

$$\mathfrak{Re}\left\{\frac{-zf'(z)}{f(z)}\right\} = \mathfrak{Re}\left\{\frac{1-\sum\limits_{n=1}^{\infty} na_n z^{n+1}}{1+\sum\limits_{n=1}^{\infty} a_n z^{n+1}}\right\} = \mathfrak{Re}\left\{\frac{1-\sum\limits_{n=1}^{\infty} n\,|a_n|\,e^{i(n+1)\theta}z^{n+1}}{1+\sum\limits_{n=1}^{\infty} |a_n|\,e^{i(n+1)\theta}z^{n+1}}\right\} > \alpha$$

for all $z \in \mathbb{U}$. Let us consider a point $z \in \mathbb{U}$ such that $z = |z|\,e^{-i\theta}$. Then we have

$$\frac{1-\sum\limits_{n=1}^{\infty} n\,|a_n|\,|z|^{n+1}}{1+\sum\limits_{n=1}^{\infty} |a_n|\,|z|^{n+1}} > \alpha.$$

Letting $|z| \to 1^-$, we obtain

$$1-\sum_{n=1}^{\infty} n\,|a_n| \geq \alpha\left(1+\sum_{n=1}^{\infty}|a_n|\right),$$

which is equivalent to the inequality (6).

From the above lemma, we immediately have
Corollary 3.2 If $f(z) \in \Sigma^*_\theta(\alpha)$, then

$$|a_n| \leq \frac{1-\alpha}{n+\alpha} \qquad (n = 1, 2, ...).$$

Remark 3.3 If $f(z) \in \Sigma_\theta^*(\alpha)$, then

$$\sum_{n=1}^{\infty} n\,|a_n| \le \sum_{n=1}^{\infty}(n+\alpha)\,|a_n| \le 1-\alpha.$$

Applying Theorem 2.1 and Lemma 3.1, we derive

Theorem 3.4 If $f(z) \in \Sigma_\theta^*(\alpha)$ and $\delta \in \mathbb{C}$ $(0 < |\delta| < 1)$. Then the function $\delta f(\delta z) \in \mathfrak{M}(\beta_1, \beta_2, \beta_3; \lambda)$ for $(0 < |\delta| \le |\delta_0(\lambda)|)$, where $|\delta_0(\lambda)|$ is the smallest positive root of the equation

$$\begin{aligned} & \frac{|\delta|^2\,|\beta_1|}{1-|\delta|^2}\sqrt{1-\alpha} + \frac{|\delta|^2\,|\beta_2|\sqrt{2(|\delta|^2+2)}}{(1-|\delta|^2)^2}\sqrt{1-\alpha} \\ & + \frac{2\,|\delta|^2\,|\beta_3|\sqrt{2\,|\delta|^6+15\,|\delta|^4+12\,|\delta|^2+1}}{(1-|\delta|^2)^3}\sqrt{1-\alpha} \\ = \; & \lambda. \end{aligned}$$

in $0 < |\delta| < 1$.

Proof. For $f(z) \in \Sigma_\theta^*(\alpha)$, we see that

$$\delta f(\delta z) = \frac{1}{z} + \sum_{n=1}^{\infty} \delta^{n+1} a_n z^n$$

and

$$\sum_{n=1}^{\infty} n\,|a_n|^2 \le 1-\alpha.$$

To show that $\delta f(\delta z) \in \mathfrak{M}(\beta_1, \beta_2, \beta_3; \lambda)$, we need to prove that

$$\sum_{n=1}^{\infty} n(|\beta_1| + (n+1)\,|\beta_2| + n(n+1)\,|\beta_3|)\,|\delta|^{n+1}\,|a_n| \le \lambda.$$

Applying Cauchy-Schwarz inequality, we note that

$$\begin{aligned} & \sum_{n=1}^{\infty} n(|\beta_1| + (n+1)\,|\beta_2| + n(n+1)\,|\beta_3|)\,|\delta|^{n+1}\,|a_n| \\ \le \; & |\delta|\,|\beta_1| \left(\sum_{n=1}^{\infty} n\,|\delta|^{2n}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} n\,|a_n|^2\right)^{\frac{1}{2}} \end{aligned}$$

$$+|\delta||\beta_2|\left(\sum_{n=1}^{\infty} n(n+1)^2|\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} n|a_n|^2\right)^{\frac{1}{2}}$$
$$+|\delta||\beta_3|\left(\sum_{n=1}^{\infty} n^3(n+1)^2|\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} n|a_n|^2\right)^{\frac{1}{2}}$$
$$\leq |\delta||\beta_1|\left(\sum_{n=1}^{\infty} n|\delta|^{2n}\right)^{\frac{1}{2}}\sqrt{1-\alpha}+|\delta||\beta_2|\left(\sum_{n=1}^{\infty} n(n+1)^2|\delta|^{2n}\right)^{\frac{1}{2}}\sqrt{1-\alpha}$$
$$+|\delta||\beta_3|\left(\sum_{n=1}^{\infty} n^3(n+1)^2|\delta|^{2n}\right)^{\frac{1}{2}}\sqrt{1-\alpha}. \tag{7}$$

For $|x|<1$, it is easy to see that

$$\sum_{n=1}^{\infty} nx^n=\frac{x}{(1-x)^2}, \tag{8}$$

$$\sum_{n=1}^{\infty} n(n+1)^2x^n=\frac{2x^2+4x}{(1-x)^4}, \tag{9}$$

and

$$\sum_{n=1}^{\infty} n^3(n+1)^2x^n=\frac{8x^4+60x^3+48x^2+4x}{(1-x)^6}. \tag{10}$$

By using (8)-(10) with $|\delta|^2=x$, from (7), we obtain

$$\sum_{n=1}^{\infty} n(|\beta_1|+(n+1)|\beta_2|+n(n+1)|\beta_3|)|\delta|^{n+1}|a_n|$$
$$\leq \frac{|\delta|^2|\beta_1|}{1-|\delta|^2}\sqrt{1-\alpha}+\frac{|\delta|^2|\beta_2|\sqrt{2(|\delta|^2+2)}}{(1-|\delta|^2)^2}\sqrt{1-\alpha}$$
$$+\frac{2|\delta|^2|\beta_3|\sqrt{2|\delta|^6+15|\delta|^4+12|\delta|^2+1}}{(1-|\delta|^2)^3}\sqrt{1-\alpha}.$$

Now, let us consider the complex number δ $(0<|\delta|<1)$ such that

$$\frac{|\delta|^2|\beta_1|}{1-|\delta|^2}\sqrt{1-\alpha}+\frac{|\delta|^2|\beta_2|\sqrt{2(|\delta|^2+2)}}{(1-|\delta|^2)^2}\sqrt{1-\alpha}$$

$$
\begin{aligned}
&+\frac{2\left|\delta\right|^{2}\left|\beta_{3}\right|\sqrt{2\left|\delta\right|^{6}+15\left|\delta\right|^{4}+12\left|\delta\right|^{2}+1}}{(1-\left|\delta\right|^{2})^{3}}\sqrt{1-\alpha} \\
&= \lambda.
\end{aligned}
$$

If we define the function $h(|\delta|)$ by

$$
\begin{aligned}
h(|\delta|) &= \left|\delta\right|^{2}\left|\beta_{1}\right|(1-\left|\delta\right|^{2})^{2}\sqrt{1-\alpha}+\left|\delta\right|^{2}\left|\beta_{2}\right|(1-\left|\delta\right|^{2})\sqrt{2(\left|\delta\right|^{2}+2)(1-\alpha)} \\
&\quad +2\left|\delta\right|^{2}\left|\beta_{3}\right|\sqrt{(2\left|\delta\right|^{6}+15\left|\delta\right|^{4}+12\left|\delta\right|^{2}+1)(1-\alpha)}-\lambda(1-\left|\delta\right|^{2})^{3},
\end{aligned}
$$

then we have $h(0)=-\lambda<0$ and $h(1)=2\left|\beta_{3}\right|\sqrt{30(1-\alpha)}\geq 0$. This means that there exists some δ_0 such that $h(|\delta_0|)=0$ $(0<|\delta_0|<1)$. This completes the proof of the theorem.

4.Radius problem for the class $\Sigma_\theta(\alpha,\beta)$

Lemma 4.1 *If* $f(z)\in\Sigma_\theta(\alpha,\beta)$, then

$$
\sum_{n=1}^{\infty}(n(1+\beta)+\alpha+\beta)\left|a_n\right|\leq 1-\alpha. \tag{11}
$$

where $0\leq\alpha<1$ *and* $\beta\geq 0$.

Proof. Let $f(z)\in\Sigma_\theta(\alpha,\beta)$. Then, we have

$$
\Re\left\{-\frac{zf'(z)}{f(z)}\right\}>\beta\left|\frac{zf'(z)}{f(z)}+1\right|+\alpha
$$

or, equivalently

$$
\Re\left\{-\frac{zf'(z)}{f(z)}\right\}>\frac{\alpha+\beta}{1+\beta}
$$

Now, using the same technique as in the proof of Lemma 3.1, we get the required result.

From Lemma 4.1, we have the following corollary.

Corollary 4.2 If $f(z)\in\Sigma_\theta(\alpha,\beta)$, then

$$
\left|a_n\right|\leq\frac{1-\alpha}{n(1+\beta)+\alpha+\beta}\qquad(n=1,2,...).
$$

Remark 4.3 If $f(z)\in\Sigma_\theta(\alpha,\beta)$, then

$$
\sum_{n=1}^{\infty}n\left|a_n\right|\leq\sum_{n=1}^{\infty}(n(1+\beta)+\alpha+\beta)\left|a_n\right|\leq 1-\alpha.
$$

Applying Theorem 2.1 and Lemma 4.1 and using the same technique as in the proof of Theorem 3.4, we derive the following theorem.

Theorem 4.4 If $f(z) \in \Sigma_\theta(\alpha, \beta)$ and $\delta \in \mathbb{C}$ $(0 < |\delta| < 1)$. Then the function $\delta f(\delta z) \in \mathfrak{M}(\beta_1, \beta_2, \beta_3; \lambda)$ for $(0 < |\delta| \leq |\delta_0(\lambda)|)$, where $|\delta_0(\lambda)|$ is the smallest positive root of the equation

$$\begin{aligned} & \frac{|\delta|^2 |\beta_1|}{1-|\delta|^2}\sqrt{1-\alpha} + \frac{|\delta|^2 |\beta_2| \sqrt{2(|\delta|^2+2)}}{(1-|\delta|^2)^2}\sqrt{1-\alpha} \\ & + \frac{2|\delta|^2 |\beta_3| \sqrt{2|\delta|^6 + 15|\delta|^4 + 12|\delta|^2 + 1}}{(1-|\delta|^2)^3}\sqrt{1-\alpha} \\ = & \ \lambda. \end{aligned}$$

in $0 < |\delta| < 1$.

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